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A Proof of Universality of Arc Length as Time Parameter in Kepler Problems

Luis Floría

Grupo de Mecánica Celeste, Depto. de Matemática Aplicada a la Ingeniería, E. T. S. de Ingenieros Industriales, Universidad de Valladolid, E-47 011 *Valladolid, Spain*

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1. Introduction and Motivation

Within the framework of the *elliptic–type two–body motion*, Ferrándiz & Ferrer (1986) and Ferrándiz, Ferrer & Sein–Echaluce (1987) replaced the physical time by a fictitious time (in fact, a family of them) as the new independent variable for the integration of perturbed Kepler problems. They used the method of *linearization* by means of a time transformation on introducing a family of generalized anomalies defined via a differential relation in which the linearizing function depends on certain parameters which can be taken as functions of some orbital elements. In the case of pure Kepler problems, they analytically integrated their time transformations, in closed form, by means of elliptic integrals and functions. Their developments were originally intended to facilitate the analytical treatment of dynamical problems (and particularly, the study of artificial Earth satellite orbits under a zonal model of geopotential on the basis of some radial intermediaries), and as a preconditioning of the problem prior to numerical integration.

Within the same framework, and for *analytical step regulation* (see Section 2 below) in numerical integration of highly eccentric orbits, E. V. Brumberg (1992) proposed the use of *orbital length of arc* as independent variable. He also replaced the physical time t by a pseudo–time introduced via a differential relation generalizing the well–known and widely used Sundman transformation (Sundman 1912, p. 127 and p. 174), and made to fit his derivation into a more general pattern of two–parameter time transformations resembling that of Ferrándiz and his collaborators. In fact, comparison with the results attained by these authors reveals that the parameter used by Brumberg does not belong to the class of the generalized elliptic anomalies considered by them, and such a parameter cannot be obtained from the proposals due to these authors, although it is related to t by an equation similar to theirs.

Brumberg himself (1992, p. 325) pointed out that his time transformation, say his Formula [14], is applicable to *any kind of Keplerian orbit.* Nevertheless, neither a proof nor the least hint was adduced in this sense. In fact, his derivation was drastically limited to, and strictly based on, the explicit consideration of geometrical and dynamical relations and properties holding for ellipses. To start with, we intend to *modify and adapt* his elliptic motion derivation and treatment of the problem in order to take into account *other cases of Keplerian orbits.*

Accordingly, the motivating point at the origin of the present research was the question *whether* (and *how*) Brumberg's (1992) study could be rendered applicable to the orbital length of arc in the case of *non–elliptic* Keplerian motion, and particularly on a hyperbola or a parabola. *These questions can be answered on the stage of a universal formulation of two–body motion.* Thus, the aim of the present paper is to make a *general* derivation of the length of orbital arc as independent variable, *universally valid* for Kepler problems.

For our current purposes, the question considered here is, in principle, of theoretical interest. Examples of application of sets of universilly valid formulae, in terms of universal variables and parameters, for use in various real astronomical or astrodynamical situations will be postponed until future work. Anyway, further analytical or semianalytical developments and numerical studies will be treated by other members of the Grupo de Mecánica Celeste at the Departamento de Matemática Aplicada a la Ingeniería (Universidad de Valladolid.)

For these reasons we consider that *a justification and a rigorous extension* of Brumberg's approach is pertinent for future practical applications, specially for the derivation of analytical expressions when constructing a perturbation theory for highly–eccentric orbits.

Since *universal–like functions* provide an adequate and powerful tool for the study of problems of orbital motion, and particularly for a *compact* representation and treatment of analytical solutions of the two–body problem, we shall devote (Section 4) some attention to a brief glance at their definition and basic properties. It must be mentioned that the use of universal functions, along with appropriate changes of integration variable, will allow us to reduce the integration of the reparametrizing transformation to that of some *algebraic functions.*

2. Why Reparametrizations of Time?

It is shown in traditional and elementary reference texts that the time dependence of the variables in the solution of the problem of two bodies cannot be expressed by closed form functions, i. e. the classical Kepler problem cannot be analytically solved in terms of closed form, explicit functions of the time. However, closed–form exact analytical solutions can be obtained with the help of adequate independent arguments other than time, although such arguments cannot be expressed as closed–form, explicit functions of time. As a general rule, obtaining the value of these parameters for a given time requires either the inversion of a transcendental equation or its numerical solution.

Moreover, in many practical cases for space research, apart from theoretical investigations concerning perturbed motion, the *improvement of numerical integrations* is also to be taken into account. In general, the integration accuracy along the orbit is not uniform. For instance, integrating highly eccentric orbits involves fast varying functions; nevertheless, some transformations of the independent variable can smooth out a part of the variations of the functions in such a way that a constant step–size method can also be used.

To this end, while seeking numerical solutions, for large eccentricities in the elliptic case (in fact, for $e > 0.2$), the spread of points on the orbit (for a constant time step) is improved after appropriate changes of the independent variable generalizing a transformation due to Sundman (1912, p. 127 and p. 174.)

In this respect, one of the main advantages of choosing other independent variables is the improvement in the distribution of the integration points, without disregarding the fact that rigorous application of certain algorithms requires other time parameters to be used as independent variables. We emphasize the practical importance of such *smoothing transformations* in order to improve the performance of standard integration methods.

In other words: to ensure a sufficiently *smooth step–size distribution* along the orbit for equidistant values of the independent argument, and avoid an unreasonable accumulation of integration steps in the neighbourhood of certain points and their excessive dissemination along other parts of the orbit, i. e. to achieve an adequate *analytical step–size regulation*, a reparametrization of time can be performed.

Consequently, in order to obtain the solutions of various problems involving gravitational dynamical systems, defining an analytical step–size regulation was one of the purposes of introducing reparametrizations of time.

As for our own interest, we are mainly concerned with analytical aspects of Orbital Dynamics of artificial Earth satellites, and leave to our colleagues the merit of computing highly eccentric Earth satellite orbits with special perturbation methods.

As a conclusion of the preceding remarks, for the numerical computation of two–body orbits, the introduction of certain adequate independent variables achieves an analytical step–size regulation. In addition to this, they have also been employed in *analytical* and *qualitative* studies.

3. Why Universal Formulae?

In this paper we aim to show that the approach taken by Brumberg to use the length of arc is not limited to the case of elliptic motion. Consequently, the present research is devoted to a *general and systematic derivation* of this kind of adapted time parameter within a *universal formulation* of the two–body problem, which leads to a *unified treatment and compact representation* of the motion. This approach is intended in the following sense: irrespective of the nature of the specific Keplerian orbit at hand, Brumberg's developments will be generalized and adapted to yield a uniform treatment of Kepler motion (also in line with the contents of Battin 1987, §4.5 and §4.6; Stiefel & Scheifele 1971, §11; Stumpff 1959, Chapter V, §41), which suggests a *nonsingular transition between different types of two–body orbits.*

In particular, we have in mind the universal–variable formulation and analytical treatment of *perturbed* Keplerian dynamical systems (e. g. the problem of perturbed highly eccentric elliptic orbits of artificial satellites), and the *transition between reference orbits of different nature while performing perturbation studies*, especially when a universal–like independent argument is put in the place of the independent variable.

In this context of studies concerning elliptic–type orbital motion, highly eccentric orbits are clearly *close to the bifurcation case represented by parabolic motion*, and it is well known (Stiefel & Scheifele 1971, §11, p. 42) that *the type of orbit is occasionally changed by perturbing forces acting during a finite interval of time.*

Thus, parabolic orbits are of great interest since they enjoy a singular property: only one highly special value of the energy results in parabolic motion. When this value is *close to zero* the actual orbit might be an ellipse (small negative value for the energy), a parabola (exactly zero energy) or a hyperbola (small positive energy value.) The trouble is that the process of determination or evaluation of dynamical variables is usually vitiated by *uncertainties* *or errors* of diverse origin (astronomical data and observations, initial conditions, numerical calculations, etc.), from which the constant of energy is determined only approximately. Therefore, from a theoretical point of view, parabolic orbits are of significant interest because they constitute examples of *non-predictable orbits*: a possible failure of predictability in establishing the nature and evolution of motion might be expected. Moreover, this lack of predictability brings up some other questions and consequences relating to qualitative properties of near parabolic orbits, such as stability.

In addition to this, the picture is considerably complicated if some *other disturbing effects* are allowed for in a model of problem of two bodies: atmospheric drag, solar radiation pressure, propulsion, perturbations due to other external bodies, etc.

We intend to *soften* some of these undesirable and troublesome effects by establishing sets of universally valid formulae within the framework of an encompassing treatment of two-body problems. It is expected that our option for future analytical developments in working out perturbation theories will benefit from the generality and intrinsic merits of a universal approach.

With this general aim in view, an essential analytical tool is provided by certain classes of *special functions*, the so-called *Stumpff c-functions* (Stumpff 1959, vol. I, §37 and §41; Stiefel & Scheifele 1971, §11, pp. 43–45) and *universal U-functions* (Battin 1987, §4.5 and §4.6). These functions can be contemplated as generalizations of the standard trigonometric and hyperbolic functions, and their application is intended to avoid having to distinguish between elliptic, parabolic or hyperbolic motion. Elliptic integrals and functions will also play a significant role in certain developments.

4. Universal Functions and Some Basic Properties

The following definitions and properties can be found in, or easily derived from, the references Stiefel & Scheifele (1971) §11, pp. 50–51, Stumpff (1959) §37 and §41, Battin (1987) §4.5 and §4.6.

A traditional technique to introduce certain families of special functions resorts to the study of *power series solutions* to linear differential equations. The *Stumpff c-functions* (Stumpff 1959, §37, §41; Stiefel & Scheifele 1971, §11) are a family of transcendental functions whose first members integrate, under a unified treatment, the model of second-order linear differential equations with constant coefficients

$$d^2y/ds^2 + \varrho y = 0,$$

whatever the sign of the parameter ϱ. It should be borne in mind that, after appropriate changes of the dependent and independent variables, this is just the type of equation to which the differential equations of motion governing the Kepler problem can be reduced, the parameter ϱ being then related to the value of the energy of the two–body system.

With the notation $z = \varrho s^2$, the general solution to the above equation can be represented as a linear combination of the Stumpff c–functions $c_0(z)$ and $c_1(z)$, this representation being independent of the sign and value of ϱ. In general, these functions obey the *defining relation*

$$(1) \qquad c_n(z) = \sum_{k=0}^{\infty}(-1)^k \frac{z^k}{(2k+n)!}, \quad n = 0,1,2,\ldots, \text{ with } z = \varrho s^2,$$

the power series being absolutely convergent for all values of the complex variable z (whence the series converge for all s regardless of ϱ.) In particular, they are real–valued functions for real values of z.

Some calculations involving these functions are simplified if the alternative *universal functions* introduced by *Battin* are used. To this end, for each $n = 0, 1, 2, \ldots$, put (Stiefel & Scheifele 1971, §11, Formula [36])

$$(2) \qquad U_n(s,\varrho) \equiv s^n c_n(\varrho s^2) = \sum_{k=0}^{\infty}(-1)^k \varrho^k \frac{s^{2k+n}}{(2k+n)!}, \quad n = 0,1,2,\ldots.$$

Other equivalent defining relations can be found, e. g. in Battin (1987), §4.5. For future reference, we quote some useful properties and identities:

$$(3) \qquad dU_n/ds = U_{n-1}, \quad n = 1,2,3,\ldots,$$

$$(4) \qquad 1 = U_0^2 + \varrho U_1^2; \quad U_1^2 = U_2\,(1+U_0)\,, \quad U_1^2 = 2U_2 - \varrho U_2^2.$$

For convenience, the quantity

$$(5) \qquad L = \mu(1-e)/2q$$

is the negative of the energy of the Keplerian orbit at issue (see Stiefel & Scheifele 1971, p. 50, Formula [64]), where μ represents the gravitational body-centric parameter of the two–body system, e is the eccentricity, and q stands for the *distance of the pericentre*. When needed, we take $\varrho = 2L$.

In terms of s as the argument of universal functions, the two–body problem admits closed form representation of its analytical *solution* in a compact, unified and universally valid (say, whatever the type of orbit) form. Indeed, using

Cartesian rectangular coordinates (x, y) in the orbital plane, a *closed form solution* for these orbital coordinates and the modulus r of the radius vector in the two–body problem in terms of the *universal eccentric–like anomaly* s as the independent argument can be presented with the help of universal–like functions, and reads:

$$x = q - \mu s^2 c_2(2Ls^2) = q - \mu U_2(s, 2L), \tag{6}$$

$$y = \sqrt{\mu q(1+e)}\, s c_1(2Ls^2) = \sqrt{\mu q(1+e)}\, U_1(s, 2L); \tag{7}$$

$$r = q + \mu e s^2 c_2\left(2Ls^2\right) = q + \mu e U_2\left(s, 2L\right); \tag{8}$$

$$t = qs + \mu e U_3\left(s, 2L\right) \quad (\textit{Kepler's equation}). \tag{9}$$

The dependence of the argument s on time is determined by the above Kepler equation. Notice that the fictitious time parameter s, proportional to the classical eccentric anomaly in the cases of elliptic and hyperbolic motion, is introduced through Stumpff's generalization (1959, §41) of Sundman's regularizing transformation (Sundman 1912, p. 127), which defines s by means of the differential relation

$$dt = r ds \quad (\textit{Sundman's transformation}), \tag{10}$$

where s vanishes at the chosen reference time. As a usual practice, this time variable is chosen so that the pericentre corresponds to $s = 0$. Observe also that s only occurs implicitly in the equation for the radial distance r, and in Kepler's equation for the physical time, through the transcendental Stumpff c_n and Battin U_n universal functions.

5. Proof of the Universality of Brumberg's Transformation

Let $d\sigma$ denote the *arc element* along a two–body conic section. Starting from Formulae (6) and (7), after differentiation with respect to s and use of Formulae (3) and (4), we obtain

$$\begin{aligned}\left(\frac{d\sigma}{ds}\right)^2 &= \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = \left(-\mu U_1(s, 2L)\right)^2 \\ &+ \left(\sqrt{\mu q(1+e)}\, U_0(s, 2L)\right)^2 = \mu^2 U_1^2(s, 2L) \\ &+ \mu q(1+e) - \mu^2\left(1 - e^2\right) U_1^2(s, 2L),\end{aligned}$$

and so, the parameters σ and s are connected with each other by the differential relation

$$d\sigma = \sqrt{\mu q(1+e) + \mu^2 e^2 U_1^2(s, 2L)}ds. \tag{11}$$

On the other hand, application of the chain rule along with Formulae (10), (11) and (4), yields

$$\frac{dt}{d\sigma} = \frac{dt}{ds}\frac{ds}{d\sigma} = \frac{(dt/ds)}{(d\sigma/ds)} = \frac{r(s)}{\sqrt{\mu q(1+e) + \mu^2 e^2 U_1^2(s, 2L)}}. \tag{12}$$

To obtain another expression for the reparametrizing function occurring in Formula (12), more easily interpretable within the framework of *two-parameter transformations of the independent variable*, we perform the sequence of substitutions $U_1 \to U_2 \to r$, namely, we express U_1 in terms of U_2 [Formulae (4)], and then, by virtue of Formula (8), we replace U_2 by r. Thus, the function under the radical sign in (12) is converted into a simple polynomial in r, say: $\mu r \{2q - (1-e)r\}/q$.

In so doing, the right-hand side of (12) can be transformed to an algebraic function of r:

$$\frac{dt}{d\sigma} = \frac{r}{\sqrt{\mu r/q}\sqrt{2q-(1-e)r}} = \sqrt{\frac{q}{\mu}}\frac{r^{1/2}}{\sqrt{2q-(1-e)r}}. \tag{13}$$

As expected, *under a universal treatment of two-body motion*, we have finally recovered the expression given by Brumberg (1992, p. 325, Formula [14]):

$$dt = Q d\sigma, \quad Q = \frac{r^{1/2}}{\sqrt{2\mu - (2L)r}}. \tag{14}$$

Comparison with the time transformations developed by Ferrándiz & Ferrer (1986) and Ferrándiz, Ferrer & Sein-Echaluce (1987) shows that the independent argument considered by Brumberg cannot be obtained from the specific formulations leading to the class of the generalized elliptic anomalies introduced in those papers, although it can be regarded as a special instance of a more general notion of two-parameter time transformation, not restricted to the exponents retained by the said authors.

It should be borne in mind that this way of proceeding can be easily adapted to the case of a certain class of perturbed Kepler problems which can be reduced to a Keplerian-like form if appropriate amended variables are used. The important feature is that the perturbation must be *compatible* with a generalized quasi-Keplerian structure.

6. On the Integration of Brumberg's Transformation

The length of arc σ of a Kepler problem, reckoned from the pericentre (at which $s = 0$), can be determined by our *universal* Formula (11). To integrate this differential transformation of time argument, we perform the *change of integration variable* $s \longrightarrow v$ given by

$$U_1(s, 2L) = v \Rightarrow dv = U_0(s, 2L)\, ds \Rightarrow ds = \frac{dv}{\sqrt{1-(2L)v^2}}, \tag{15}$$

taking into account Formulae (4). In this way, the integration of the time transformation is reduced to that of an *algebraic integrand* with respect to v:

$$d\sigma = \sqrt{q}\sqrt{\frac{\mu q(1+e)+\mu^2 e^2 v^2}{q-\mu(1-e)v^2}}\,dv, \tag{16}$$

where Formula (5) has also been employed.

The intermediate calculations and the final results depend on the *roots* of the polynomials occurring in the above expression. In certain cases, elliptic integrals and functions will be required to complete the integration. Details of the conclusions and applications of this study will be communicated elsewhere.

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