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Solving non-Holonomic Lagrangian Dynamics in Terms of Almost Product Structures

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Given a Lagrangian system with non-holonomic constraints we construct an almost product structure on the tangent bundle of the configuration manifold such that the projection of the Euler-Lagrange vector field gives the dynamics of the system. In a degenerate case, we develop a constraint algorithm which determines a final constraint submanifold where a completely consistent dynamics of the initial system exists.

1. Introduction

The theory of classical mechanical systems with constraints comes back to the last century, but it has deserved a continuous attention up to the present days because its importance to solve practical problems as in control theory, for instance.

There are two different meanings for constrained system in Lagrangian mechanics: "internal" constraints imposed by the singularity of the Lagrangian function, or "external" constraints imposed by forces of constraint acting on the regular system. The first type of constraints was studied in the seminal works of Dirac and Bergmann [8], whose ideas were later developed in the framework of presymplectic and precosymplectic geometry for many authors [13, 11, 12, 17, 5]. Almost tangent geometry also deserves a key role in the theory.

Concerning the second type of constraints, the theory was extensively studied in the framework of Calculus of Variations (see [9, 29, 31] and the references therein, for instance). Recently, several papers have appeared developing a geometrical setting for this type of constraints: Weber [35, 36], Marle [20], Bates & Śniatycki [1], Koiller [15], Dazord [7], Massa & Pagani [21] and others. Another papers concerning with the particular aspects of the Lagrangian formulation in terms of almost tangent geometry are the following: Cariñena & Rañada [3], Rañada [28], Cariñena & Rañada [4], Sarlet, Cantrijn & Saunders [32] and Sarlet [33]. (We also refer to Giachetta [10] for a formulation in terms of jet bundles.)

The external constraints considered here are usually termed non-holonomic. This means that there are a family of m affine functions on the velocities $\phi_i(q,v) = (\mu_i)_A(q)v^A + h_i(q)$, $(1 \leq i \leq m,\ 1 \leq A \leq n,\ m < n)$, which constraints the motion to be only allowable for some values of positions and velocities. In other words, the solutions of the modified Euler-Lagrange equations have to be tangent to the submanifold M_1 defined by the constraints. A holonomic constraint is a function $g(q)$ on the configuration space which can be interpreted as a couple of non-holonomic constraints given by the function $g(q)$ and its derivative, i.e., $\phi_1 = \frac{\partial g}{\partial q} v$ and $\phi_2 = g$. The distinction between holonomic (integrable or geometric) and non-holonomic (non-integrable or kinematic) constraints is due to H. R. Hertz [14] (see also [22]).

The classical way to treat with non-holonomic constraints is to use Lagrange multipliers. In [3, 28] this technique was developed from a geometrical point of view. In this paper, our aim is to go further in that direction. Thus, we construct an almost product structure on the phase space TQ of velocities of a Lagrangian system with non-holonomic constraints such that the dynamics are obtained by projecting the Euler-Lagrange vector field which solves the motion equations of the free problem. (We notice that almost product structures in Lagrangian mechanics were early used by de León & Rodrigues [16, 18], Pitanga & Mundin [24, 25] and de León, de Diego & Pitanga [19]).

In our analysis we have two different kinds of behaviours. From the tangency condition, we obtain a system of m equations with m unknowns (the Lagrange multipliers) whose solutions give the appropiate values for the Lagrange multipliers. By scrutinizing these equations we have two possibilities: the system has a unique solution (when the matrix $\mathcal{C}$ of the coefficients of the system of equations which gives the Lagrange multipliers has maximal rank $k = m$), in such a case we obtain a well-defined vector field on TQ (at least, on M_1); or the matrix $\mathcal{C}$ has rank $k < m$. In the latter case, assuming the compatibility of the system of equations, we get a dynamics up to the choice of $m - k$ arbitrary Lagrange multipliers. Otherwise, if the system of equations is not compatible, we take the points of M_1 where the system has a solution, and then we obtain new constraints which define a submanifold M_2 of M_1. If M_2 has zero-dimension or $M_2 = \emptyset$, then there are no dynamics. Otherwise, we incorporate the new constraints to the analysis and obtain, from the additional tangency conditions, a new constrained submanifold M_3, and so on. This algorithm produces a sequence of submanifolds $\cdots M_3 \subset M_2 \subset M_1 \subset TQ$. If the algorithm stabilizes in some final constraint submanifold, we can determine a well-defined dynamics by constructing an adequate almost product structure, or, in the worst case, the dynamics is fully undetermined. We remark the similarity with the Dirac-Bergmann-Gotay-Nester algorithm ([11, 12]).

The paper is structured as follows. In Section 2 we recall the geometric formulation of Lagrangian systems subjected to non-holonomic constraints and construct an almost product structure on the phase space which permits us to obtain the true dynamics by projecting the Euler-Lagrange vector field corresponding to the free problem. A constraint algorithm is developed in Section 3 for the non-regular case. If the algorithm stabilizes in some final constraint submanifold, a suitable almost product structure is constructed on it. The holonomic case is considered in Section 4 and, in Section 5, a discussion of the symmetries and constant of the motion of non-holonomic Lagrangian systems is made. The Hamiltonian counterpart is studied in Section 6.

2. Non-Holonomic Lagrangian Systems

Let Q be an n-dimensional manifold, TQ its tangent bundle, and $\tau_Q : TQ \longrightarrow Q$ the canonical projection. We will denote by $\{q^A; 1 \leq A \leq n\}$ a set of local coordinates in Q and by $\{q^A, v^A; 1 \leq A \leq n\}$ the induced coordinates in TQ.

We denote by J the canonical almost tangent structure on TQ. Let us recall that J is a (1,1) tensor field on TQ defined by

$$J = \frac{\partial}{\partial v^A} \otimes dq^A \, .$$

Hence $J^2 = 0$ and rank $J = n$ (see [17] and the references therein). The other geometric ingredient of TQ is the Liouville vector field C, which is the infinitesimal generator of the dilations along the fibres. We locally have

$$C = v^A \frac{\partial}{\partial v^A} \, .$$

Suppose that a regular Lagrangian is given, that is, a function $L : TQ \longrightarrow$

$\mathbb{R}$ such that its Hessian matrix

$$(W_{AB}) = \left(\frac{\partial^2 L}{\partial v^A \partial v^B} \right)$$

is regular. Then we can construct an energy function E_L, a one-form α_L (the Poincaré-Cartan 1-form), and a 2-form ω_L (the Poincaré-Cartan 2-form) on TQ by

$$E_L = CL - L \; , \; \alpha_L = J^*(dL) \; , \; \omega_L = -d\alpha_L \; .$$

We recall that, since L is regular, ω_L is symplectic and, in this case the motion equation for the free Lagrangian system is

$$i_X \omega_L = dE_L \; . \tag{1}$$

These equations have a unique solution ξ_L (the Euler-Lagrange vector field). Moreover, ξ_L turns out to be a second order differential equation (hereafter shortened as SODE), that is, ξ_L verifies that $J\xi_L = C$.

The local expression of ξ_L is

$$\xi_L = v^A \frac{\partial}{\partial q^A} + \xi^A(q,v) \frac{\partial}{\partial v^A} \; ,$$

where

$$\xi^A = W^{AB} \left[\frac{\partial L}{\partial q^B} - \frac{\partial^2 L}{\partial q^C \partial v^B} v^C \right] \; ,$$

being (W^{AB}) the inverse matrix of the Hessian matrix. Therefore, the solutions of ξ_L are the solutions of the Euler-Lagrange equations:

$$\left\{ \begin{array}{l} \dfrac{d}{dt} \left(\dfrac{\partial L}{\partial v^A} \right) - \dfrac{\partial L}{\partial q^A} = 0 \; , \\ v^A = \dfrac{dq^A}{dt} \; (1 \leq A \leq n) \; . \end{array} \right.$$

Since ω_L is symplectic it induces a Poisson bracket on $C^\infty(TQ)$ defined by

$$\{f, g\}_L = \omega_L(X_f, X_g) \; , \; \forall f, g \in C^\infty(TQ) \; ,$$

where X_f denotes the Hamiltonian vector field with Hamiltonian energy f, that is, $i_{X_f}\omega_L = df$. Hence $\xi_L = X_{E_L}$ and $\{f, E_L\}_L = \xi_L(f)$.

We assume that L is subjected to a system of m independent non-holonomic constraints $\{\phi_i; 1 \leq i \leq m\}$, where $m < n$ which are affine in the velocities; that is, $\phi_i : TQ \longrightarrow \mathbb{R}$ is a function which can be locally expressed as follows:

$$\phi_i = (\mu_i)_A(q) v^A + h_i(q) \; , \tag{2}$$

where $(\mu_i)_A$ and h_i are functions on Q. Hence, only particular motions are allowed: those such that (2) vanishes.

Let us recall that an affine function $\phi = \mu_A(q)v^A + h$ on TQ may be globally defined as follows:

$$\phi = \hat{\mu} + h^v ,$$

where $\mu = \mu_A(q)dq^A$ is a 1-form on Q and $h^v = h \circ \tau_Q$. Here $\hat{\mu} : TQ \longrightarrow \mathbb{R}$ denotes the function defined by

$$\hat{\mu}(X_q) = \langle \mu(q), X_q \rangle \ , \forall X_q \in T_qQ \ .$$

Thus, there are m 1-forms $\{\mu_i\}$ and m functions $\{h_i\}$ defined on Q such that

$$\phi_i = \hat{\mu}_i + h_i^v \ , \ (1 \le i \le m) \ ,$$

with $\mu_i = (\mu_i)_A dq^A$.

We have to restrict the dynamics to the submanifold M_1 of TQ defined by the vanishing of the functions ϕ_i. The Euler-Lagrange vector field ξ_L is the unique solution of the free Lagrangian system, but, in general, ξ_L is not tangent to the submanifold M_1. However, the dynamics of the constrained system must be represented by a vector field which is a solution of the dynamics and, moreover, it is tangent to M_1. Therefore, we modify the motion equations (1) to obtain the following system of equations:

$$\text{(3)} \qquad \left\{ \begin{array}{l} (i_X\omega_L = dE_L + \lambda^i \mu_i^v)_{/M_1} \ , \\ (d\phi_i(X) = 0)_{/M_1} \ , \end{array} \right.$$

where $\mu_i^v = \tau_Q^* \mu_i$. The functions λ^i are Lagrange multipliers.

Hence, the Euler-Lagrange equations are:

$$\left\{ \begin{array}{l} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial v^A}\right) - \dfrac{\partial L}{\partial q^A} = -\lambda^i (\mu_i)_A \ , \\ v^A = \dfrac{dq^A}{dt} \ , \ (1 \le A \le n) \ . \end{array} \right.$$

For an m-tuple $\lambda = (\lambda^1, \ldots, \lambda^m)$, we take the vector field Y_λ given by

$$Y_\lambda = \xi_L + \lambda^i Z_i \ ,$$

where Z_i are the vertical vector fields (see [3]) defined by:

$$i_{Z_i}\omega_L = \mu_i^v \ .$$

Notice that Y_λ verifies the first equation of the system (3). We also demand that Y_λ satisfies the second condition, thus, Y_λ has to be tangent to M_1. Therefore, we obtain that:

$$\begin{aligned} 0 = d\phi_j(Y_\lambda) &= d\phi_j(\xi_L + \lambda^i Z_i) \\ &= \{\phi_j, E_L\}_L + \lambda^i Z_i(\phi_j) \ . \end{aligned}$$

Denote by $\mathcal{C}$ the matrix of order m whose entries are $\mathcal{C}_{ij} = Z_i(\phi_j)$. If $\mathcal{C}$ is regular, then, the Lagrange multipliers λ_i are uniquely determined on TQ. For example, if L is a natural Lagrangian (i.e., $L = T - V$, where $T = \frac{1}{2} g_{AB} dq^A dq^B$ is the kinetic energy of a Riemannian metric g on Q and $V : Q \longrightarrow \mathbb{R}$ is a potential energy) then the matrix $\mathcal{C}$ is regular (see [1, 3]).

First, we assume that the matrix $\mathcal{C}$ is regular on TQ. Our purpose is to construct an almost product structure on TQ such that the projection of the Euler-Lagrange vector field ξ_L gives the dynamics of the system. We recall that an almost product structure on a manifold M is a tensor field F of type $(1,1)$ on M such that $F^2 = \mathrm{id}$. The manifold M will be called an almost product manifold (see [17]). Associated with F there are two complementary projectors $\mathcal{P} = \frac{1}{2}(id + F)$ and $\mathcal{Q} = \frac{1}{2}(id - F)$. Thus, an almost product structure consists in to give two complementary distributions.

Consider the (1,1) tensor field $\mathcal{Q}$ given by:

$$\mathcal{Q} = \mathcal{C}^{ij} Z_j \otimes d\phi_i \ ,$$

where $\mathcal{C}^{ij}$ are the entries of the inverse matrix of $\mathcal{C}$, that is, $\mathcal{C}^{ij}\mathcal{C}_{jk} = \delta^i_k$. A direct computation shows that $\mathcal{Q}^2 = \mathcal{Q}$. If we set $\mathcal{P} = id - \mathcal{Q}$, then $(\mathcal{P}, \mathcal{Q})$ is an almost product structure on the phase space TQ. Moreover, we have that the vector field $\mathcal{P}(\xi_L)$ is the unique solution of equations (3). In fact,

$$\mathcal{P}(\xi_L) = \xi_L - \mathcal{C}^{ij} \xi_L(\phi_i) Z_j \ ,$$

which implies that $\mathcal{P}(\xi_L)$ verifies the equation (3) for $\lambda^i = -\mathcal{C}^{ji}\xi_L(\phi_j)$. Furthermore, for all ϕ_l, $(1 \leq l \leq m)$, we have

$$\begin{aligned} \mathcal{P}(\xi_L)(\phi_l) &= \xi_L(\phi_l) - \mathcal{C}^{ij}\xi_L(\phi_i) Z_j(\phi_l) \\ &= \xi_L(\phi_l) - \delta^i_l \xi_L(\phi_i) = 0 \ . \end{aligned}$$

REMARK 2.1. Take now another combination of the non-holonomic constraints, that is, we consider the new non-holonomic constraints $\bar{\phi}_j = D^i_j \phi_i$ where $D^i_j \in \tau^*_Q(C^\infty(Q))$ being the matrix (D^i_j) regular on TQ. Applying the previous technique to the Lagrangian L and the constraints $\bar{\phi}_i$, we can construct a new almost product structure $(\bar{\mathcal{P}}, \bar{\mathcal{Q}})$ such that the projection of the Euler-Lagrange vector field, say $\bar{\mathcal{P}}(\xi_L)$, determines the dynamics of the constrained system. But, a straightforward computation shows us that the projectors $\mathcal{P}$ and $\bar{\mathcal{P}}$ are equal on the phase space M_1, that is

$$\mathcal{P}_{/M_1} = \bar{\mathcal{P}}_{/M_1} .$$

REMARK 2.2. A more general kind of constraints were studied in the litterature. The constraint functions are not linear in the velocities, but satisfy the so-called Chetaev conditions (see [23, 30, 35]). The study of this case by using the geometrical procedure introduced in this paper is a matter of a future research.

EXAMPLE 2.1. (see for instance [22, 31, 3, 32]) Consider a rolling disk of radius R constrained to remain vertical on a plane. The standard coordinates of the configuration space $\mathbb{R} \times S^1 \times S^1$ are: x, y the Cartesian coordinates of the center of mass, θ_1 is the angle between the tangent of the disc at the point of contact and the x axis and θ_2 is the angle given by some diameter and the vertical.

The dynamics of this mechanical system is described by:

1. The regular Lagrangian:

$$L = \frac{1}{2}\left(m\dot{x}^2 + m\dot{y}^2 + I_1\dot{\theta}_1^2 + I_2\dot{\theta}_2^2\right) ,$$

where m is the mass, and I_1 and I_2 are moments of inertia.

2. The non-holonomic constraints:

$$\begin{aligned} \phi_1 &= \dot{x} - (R\cos\theta_1)\dot{\theta}_2 = 0 , \\ \phi_2 &= \dot{y} - (R\sin\theta_1)\dot{\theta}_2 = 0 . \end{aligned}$$

The Poincaré-Cartan 2-form of the Lagrangian L is:

$$\omega_L = mdx \wedge d\dot{x} + mdy \wedge d\dot{y} + I_1 d\theta_1 \wedge d\dot{\theta}_1 + I_2 d\theta_2 \wedge d\dot{\theta}_2 ,$$

and the Euler-Lagrange vector field is:

$$\xi_L = \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{\theta}_1\frac{\partial}{\partial \theta_1} + \dot{\theta}_2\frac{\partial}{\partial \theta_2} .$$

We have that

$$\begin{aligned} Z_1 &= -\frac{1}{m}\frac{\partial}{\partial \dot{x}} + \frac{R}{I_2}\cos\theta_1 \frac{\partial}{\partial \dot{\theta}_2} , \\ Z_2 &= -\frac{1}{m}\frac{\partial}{\partial \dot{y}} + \frac{R}{I_2}\sin\theta_1 \frac{\partial}{\partial \dot{\theta}_2} , \end{aligned}$$

where $\mu_1 = dx - (R\cos\theta_1)d\theta_2$ and $\mu_2 = dy - (R\sin\theta_1)d\theta_2$ are the 1-forms such that $\hat{\mu}_i = \phi_i$, $i = 1, 2$. We construct the matrix $\mathcal{C}$ given by

$$\mathcal{C} = (\mathcal{C}_{ij}) = \begin{pmatrix} Z_1(\phi_1) & Z_1(\phi_2) \\ Z_2(\phi_1) & Z_2(\phi_2) \end{pmatrix} = \begin{pmatrix} -\frac{1}{m} - \frac{R^2}{I_2}(\cos\theta_1)^2 & -\frac{R^2}{I_2}\cos\theta_1 \sin\theta_1 \\ -\frac{R^2}{I_2}\cos\theta_1\sin\theta_1 & -\frac{1}{m} - \frac{R^2}{I_2}(\sin\theta_1)^2 \end{pmatrix}$$

and the projector $\mathcal{Q}$ becomes

$$\mathcal{Q} = \mathcal{C}^{ij} Z_j \otimes d\phi_i .$$

Observe that $\mathcal{C}$ is regular over all $T(\mathbb{R}^2 \times S^1 \times S^1)$.

Now, we compute $\mathcal{P}(\xi_L)$ with $\mathcal{P} = id - \mathcal{Q}$:

$$\begin{aligned} \mathcal{P}(\xi_L) &= \xi_L - \mathcal{Q}(\xi_L) \\ &= \xi_L + (mR\dot{\theta}_1\dot{\theta}_2\sin\theta_1)Z_1 - (mR\dot{\theta}_1\dot{\theta}_2\cos\theta_1)Z_2 . \end{aligned}$$

If we can only state that the matrix $\mathcal{C} = (Z_i(\phi_j))$ is regular on M_1 then we construct a (1,1) tensor field along M_1, i.e.,

$$\mathcal{P}(x) : T_x(TQ) \longrightarrow T_x(TQ) , \quad \forall x \in M_1 ,$$

defined as follows:

$$\mathcal{P} = \left(id - \mathcal{C}^{ij} Z_j \otimes d\phi_i\right)_{/M_1} .$$

In this case, we can determine the Lagrange multipliers λ^i only for points x on M_1. The projection $\mathcal{P}((\xi_L)_{/M_1})$ gives us a vector field tangent to M_1 which determines uniquely the dynamics of the Lagrangian system subjected to the constraints ϕ_i. It is clear that if $\mathcal{C}$ is regular on M_1, then it is regular on an open neighbourhood of it.

3. The non-Regular Case

Now, we shall study the remaining case: the matrix $\mathcal{C}$ is non regular. It will be assumed that over all M_1 the matrix $\mathcal{C}$ has constant rank k, with $k < m$. We examine the equations:

$$0 = \{\phi_j, E_L\}_L + \lambda^i Z_i(\phi_j)\ , \quad (1 \leq j \leq m)\ . \tag{4}$$

It is possible that (4) give directly an inconsistency (type $0 = 1$), and then we say that equations (3) are inconsistent. In order to avoid this problem, we impose, as in the Dirac-Bergmann algorithm [8] (see also [11, 12]), the condition that these equations do not involve an inconsistency. In such a case, the number of Lagrange multipliers λ^i determined by the equations (3) is k. Moreover, new constraints $\psi_{i'}$, $(1 \leq i' \leq m')$, with $m' \leq m - k$, may arise. We obtain a new constraint submanifold M_2 determined by the vanishing of the constraints ϕ_i and $\psi_{i'}$. Moreover, we must incorporate these new constraints $\psi_{i'}$ to the equations (3) to get a new set of motion equations:

$$\left\{ \begin{array}{l} (i_X\omega_L = dE_L + \lambda^i\mu_i^v)_{/M_2}\ , \\ (d\phi_i(X) = 0)_{/M_2}\ , \\ (d\psi_{i'}(X) = 0)_{/M_2}\ . \end{array} \right. \tag{5}$$

The tangency of X to M_2 gives us the equations:

$$0 = \{\psi_{i'}, E_L\}_L + \lambda^i Z_i(\psi_{i'})\ .$$

We incorporate these equations to (4) and this system of equations has to be treated on the same footing as (4) and, probably, we determine new Lagrange multipliers λ^i and other constraints may arise. This procedure finishes when we have exhausted all the consistency conditions. If the initial problem is solvable, we arrive at some final constraint submanifold M_f, determined by the vanishing of all the constraints, where "consistent" solutions exist. In this process, some Lagrange multipliers may remain undetermined, in such a case we say that there exists an ambiguity in the description of the dynamics.

In order to illustrate the above procedure, we particularize for the case of a Lagrangian system subjected to a unique linear constraint $\phi = \hat{\mu} + h^v$, where μ is an 1-form on Q and $h \in C^\infty(Q)$. The modified motion equations are:

$$\left\{ \begin{array}{rcl} i_X\omega_L & = & dE_L + \lambda\mu^v\ , \\ d\phi(X) & = & 0\ . \end{array} \right.$$

Since $X\phi = 0$ then we obtain that:

$$0 = \{\phi, E_L\}_L + \lambda Z(\phi) \; . \tag{6}$$

If $Z(\phi) \neq 0$ we deduce that the value of the Lagrange multiplier is

$$\lambda = -\frac{\{\phi, E_L\}_L}{Z(\phi)} \; ,$$

and the dynamics is given by the SODE

$$Y_\lambda = \xi_L - \frac{\{\phi, E_L\}_L}{Z(\phi)} Z \; .$$

If $Z(\phi) = 0$ and $\{\phi, E_L\}_L = 0$ on M_1 then we obtain that the dynamics of the system is determined on M_1 by

$$Y_\lambda = \xi_L + \lambda Z \; ,$$

for all arbitrary values of λ. However, if $Z(\phi) = 0$ and $\{\phi, E_L\}_L \neq 0$ on M_2 equation (6) would be satisfied only on a submanifold M_2 of M_1 given by the constraint $\psi = \{\phi, E_L\}_L$. Preservation of the constraint ψ requires that $X(\psi) = 0$. Thus we obtain the equation:

$$0 = \{\psi, E_L\}_L + \lambda Z(\psi) \; .$$

As above, if $Z(\psi) \neq 0$, then, the Lagrange multiplier is fixed and the dynamics is determined on M_2 by

$$Y_\lambda = \xi_L - \frac{\{\psi, E_L\}_L}{Z(\psi)} Z \; .$$

On the other hand, if $Z(\psi) = 0$ and $\{\psi, E_L\}_L = 0$ on M_2 then the dynamics is fully undetermined. In other case, we obtain a new constraint. One then iterates this procedure, arriving at some final submanifold M_f (if the problem is solvable) where there exists at least a solution of the initial problem.

EXAMPLE 3.1. Let $L : T\mathbb{R}^3 \longrightarrow \mathbb{R}$ the regular Lagrangian function given by

$$L = \frac{1}{2}\left((v^1)^2 + (v^2)^2 - (v^3)^2\right) \; ,$$

subjected to the linear constraint

$$\phi(q, v) = v^2 + v^3 \; .$$

Here $\{q^1, q^2, q^3, v^1, v^2, v^3\}$ denote the bundle coordinates in $T\mathbb{R}^3$. We have

$$\begin{aligned} E_L &= \frac{1}{2}\left((v^1)^2 + (v^2)^2 - (v^3)^2\right) = L\,, \\ \omega_L &= dq^1 \wedge dv^1 + dq^2 \wedge dv^2 - dq^3 \wedge dv^3\,, \\ \xi_L &= v^1\frac{\partial}{\partial q^1} + v^2\frac{\partial}{\partial q^2} + v^3\frac{\partial}{\partial q^3} = C\,. \end{aligned}$$

The 1-form $\mu = dq^2 + dq^3$ on $\mathbb{R}^3$ verifies that $\hat{\mu} = \phi$ and, the vector field Z such that

$$i_Z\omega_L = \mu^v\,,$$

is given by

$$Z = -\frac{\partial}{\partial v^2} + \frac{\partial}{\partial v^3}\,.$$

Since $Z(\phi) = 0$ we must consider the new constraint $\psi = \xi_L(\phi)$. But ψ identically vanishes and we conclude that we can not determine the Lagrange multiplier λ. Remark that for each λ we obtain a solution $Y_\lambda = \xi_L + \lambda Z$ of the motion equations. Consequently, the dynamics is fully undetermined.

Notice that L may be considered as the kinetic energy defined by a Lorentz metric $G = (dq^1)^2 + (dq^2)^2 - (dq^3)^2$ on $\mathbb{R}^3$. The Euler-Lagrange equations for L are just the differential equations yielding the geodesics of the Levi-Civita connection. In fact, the geodesic spray is just ξ_L. For the constrained problem there is not a unique "geodesic spray", any SODE $\xi_L + \lambda Z$ yields an admissible dynamics.

EXAMPLE 3.2. Consider the regular Lagrangian on $T\mathbb{R}^3$ given by

$$L = \frac{1}{2}\left((v^1)^2 + (v^2)^2 - (v^3)^2\right) + \frac{1}{2}(q^1)^2\,,$$

subjected to the linear constraint

$$\phi(q, v) = v^1 + v^3\,.$$

Then $\phi = \hat{\mu}$, with $\mu = dq^1 + dq^3$.

A direct computation shows that:

$$\begin{aligned} E_L &= \frac{1}{2}\left((v^1)^2 + (v^2)^2 - (v^3)^2\right) - \frac{1}{2}(q^1)^2\,, \\ \omega_L &= dq^1 \wedge dv^1 + dq^2 \wedge dv^2 - dq^3 \wedge dv^3\,, \\ \xi_L &= v^1\frac{\partial}{\partial q^1} + v^2\frac{\partial}{\partial q^2} + v^3\frac{\partial}{\partial q^3} + q^1\frac{\partial}{\partial v^1}\,. \end{aligned}$$

The vector field Z, such that $i_Z\omega_L = \mu^v$ is

$$Z = -\frac{\partial}{\partial v^1} + \frac{\partial}{\partial v^3} .$$

Since $Z(\phi) = 0$ we obtain the new constraint $\psi = \xi_L(\phi) = q^1$. Hence we obtain a submanifold:

$$M_2 = \{(q^1, q^2, q^3, v^1, v^2, v^3) \mid v^1 + v^3 = 0, q^1 = 0\} .$$

Since $Z(\psi) = 0$, a new constraint arises; $\psi' = \xi_L(\psi) = v^1$, and we obtain the submanifold M_3 defined by

$$M_3 = \{(q^1, q^2, q^3, v^1, v^2, v^3) \mid q^1 = 0 \ , \ v^1 = 0 \ , \ v^3 = 0\} .$$

Moreover $Z(\psi') = -1$, and the algorithm stabilizes on the submanifold M_3. Now, we determine the Lagrange multiplier λ:

$$\lambda = -\frac{\xi_L(\psi')}{Z(\psi')} = q^1 .$$

Therefore, we have that the vector field $\xi_L + q^1 Z$ determines the dynamics on the submanifold M_3.

As we have seen, if the matrix $\mathcal{C}$ is regular on TQ, we can construct an almost product structure $(\mathcal{P}, \mathcal{Q})$ on TQ such that it gives, by projection the dynamics of the Lagrangian system subjected to constraints. Our aim is to generalize this technique to the singular case using the above algorithm.

Assume that the matrix $\mathcal{C} = (Z_i(\phi_j))$, $(1 \leq i, j \leq m)$, is singular and rank $\mathcal{C} = k < m$. Without lost of generality, we can suppose that the submatrix $\mathcal{C}_1 = (\mathcal{C}_{ij}) = (Z_i(\phi_j))$, $(1 \leq i, j \leq k)$, of $\mathcal{C}$ is regular.

The problem is solvable on M_1 if

$$\text{(7)} \qquad \Big(\text{rank } (Z_i(\phi_j)) = \text{rank } (Z_i(\phi_j) \ ; \ \{E_L, \phi_j\}_L)\Big)_{/M_1} .$$

In that case, we can construct the almost product structure $(\mathcal{P}, \mathcal{Q})$ defined by

$$\mathcal{Q} = \mathcal{C}^{ij} Z_j \otimes d\phi_i \ , \quad (1 \leq i, j \leq k) \ ,$$

and $\mathcal{P} = \mathrm{id} - \mathcal{Q}$, where $\mathcal{C}^{ij}$ is the ij-entry of the inverse matrix of $\mathcal{C}_1$. We shall prove that the projection $\mathcal{P}(\xi_L)$ gives a solution of the constrained dynamics.

In fact, it is only necessary to see that $\mathcal{P}(\xi_L)(\phi_i) = 0$, for any $1 \leq i \leq m$. First, for each $1 \leq l \leq k$, we have that

$$\begin{aligned}\mathcal{P}(\xi_L)(\phi_l) &= \xi_L(\phi_l) - \mathcal{C}^{ij}\xi_L(\phi_i)Z_j(\phi_l) \\ &= \xi_L(\phi_l) - \mathcal{C}^{ij}\mathcal{C}_{jl}\xi_L(\phi_i) = 0\,.\end{aligned}$$

Now, if we consider ϕ_α ,$(k+1 \leq \alpha \leq m)$, we obtain

$$\mathcal{P}(\xi_L)(\phi_\alpha) = \xi_L(\phi_\alpha) - \mathcal{C}^{ij}\xi_L(\phi_i)Z_j(\phi_\alpha)\,.$$

From (7) we deduce that $Z_j(\phi_\alpha) = f_\alpha^l Z_j(\phi_l)$,$(1 \leq l \leq k)$, for some functions $f_\alpha^l \in C^\infty(TQ)$. Thus,

$$\begin{aligned}\mathcal{P}(\xi_L)(\phi_\alpha) &= \xi_L(\phi_\alpha) - \mathcal{C}^{ij}\xi_L(\phi_i)f_\alpha^l Z_j(\phi_l) \\ &= \xi_L(\phi_\alpha) - \mathcal{C}^{ij}\xi_L(\phi_i)f_\alpha^l \mathcal{C}_{jl} \\ &= \xi_L(\phi_\alpha) - f_\alpha^l\xi_L(\phi_l) = 0\,,\end{aligned}$$

since we have assumed that rank $(Z_i(\phi_j)\ ;\ \{E_L, \phi_j\}_L)_{/M_1} = k$.

Otherwise, we have that

$$\left(\text{rank}\ (Z_i(\phi_j)) < \text{rank}\ (Z_i(\phi_j)\ ;\ \{E_L, \phi_j\}_L)\right)_{/M_1}, \tag{8}$$

and, then, we obtain a set of additional constraints. By applying the algorithm, we get a final constraint submanifold M_f (if the problem is solvable) given by the vanishing of constraints $\chi_{i'}$, $(1 \leq i' \leq m')$, where $m' \geq m$. On M_f we have

$$\left(\text{rank}\ (Z_{i'}(\chi_{j'})) = \text{rank}\ (Z_{i'}(\chi_{j'})\ ;\ \{E_L, \chi_{j'}\}_L)\right)_{/M_f}. \tag{9}$$

We suppose that the rank of the matrix $\mathcal{C}' = (Z_{i'}(\chi_{j'}))$,$(1 \leq i' \leq m)$, $(1 \leq j' \leq m')$, is constant, i.e., rank $\mathcal{C}' = k' < m$ $(k \leq k')$. For simplicity, we suppose that the matrix $\mathcal{C}_1' = (\mathcal{C}_{i'j'}') = (Z_{i'}(\chi_{j'}))$, $(1 \leq i', j' \leq k')$, is non-singular. As above, we construct an almost product structure $(\mathcal{P}, \mathcal{Q})$ given by

$$\mathcal{Q} = (\mathcal{C}')^{i'j'} Z_{j'} \otimes d\chi_{i'}\,, \quad (1 \leq i', j' \leq k' < m)\,,$$

and $\mathcal{P} = id - \mathcal{Q}$, where $(\mathcal{C}')^{i'j'}$ is the $i'j'$-entry of the inverse matrix of $\mathcal{C}'$. For each solution $X = \xi_L + \lambda^l Z_l + \lambda^\alpha Z_\alpha$, $(1 \leq l \leq k',\ k'+1 \leq \alpha \leq m)$ of the equation

$$i_X\omega_L = dE_L + \lambda^i\mu_i^v\ (1 \leq i \leq m)\,,$$

we obtain that $\mathcal{P}(X)$ is a solution of the motion equations. In fact, we have

$$\begin{aligned} \mathcal{P}(X) &= \xi_L + \lambda^l Z_l + \lambda^\alpha Z_\alpha - (\mathcal{C}')^{i'j'}\xi_L(\chi_{i'})Z_{j'} \\ &\quad -\lambda^l(\mathcal{C}')^{i'j'}Z_l(\chi_{i'})Z_{j'} - \lambda^\alpha(\mathcal{C}')^{i'j'}Z_\alpha(\chi_{i'})Z_{j'} \\ &= \xi_L - (\mathcal{C}')^{i'j'}\left(\xi_L(\chi_{i'}) + \lambda^\alpha Z_\alpha(\chi_{i'})\right)Z_{j'} + \lambda^\beta Z_\beta \ , \end{aligned}$$

with $k'+1 \leq \beta \leq m$. We then conclude that

$$\left\{ \begin{array}{rcl} i_{\mathcal{P}(X)}\omega_L &=& dE_L + \lambda^i \mu_i^v \ , \\ ((\mathcal{P}(X))(\chi_{i'}) &=& 0)_{/M_f} \ . \end{array} \right.$$

4. Holonomic Constraints

We suppose that a Lagrangian system is subjected to holonomic constraints $h_i \in C^\infty(Q)$, $(1 \leq i \leq m)$, with $m < n$. That is, the velocities do not enter into the constraint equations. In geometrical terms the motion equations are:

$$(10) \qquad \left\{ \begin{array}{rcl} i_X\omega_L &=& dE_L + \lambda^i dh_i^v \ , \\ dh_i^c(X) &=& 0 \ , \\ dh_i^v(X) &=& 0 \ . \end{array} \right.$$

Since $dh_i^v(X) = h_i^c$ because X is a SODE, we can first study the system as a Lagrangian system subjected to the non-holonomic constrains h_i^c and solve the motion equations

$$(11) \qquad \left\{ \begin{array}{rcl} i_X\omega_L &=& dE_L + \lambda^i dh_i^v \ , \\ dh_i^c(X) &=& 0 \ , \end{array} \right.$$

and later, we impose the constraints $h_i^v = 0$.

We shall show how a holonomic system is, in some sense, an special case of a free system.

First of all, notice that $M_1 = TQ_1$, where Q_1 denotes the submanifold of Q defined by the vanishing of the functions h_i. Next, assume that the matrix $\mathcal{C}$ with entries $\mathcal{C}_{ij} = Z_i(h_j^c)$ is regular on M_1 (and hence, on an open neighborhood of it). The dynamics is then given by the vector field on M_1

$$\mathcal{P}(\xi_L) = \xi_L - \mathcal{C}^{ji}\xi_L(h_j^c)Z_i \ .$$

Put

$$\Lambda^i = -\mathcal{C}^{ji}\xi_L(h_j^c) \ ,$$

and define a new Lagrangian function

$$\mathbb{L} = L - \sum_i \Lambda^i h_i^v .$$

A direct computation shows that

$$\begin{aligned} E_{\mathbb{L}} &= E_L - h_i^v E_{\Lambda^i} \\ \omega_{\mathbb{L}} &= \omega_L - \alpha_{\Lambda^i} \wedge dh_i^v - h_i^v \omega_{\Lambda^i} . \end{aligned}$$

Hence we get

$$i_{\mathcal{P}(\xi_L)} \omega_{\mathbb{L}} = dE_{\mathbb{L}} - h_i^v (i_{\mathcal{P}(\xi_L)} \omega_{\Lambda^i} - dE_{\Lambda^i}) + h_i^c \alpha_{\Lambda^i} .$$

But $M_1 = TQ_1$ is defined by the vanishing of h_i^v and h_i^c which implies that

$$i_{\mathcal{P}(\xi_L)} \omega_{\mathbb{L}} = dE_{\mathbb{L}}$$

on TQ_1. This result tells us that the constrained system may be considered as a free Lagrangian system with admissible Lagrangian function $\mathbb{L}$ and with a global dynamics $\mathcal{P}(\xi_L)$.

EXAMPLE 4.1. Consider the system given by two ponderous particles P_1 and P_2 of identical mass $m = 1$ which are joined by a rod of constant length l and negligibly small mass. Also, the system is constrained to move in the vertical plane and only on such manner that the velocity of the midpoint of the rod is directed along it (see [9]).

Let (x_1, x_2) and (y_1, y_2) be the coordinates of P_1 and P_2, respectively. Then the motion of this system is described by:

1. The regular Lagrangian:

$$L = \frac{1}{2} (\dot{x}_1^2 + \dot{x}_2^2 + \dot{y}_1^2 + \dot{y}_2^2) - g(y_1 + y_2) ;$$

2. The holonomic constraint:

$$h_1 = \frac{1}{2} \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 - l^2 \right] ;$$

3. The non-holonomic constraint:

$$\phi_2 = (x_2 - x_1)(\dot{y}_2 + \dot{y}_1) - (\dot{x}_2 + \dot{x}_1)(y_2 - y_1) .$$

The holonomic constraint h_1 gives rise to a non-holonomic constraint:

$$\phi_1 = \widehat{dh_1} = (x_2 - x_1)(\dot{x}_2 - \dot{x}_1) - (y_2 - y_1)(\dot{y}_1 - \dot{y}_2) \, .$$

From the free Lagrangian L we obtain

$$\begin{aligned} E_L &= \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{y}_1^2 + \dot{y}_2^2\right) + g(y_1 + y_2) \, , \\ \omega_L &= dx_1 \wedge d\dot{x}_1 + dx_2 \wedge d\dot{x}_2 + dy_1 \wedge d\dot{y}_1 + dy_2 \wedge d\dot{y}_2 \, , \\ \xi_L &= \dot{x}_1 \frac{\partial}{\partial x_1} + \dot{x}_2 \frac{\partial}{\partial x_2} + \dot{y}_1 \frac{\partial}{\partial y_1} + \dot{y}_2 \frac{\partial}{\partial y_2} - g \frac{\partial}{\partial \dot{y}_1} - g \frac{\partial}{\partial \dot{y}_2} \, . \end{aligned}$$

Also, we have that the vector fields Z_1 and Z_2 such that $i_{Z_i}\omega_L = \mu_i^v$ where $\mu_i^v = J^* d\phi_i$, $i = 1, 2$, are given by

$$\begin{aligned} Z_1 &= (x_2 - x_1)\frac{\partial}{\partial \dot{x}_1} - (x_2 - x_1)\frac{\partial}{\partial \dot{x}_2} + (y_2 - y_1)\frac{\partial}{\partial \dot{y}_1} - (y_2 - y_1)\frac{\partial}{\partial \dot{y}_2} \, , \\ Z_2 &= (y_2 - y_1)\frac{\partial}{\partial \dot{x}_1} + (y_2 - y_1)\frac{\partial}{\partial \dot{x}_2} - (x_2 - x_1)\frac{\partial}{\partial \dot{y}_1} - (x_2 - x_1)\frac{\partial}{\partial \dot{y}_2} \, . \end{aligned}$$

Now, if we evaluate the matrix $\mathcal{C} = (\mathcal{C}_{ij})$ on

$$M_1 = \{(x_1, x_2, y_1, y_2, \dot{x}_1, \dot{x}_2, \dot{y}_1, \dot{y}_2) \; / \; h_1 = 0, \phi_1 = 0, \phi_2 = 0\}$$

we obtain

$$\mathcal{C}_{/M_1} = \begin{pmatrix} Z_1(\phi_1) & Z_1(\phi_2) \\ Z_2(\phi_1) & Z_2(\phi_2) \end{pmatrix}_{/M_1} = \begin{pmatrix} -2l^2 & 0 \\ 0 & -2l^2 \end{pmatrix} .$$

After simple manipulations, we deduce that the projector $\mathcal{P}$ is given on M_1 by the matrix P (see the next page). The projection $\mathcal{P}(\xi_L)$ of ξ_L gives the dynamics of the constrained system on M_1

$$\begin{aligned} \mathcal{P}(\xi_L) = \; & \dot{x}_1 \frac{\partial}{\partial x_1} + \dot{x}_2 \frac{\partial}{\partial x_2} + \dot{y}_1 \frac{\partial}{\partial y_1} + \dot{y}_2 \frac{\partial}{\partial y_2} - \frac{g}{l^2}(x_2 - x_1)(y_2 - y_1)\frac{\partial}{\partial \dot{x}_1} \\ & - \frac{g}{l^2}(x_2 - x_1)(y_2 - y_1)\frac{\partial}{\partial \dot{x}_2} - \left(\frac{g}{2} - \frac{g}{l^2}(2(x_2 - x_1)^2 - l^2)\right)\frac{\partial}{\partial \dot{y}_1} \\ & - \left(\frac{g}{2} - \frac{g}{l^2}(2(x_2 - x_1)^2 - l^2)\right)\frac{\partial}{\partial \dot{y}_2} \, . \end{aligned}$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{l^2}\dot{y}_1(y_2-y_1) & -\frac{1}{l^2}\dot{y}_2(y_2-y_1) & \frac{1}{l^2}\dot{x}_1(y_2-y_1) & \frac{1}{l^2}\dot{x}_2(y_2-y_1) & \frac{1}{2} & \frac{1}{2l^2}\left(2(x_2-x_1)^2-l^2\right) & 0 & \frac{1}{l^2}(x_2-x_1)(y_2-y_1) \\ \frac{1}{l^2}\dot{y}_1(y_2-y_1) & \frac{1}{l^2}\dot{y}_2(y_2-y_1) & -\frac{1}{l^2}\dot{x}_1(y_2-y_1) & -\frac{1}{l^2}\dot{x}_2(y_2-y_1) & \frac{1}{2l^2}\left(2(x_2-x_1)^2-l^2\right) & \frac{1}{2} & \frac{1}{l^2}(x_2-x_1)(y_2-y_1) & 0 \\ \frac{1}{l^2}\dot{y}_1(x_2-x_1) & \frac{1}{l^2}\dot{y}_2(x_2-x_1) & -\frac{1}{l^2}\dot{x}_1(x_2-x_1) & -\frac{1}{l^2}\dot{x}_2(y_2-y_1) & 0 & \frac{1}{l^2}(x_2-x_1)(y_2-y_1) & \frac{1}{2} & -\frac{1}{2l^2}\left(2(x_2-x_1)^2-l^2\right) \\ -\frac{1}{l^2}\dot{y}_1(x_2-x_1) & -\frac{1}{l^2}\dot{y}_2(x_2-x_1) & \frac{1}{l^2}\dot{x}_1(x_2-x_1) & \frac{1}{l^2}\dot{x}_2(y_2-y_1) & \frac{1}{l^2}(x_2-x_1)(y_2-y_1) & 0 & -\frac{1}{2l^2}\left(2(x_2-x_1)^2-l^2\right) & \frac{1}{2} \end{pmatrix}$$

5. Symmetries and Constants of the Motion

In discussing symmetries and constants of the motion of a Lagrangian system subjected to non-holonomic constraints, we shall follow the classification of Prince [26, 27] (see also [6]).

We first suppose that the matrix $\mathcal{C} = (Z_i(\phi_j))$ is regular. Then, there exists a unique solution Λ of the constrained dynamics on M_1 where

$$\Lambda = (\mathcal{P}(\xi_L))_{/M_1} .$$

Thus, we have the following definitions:

Definition 5.1. A function f on $C^\infty(M_1)$ is said to be a constant of the motion of Λ if $\Lambda f = 0$.

Definition 5.2. A dynamical symmetry of Λ is a vector field $\tilde{X}$ on M_1 such that $[\tilde{X}, \Lambda] = 0$.

It is clear that if f is a constant of the motion of Λ, then $\tilde{X} f$ is also a constant of the motion of Λ for any dynamical symmetry $\tilde{X}$.

Denote by X^c the complete lift to TQ of a vector field X on Q. We shall introduce the notion of Lie symmetry.

Definition 5.3. A Lie symmetry of Λ is a vector field X on Q such that X^c is tangent to M_1 and $(X^c)_{/M_1}$ is a dynamical symmetry of Λ.

As it is well-known, there exists a biyective relationship between the constants of the motion of the Euler-Lagrange vector field ξ_L of the free Lagrangian L and certain class of dynamical symmetries, the Cartan symmetries. The Cartan symmetries for free Lagrangians are Hamiltonian vector fields X_F such that $X_F E_L = 0$ and, in this case, F is a constant of the motion.

Consider now a constant of the motion $F : TQ \longrightarrow \mathbb{R}$ of ξ_L, i.e., $\xi_L F = 0$. One easily proves that, if $(Z_i F)_{/M_1} = 0$ $(1 \leq i \leq m)$ then, since

$$\Lambda = (\xi_L - \mathcal{C}^{ij} \xi_L(\phi_i) Z_j)_{/M_1} ,$$

we have that $\Lambda(F_{/M_1}) = 0$ and, thus $F_{/M_1}$ is a constant of the motion of Λ (see [3]). It follows that

$$Z_i F = -\mu_i^v(X_F) = -(J^* d\phi_i)(X_F) = -d\phi_i(JX_F) = -L_{JX_F}\phi_i .$$

Therefore, if $(L_{JX_F}\phi_i)_{/M_1} = 0$ we conclude that $F_{/M_1}$ is a constant of the motion of Λ. Notice that, in general, $(X_F)_{/M_1}$ is not a dynamical symmetry of Λ and, in fact, we can not assure that X_F be tangent to M_1 (see [2]).

EXAMPLE 5.1. We continue with Example 2.1. Since x, y, θ_1, θ_2 are cyclic coordinates, there are four Noether symmetries of the free Lagrangian system:

$$X = \frac{\partial}{\partial x} \;,\; Y = \frac{\partial}{\partial y} \;,\; \Theta_1 = \frac{\partial}{\partial \theta_1} \;,\; \Theta_2 = \frac{\partial}{\partial \theta_2} \;,$$

which yield four constants of the motion:

$$f_1 = m\dot{x} \;,\; f_2 = m\dot{y} \;,\; f_3 = I_1\dot{\theta}_1 \;,\; f_4 = I_2\dot{\theta}_2 \;.$$

Since $Z_1(f_3) = Z_2(f_3) = 0$ we conclude that f_3 is a constant of the motion for the constrained system.

Our next objective is study the degenerate case, i.e., $\mathcal{C}$ is singular. If we apply the algorithm developed in the previous section, we obtain a final constraint submanifold M_f where solutions of the dynamics exist. Thus, for any λ_α, $(k'+1 \leq \alpha \leq m)$, each vector field X such that

$$X = \left(\xi_L - (\mathcal{C}')^{i'j'} \left(\xi_L(\chi_{i'}) + \lambda^\alpha Z_\alpha(\chi_{i'})\right) Z_{j'} + \lambda^\alpha Z_\alpha\right)_{/M_1} \;,$$

is a solution of the motion equations. As in the regular case, if F is a constant of the motion of ξ_L and $\left(L_{JX_f}\phi_i\right)_{/M_f} = 0$ $(1 \leq i \leq m)$ then $F_{/M_f}$ is a constant of the motion of any solution X.

6. HAMILTONIAN FORMALISM

Let T^*Q be the cotangent bundle of Q with canonical projection $\pi : T^*Q \longrightarrow Q$.

Let $L : TQ \longrightarrow \mathbb{R}$ be a regular Lagrangian subjected to a set of independent non-holonomic constraints $\phi_i = \hat{\alpha}_i + h_i^v$. We will denote by $Leg : TQ \longrightarrow T^*Q$ the Legendre map defined by L:

$$Leg : (q^A, \dot{q}^A) \longrightarrow (q^A, p_A) \;,$$

with $p_A = \partial L/\partial \dot{q}^A$. Since L is regular, Leg is a local diffeomorphism. For the sake of simplicity, we suppose that L is hyper-regular, that is, Leg is a global diffeomorphism. If ω_Q is the canonical symplectic form on T^*Q, we have that: $Leg^*\omega_Q = \omega_L$. We define the Hamiltonian function H on T^*Q by $H = E_L \circ Leg^{-1}$.

Since $Leg : TQ \longrightarrow T^*Q$ is a diffeomorphism, the Lagrangian and Hamiltonian descriptions are completely equivalent. Thus, the constrained Hamilton equations are obtained by a direct transportation of the Euler-Lagrange ones.

If the constrained system is singular, we also develop a constraint algorithm on the hamiltonian side, and both algorithms are related by the Legendre transformation, as in the case of singular lagrangians [8, 13, 11, 12].

Since (TQ, ω_L) and (T^*Q, ω_Q) are symplectic manifolds, we have the Poisson brackets $\{\ ,\ \}_L$ and $\{\ ,\ \}$ on TQ and T^*Q, respectively. Moreover, since $Leg^*\omega_Q = \omega_L$, the Legendre transformation is a Poisson map, i.e.,

$$Leg^*\{f, g\} = \{Leg^*f, Leg^*g\}_L\ , \forall f, g \in C^\infty(T^*Q)\ .$$

Denote by M_f and $\tilde{M}_f$ the corresponding final constraint submanifolds. Suppose now that M_f is given by the vanishing of the constraints $\chi_{i'}$,$(1 \leq i' \leq m')$. We can classify these constraints in two different types. A constraint χ of M_f is said to be first class if $\{\chi, \chi_{i'}\}_{/M_f} = 0$ for each constraint $\chi_{i'}$ of M_f, and second class otherwise. In a similar way, we obtain a classification of the constraints of $\tilde{M}_f$.

Denote by ω_{M_f} and $\omega_{\tilde{M}_f}$ the restrictions of ω_L and ω_Q to M_f and $\tilde{M}_f$. Thus, if all the constraints of M_f (resp. $\tilde{M}_f$) are first class then M_f (resp. $\tilde{M}_f$) is a coisotropic submanifold of TQ (resp. T^*Q). Also, if all the constraints of M_f (resp. $\tilde{M}_f$) are second class then (M_f, ω_{M_f}) (resp. $(\tilde{M}_f, \omega_{\tilde{M}_f})$) is a symplectic manifold.

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