EXTRACTA MATHEMATICAE Vol. 11, Núm. 2, 282 – 287 (1996)

On a Property of Gram's Determinant

S.S. Dragomir and B. Mond

Dept. of Mathematics, Timisoara Univ., B-dul. V. Parvan 4, *Ro*-1900 *Timisoara, Romania*

School of Mathematics, La Trobe University, Bundoora, Victoria, 3983 , *Australia*

(Presented by C. Benítez)

AMS *Subject Class.* (1991): 26D99, 46C99 *Received March* 2, 1995

1. Introduction

Let $(H;(\cdot\,,\,\cdot))$ be an inner product space over the real or complex number field $\mathbb{K}$ and $\{x_1,\dots,x_n\}$ a system of vectors in H. Consider the Gram matrix $G(x_1,\dots,x_n) := [(x_i,x_j)]_{i,j=\overline{1,n}}$ and the Gram determinant

$$\Gamma(x_1,\dots,x_n) := \det G(x_1,\dots,x_n)\,.$$

The following inequality is well known in the literature as Gram's inequality (see, e.g., [4, p. 595]):

$$\Gamma(x_1,\dots,x_n) \geq 0\,. \tag{1.1}$$

Note that equality holds in (1.1) iff the system of vectors $\{x_1,\dots,x_n\}$ is linearly dependent.

A well known converse of this inequality is the following:

$$\Gamma(x_1,\dots,x_n) \leq \Pi_{i=1}^n \|x_i\|^2, \quad x_i \in H\ (i=\overline{1,n}) \tag{1.2}$$

known as Hadamard's inequality. Equality holds in (1.2) if and only if $(x_i,x_j) = \delta_{i,j}\|x_i\|\|x_j\|$ for all $i,j \in \{1,\dots,n\}$ (see [3]).

Some special but very interesting inequalities which involve Gram determinants are the following (see [4, p. 597]):

$$\frac{\Gamma(x_1,\dots,x_n)}{\Gamma(x_1,\dots,x_k)} \leq \frac{\Gamma(x_2,\dots,x_n)}{\Gamma(x_2,\dots,x_k)} \leq \cdots \leq \Gamma(x_{k+1},\dots,x_n) \tag{1.3}$$

$$\Gamma(x_1,\dots,x_n) \leq \Gamma(x_1,\dots,x_k)\Gamma(x_{k+1},\dots,x_n) \tag{1.4}$$

$$(1.5)\qquad \left[\Gamma(x_1+y_1,x_2,\dots,x_n)\right]^{1/2} \le \left[\Gamma(x_1,x_2,\dots,x_n)\right]^{1/2}+\left[\Gamma(y_1,x_2,\dots,x_n)\right]^{1/2}.$$

2. Results

For a fixed inner product $(\cdot\,,\,\cdot)$ on linear space H over $\mathbb{K}$, consider the Gram determinant

$$\Gamma((\cdot\,,\,\cdot);x_1,\dots,x_n)=\det\left[(x_i,x_j)\right]_{i,j=\overline{1,n}}\,.$$

We now state and prove our main result.

THEOREM 2.1. *Let $(\cdot\,,\,\cdot)_1$, $(\cdot\,,\,\cdot)_2$ be two inner products on the linear space H. Then one has the inequality:*

$$(2.1)\qquad \left[\Gamma((\cdot\,,\,\cdot)_1+(\cdot\,,\,\cdot)_2;x_1,\dots,x_n)\right]^{1/2} \ge \left[\Gamma((\cdot\,,\,\cdot)_1;x_1,\dots,x_n)\right]^{1/2}+\left[\Gamma((\cdot\,,\,\cdot)_2;x_1,\dots,x_n)\right]^{1/2}\ge 0$$

for all $x_i\in H$ $(i=\overline{1,n})$ and $n\ge 2$.

Proof. If $\{x_1,\dots,x_n\}$ is a system of linearly dependent vectors in H, the inequality becomes an identity.

Suppose that $\{x_1,\dots,x_n\}$ is linearly independent. With this assumption we can consider the map:

$$\gamma((\cdot\,,\,\cdot);x_1,\dots,x_n):=\frac{\Gamma((\cdot\,,\,\cdot);x_1,\dots,x_n)}{\Gamma((\cdot\,,\,\cdot);x_2,\dots,x_n)},\quad n\ge 2,$$

where $(\cdot\,,\,\cdot)$is an inner product on H.

It is well known that (see also [3] or [4]) we have the representation

$$\gamma((\cdot\,,\,\cdot);x_1,\dots,x_n)=d^2(x_1,H_{\overline{2,n}})=\inf_{x\in H_{\overline{2,n}}}\|x_1-x\|^2$$

where $H_{\overline{2,n}}$ is the linear space spanned by the linearly independent system of vectors $\{x_2,\dots,x_n\}$.

Let us prove the inequality:

$$(2.2)\qquad \gamma((\cdot\,,\,\cdot)_1+(\cdot\,,\,\cdot)_2;x_1,\dots,x_n) \ge \gamma((\cdot\,,\,\cdot)_1;x_1,\dots,x_n)+\gamma((\cdot\,,\,\cdot)_2;x_1,\dots,x_n),$$

where $(\cdot\,,\,\cdot)_i$ is as above, which is interesting in itself (see also [1]). We have:

$$\begin{aligned}\gamma((\cdot\,,\,\cdot)_1+(\cdot\,,\,\cdot)_2;x_1,\dots,x_n) &= \inf_{x\in H_{\overline{2,n}}}\left[\|x_1-x\|_1^2+\|x_1-x\|_2^2\right]\\ &\geq \inf_{x\in H_{\overline{2,n}}}\|x_1-x\|_1^2+\inf_{x\in H_{\overline{2,n}}}\|x_1-x\|_2^2\\ &= \gamma((\cdot\,,\,\cdot)_1;x_1,\dots,x_n)+\gamma((\cdot\,,\,\cdot)_2;x_1,\dots,x_n),\end{aligned}$$

i.e., the inequality (2.2).

We now prove another inequality which is also interesting in itself [1]:

$$\begin{aligned}\text{(2.3)}\qquad &\Gamma((\cdot\,,\,\cdot)_1+(\cdot\,,\,\cdot)_2;x_1,\dots,x_n)\\ &\qquad\geq \Gamma((\cdot\,,\,\cdot)_1;x_1,\dots,x_n)+\Gamma((\cdot\,,\,\cdot)_2;x_1,\dots,x_n)\end{aligned}$$

for all $(\cdot\,,\,\cdot)_i$ $(i=1,\,2)$ two inner products on H and $\{x_1,\dots,x_n\}\subset H$, $n\geq 2$.

Of course, we must only prove the result in the case when $\{x_1,\dots,x_n\}$ is linearly independent. We give a proof based on mathematical induction.

Let $n=2$. Then we have:

$$\begin{aligned}\Gamma((\cdot\,,\,\cdot)_1+(\cdot\,,\,\cdot)_2;x_1,x_2) &= (\|x_1\|_1^2+\|x_1\|_2^2)(\|x_2\|_1^2+\|x_2\|_2^2)\\ &\quad-|(x_1,x_2)_1+(x_1,x_2)_2|^2\\ &\geq \|x_1\|_1^2\|x_2\|_1^2+\|x_1\|_2^2\|x_2\|_2^2+\|x_1\|_1^2\|x_2\|_2^2+\|x_1\|_2^2\|x_2\|_1^2\\ &\quad-|(x_1,x_2)_1|^2-2|(x_1,x_2)_1(x_1,x_2)_2|-|(x_1,x_2)_2|^2\\ &= \Gamma((\cdot\,,\,\cdot)_1;x_1,x_2)+\Gamma((\cdot\,,\,\cdot)_2;x_1,x_2)+\|x_1\|_1^2\|x_2\|_2^2\\ &\quad+\|x_1\|_2^2\|x_2\|_1^2-2|(x_1,x_2)_1|\,|(x_1,x_2)_2|\\ &\geq \Gamma((\cdot\,,\,\cdot)_1;x_1,x_2)+\Gamma((\cdot\,,\,\cdot)_2;x_1,x_2)\\ &\quad+\left(\|x_1\|_1\|x_2\|_2-\|x_1\|_2\|x_2\|_1\right)^2\\ &\geq \Gamma((\cdot\,,\,\cdot)_1;x_1,x_2)+\Gamma((\cdot\,,\,\cdot)_2;x_1,x_2)\end{aligned}$$

since, by Schwarz's inequality,

$$\|x_1\|_1\|x_2\|_1\geq|(x_1,x_2)_1|\,,\quad \|x_1\|_2\|x_2\|_2\geq|(x_1,x_2)_2|$$

for all $x_1,\,x_2\in H$ and $(\cdot\,,\,\cdot)_i$ $(i=\overline{1,2})$ as above.

Now, suppose that the inequality (2.3) is true for all $(n-1)$ linearly independent vectors in H. If $\{x_1,\dots,x_n\}\subset H$ is linearly independent, then by the inequality (2.2) we have:

$$\begin{aligned}
&\Gamma((\cdot,\cdot)_1+(\cdot,\cdot)_2;x_1,\dots,x_n)\\
&\quad=\gamma((\cdot,\cdot)_1+(\cdot,\cdot)_2;x_1,\dots,x_n)\Gamma((\cdot,\cdot)_1+(\cdot,\cdot)_2;x_2,\dots,x_n)\\
&\quad\geq\gamma((\cdot,\cdot)_1;x_1,\dots,x_n)\Gamma((\cdot,\cdot)_1+(\cdot,\cdot)_2;x_2,\dots,x_n)\\
&\qquad+\gamma((\cdot,\cdot)_2;x_1,\dots,x_n)\Gamma((\cdot,\cdot)_1+(\cdot,\cdot)_2;x_2,\dots,x_n)\\
&\quad=\Gamma((\cdot,\cdot)_1;x_1,\dots,x_n)\frac{\Gamma((\cdot,\cdot)_1+(\cdot,\cdot)_2;x_2,\dots,x_n)}{\Gamma((\cdot,\cdot)_1;x_2,\dots,x_n)}\\
&\quad=\Gamma((\cdot,\cdot)_2;x_1,\dots,x_n)\frac{\Gamma((\cdot,\cdot)_1+(\cdot,\cdot)_2;x_2,\dots,x_n)}{\Gamma((\cdot,\cdot)_2;x_2,\dots,x_n)}\,.
\end{aligned}\tag{2.4}$$

But, by the inductive hypothesis we have

$$\Gamma((\cdot,\cdot)_1+(\cdot,\cdot)_2;x_2,\dots,x_n)\geq\Gamma((\cdot,\cdot)_1;x_2,\dots,x_n)+\Gamma((\cdot,\cdot)_2;x_2,\dots,x_n)$$

which shows us that

$$\frac{\Gamma((\cdot,\cdot)_1+(\cdot,\cdot)_2;x_2,\dots,x_n)}{\Gamma((\cdot,\cdot)_i;x_2,\dots,x_n)}\geq 1\,,\quad i=1,2$$

and thus, by the inequality (2.4), we obtain the superadditivity of $\Gamma(\cdot;x_1,\dots,x_n)$, i.e., the inequality (2.3).

Now, by the inequality (2.4) and by this superadditivity we can prove more, i.e.

$$\begin{aligned}
&\Gamma((\cdot,\cdot)_1+(\cdot,\cdot)_2;x_1,\dots,x_n)\\
&\quad\geq\Gamma((\cdot,\cdot)_1;x_1,\dots,x_n)\frac{\Gamma((\cdot,\cdot)_1;x_2,\dots,x_n)+\Gamma((\cdot,\cdot)_2;x_2,\dots,x_n)}{\Gamma((\cdot,\cdot)_1;x_2,\dots,x_n)}\\
&\qquad+\Gamma((\cdot,\cdot)_2;x_1,\dots,x_n)\frac{\Gamma((\cdot,\cdot)_1;x_2,\dots,x_n)+\Gamma((\cdot,\cdot)_2;x_2,\dots,x_n)}{\Gamma((\cdot,\cdot)_2;x_2,\dots,x_n)}\\
&\quad=\Gamma((\cdot,\cdot)_1;x_1,\dots,x_n)+\Gamma((\cdot,\cdot)_2;x_1,\dots,x_n)\\
&\qquad+\Gamma((\cdot,\cdot)_1;x_1,\dots,x_n)\frac{\Gamma((\cdot,\cdot)_2;x_2,\dots,x_n)}{\Gamma((\cdot,\cdot)_1;x_2,\dots,x_n)}\\
&\qquad+\Gamma((\cdot,\cdot)_2;x_1,\dots,x_n)\frac{\Gamma((\cdot,\cdot)_1;x_2,\dots,x_n)}{\Gamma((\cdot,\cdot)_2;x_2,\dots,x_n)}\,.
\end{aligned}\tag{2.5}$$

But a simple inequality for real numbers shows us:

$$\begin{aligned}\Gamma((\cdot,\cdot)_1;x_1,\dots,x_n)\frac{\Gamma((\cdot,\cdot)_2;x_2,\dots,x_n)}{\Gamma((\cdot,\cdot)_1;x_2,\dots,x_n)} \\ +\Gamma((\cdot,\cdot)_2;x_1,\dots,x_n)\frac{\Gamma((\cdot,\cdot)_1;x_2,\dots,x_n)}{\Gamma((\cdot,\cdot)_2;x_2,\dots,x_n)} \\ \geq 2\left[\Gamma((\cdot,\cdot)_1;x_1,\dots,x_n)\Gamma((\cdot,\cdot)_2;x_1,\dots,x_n)\right]^{1/2}\end{aligned}$$

which gives, from inequality (2.5), that

$$\begin{aligned}\Gamma((\cdot,\cdot)_1+(\cdot,\cdot)_2;x_1,\dots,x_n) \\ \geq \left(\left[\Gamma((\cdot,\cdot)_1;x_1,\dots,x_n)\right]^{1/2}+\left[\Gamma((\cdot,\cdot)_2;x_1,\dots,x_n)\right]^{1/2}\right)^2\end{aligned}$$

i.e., the desired inequality (2.1). ∎

Remark 2.2. If the system $\{x_1,\dots,x_n\}$ is linearly independent and $(\cdot,\cdot)_2 > (\cdot,\cdot)_1$, i.e. $\|x\|_2 > \|x_1\|$ for all $x \in H\setminus\{0\}$, we have the monotonicity property [1]:

$$\gamma((\cdot,\cdot)_2;x_1,\dots,x_n) \geq \gamma((\cdot,\cdot)_1;x_1,\dots,x_n) \geq 0\,. \tag{2.6}$$

Indeed, by the inequality (2.2), we have:

$$\begin{aligned}\gamma((\cdot,\cdot)_2;x_1,\dots,x_n) &= \gamma((\cdot,\cdot)_{2,1}+(\cdot,\cdot)_1;x_1,\dots,x_n) \\ &\geq \gamma((\cdot,\cdot)_{2,1};x_1,\dots,x_n)+\gamma((\cdot,\cdot)_1;x_1,\dots,x_n)\end{aligned}$$

where $(\cdot,\cdot)_{2,1} = (\cdot,\cdot)_2 - (\cdot,\cdot)_1$ is an inner product on H.

Remark 2.3. If $(\cdot,\cdot)_2 \geq (\cdot,\cdot)_1$, i.e., $\|x\|_2 \geq \|x\|_1$ for all $x \in H$, and $\{x_1,\dots,x_n\} \subset H$, then

$$\Gamma((\cdot,\cdot)_2;x_1,\dots,x_n) \geq \Gamma((\cdot,\cdot)_1;x_1,\dots,x_n) \geq 0 \tag{2.7}$$

i.e. the monotonocity of Gram's determinant [1].

For other related results, see the papers [1] and [2] where further applications and consequences are given.

Acknowledgements

The first author is grateful to La Trobe University for its support on this research during the summer of 1994.

References

[1] DRAGOMIR, S.S., MOND, B., On the superadditivity and monotonicity of Gram's inequality and related results, *Acta Math. Hungarica*, in press.

[2] DRAGOMIR, S.S., MOND, B., PALES, Z.S., On a superadditivity property of Gram's determinant, preprint.

[3] FURUTA, T., An elementary proof oh Hadamard theorem, *Math. Vesnik* **8** (23) (1971), 267–269.

[4] MITRINOVIĆ, D.S. PEČARIĆ, J.E., FINK, A.M., "Classical and New Inequalities in Analysis", Kluwer Academic Publisher, Dordrecht-Boston-London, 1992.