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Weak Homological Bidimension and its Values in the Class of Biflat Banach Algebras†

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1. INTRODUCTION

The weak (homological) bidimension, $\mathrm{w.db}\, A$, of a Banach algebra A is a number that measures how much this algebra is "homologically worse" than *amenable*; it is equal to zero precisely when A is amenable. The class of amenable Banach algebras was introduced by Johnson [10]. He selected these algebras by the condition $\mathcal{H}^1(A, X^*) = 0$ for all Banach A-bimodules X. Here X^* is the Banach A-bimodule dual to X, in which the module multiplications are given by $(a \cdot f)(x) = f(x \cdot a)$, $(f \cdot a)(x) = f(a \cdot x)(a \in A$, $f \in X^*, x \in X), \mathcal{H}^1(A, X^*)$ is the (continuous) one-dimensional cohomology group of A with coefficients in X^*. The motivation for the term "amenable algebra" was [10, Theorem 2.5]. This theorem goes as follows: *the algebra* $L^1(G)$ *is amenable if and only if the locally compact group* G *is amenable in the traditional meaning of harmonic analysis* (see [3]).

The concept of a *biflat* Banach algebra is due to Helemskii [6]. A Banach algebra A is said to be biflat if it is a flat Banach A-bimodule or, equivalenty, if A^* is an injective Banach A-bimodule. If A has a bounded approximate identity (b.a.i. for short), then the above property is equivalent to the amenability of A (see [9, Theorem 1]). At the same time, the class of biflat Banach algebras, not necessarily with b.a.i., is wider. All the algebras $\mathcal{N}(E)$ of nuclear operators on E, where E is an infinite-dimensional Banach space with the approximation property, provide examples of biflat non-amenable Banach algebras (see [15, Corollaries 2 and 6]).

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The purpose of the present paper is to compute the weak bidimension for an arbitrary biflat Banach algebra and, in particular, to find the set of values taken by this dimension in the class of all such algebras. It turns out that *a biflat Banach algebra* A *has always* $\mathrm{w.db}\, A \le 2$ (Theorem 2), and *it has* $\mathrm{w.db}\, A \le 1$ *if and only if it has a "one-sided" (left or right) b.a.i.* (Corollary 4). Thus we see that *the weak bidimension of biflat Banach algebras can only take the values* 0, 1 *or* 2 (in detail see Theorem 6). For example, if E is an arbitrary infinite-dimensional Banach space with the approximation property, then $\mathrm{w.db}\, \mathcal{N}(E) = 2$ (Corollary 5).

Effros and Kishimoto [2] raised the question of whether there exist non-amenable Banach algebras A such that $\mathcal{H}^2(A, X^*) = 0$ for all dual bimodules X^*. The same question was posed earlier in [10] by Johnson. *If we are talking about all Banach algebras, then the answer is positive.* In fact, it was shown in [12, Theorem 1] by the present author that for any Banach space $E \neq \mathbb{C}$ there exists a *biprojective* (and, in particular, biflat) Banach algebra $A = A(E)$ with underlying space E, which has a left, but no two-sided, identity; hence $\mathrm{w.db}\, A(E) = 1$, and so $\mathcal{H}^2(A(E), X^*) = 0$ for all dual X^*. On the other hand, *the answer to this question is negative in the class of commutative biflat Banach algebras*, since $\mathrm{w.db}\, A = 2$ for each non-amenable algebra in this class (Corollary 6).

2. The Weak Bidimension of Banach Algebras and its Relation with the Bidimension

Let A be a Banach algebra, not necessarily with an identity, and let A_+ be its unitization. The categories of left Banach A-modules, right Banach A-modules and Banach A-bimodules will be denoted respectively by A-**mod**, **mod**-A and A-**mod**-A; the corresponding sets of (continuous) morphisms from X to Y will be denoted by ${}_A\mathbf{h}(X,Y)$, $\mathbf{h}_A(X,Y)$ and ${}_A\mathbf{h}_A(X,Y)$. The fundamental homological concepts for the categories A-**mod**, **mod**-A and A-**mod**-A (projectivity, flatness, resolutions, the groups "Ext" and "Tor", the (co)homology groups of A and others) are assumed to be known; they are set out in detail in Helemskii's book [7]. We recall only that, historically, the *cohomology groups* $\mathcal{H}^n(A,X)$ (respectively, the *homology groups* $\mathcal{H}_n(A,X)$), $X \in A$-**mod**-A, $n = 0, 1, \ldots$, were defined in terms of the so-called standard cohomology (respectively, homology) complex. At the same time, both of these homology invariants (which are complete seminormed spaces) can be defined as a particular case of the Ext or Tor functors for bimodules. The corresponding formulae (which are valid up to a topological isomorphism of seminormed spaces) are

$$\mathcal{H}^n(A,X) = \mathrm{Ext}^n_{A-A}(A_+,X) = \mathrm{Ext}^n_{A^{\mathrm{env}}}(A_+,X) \tag{1}$$

and

$$\mathcal{H}_n(A,X) = \mathrm{Tor}^{A-A}_n(X,A_+) = \mathrm{Tor}^{A^{\mathrm{env}}}_n(X,A_+),$$

where A^{env} is the enveloping Banach algebra of A (see [7, Theorems III.4.9 and III.4.24]).

We now recall that Johnson [10, Corollary 1.3] (see also [7, Proposition II.5.29]) observed the following connection between questions about the triviality of homology and cohomology groups. Namely, let X be a Banach A-bimodule, and let $n > 0$ be an integer. Then $\mathcal{H}_m(A,X) = 0$ for $m \geq n$ and $\mathcal{H}_{n-1}(A,X)$ is Hausdorff if and only if $\mathcal{H}^m(A,X^*) = 0$ for $m \geq n$.

DEFINITION 1. The *weak homological bidimension* (later simply the *weak bidimension*) of the Banach algebra A is the smallest integer n such that $\mathcal{H}^m(A,X^*) = 0$ for all $X \in A$-**mod**-A and $m > n$, or ∞ if there are no such n.

The above number is denoted by $\mathrm{w.\,db}\,A$. The following proposition follows inmediately from the observation just mentioned.

PROPOSITION 1. *For each integer $n \geq 0$, the following conditions are equivalent:*

(i) $\mathrm{w.\,db}\,A \leq n$;

(ii) $\mathcal{H}^m(A,X^*) = 0$ *for all* $X \in A$-**mod**-A *and* $m > n$;

(iii) $\mathcal{H}_m(A,X) = 0$ *and* $\mathcal{H}_n(A,X)$ *is Hausdorff for all* $X \in A$-**mod**-A *and* $m > n$.

Algebras A for which $\mathrm{w.\,db}\,A = 0$ are of special interest. This condition is equivalent to the condition that $\mathcal{H}^1(A,X^*) = 0$ for all $X \in A$-**mod**-A (see, for example, [7, Theorem VII.2.19]), that is, to the amenability of A. For example, $\mathrm{w.\,db}\,C_0(\Omega) = 0$ for any locally compact space Ω, and $\mathrm{w.\,db}\,L^1(G) = 0$ for any amenable locally compact group G.

THEOREM 1. *For each integer $n \geq 0$, the following conditions are equivalent:*

(i) $\mathrm{w.db}\, A \leq n$;

(ii) $\mathcal{H}^{n+1}(A, X^*) = 0$ *for all* $X \in A$-**mod**-A;

(iii) *if*

$$\aleph\colon 0 \longleftarrow A_+ \longleftarrow X_0 \longleftarrow \cdots \longleftarrow X_{n-1} \longleftarrow Y \longleftarrow 0$$

is a resolution of the A-bimodule A_+ *in which all the (Banach) A-bimodules* X_i *are flat, then so is* Y;

(iv) *the A-bimodule* A_+ *has a flat resolution of length* n

$$0 \longleftarrow A_+ \longleftarrow X_0 \longleftarrow \cdots \longleftarrow X_{n-1} \longleftarrow X_n \longleftarrow 0.$$

Proof. Trivially, (i) implies (ii). To show that (ii) implies (iii), suppose (ii) holds. Decompose the resolution $\aleph$ into the product of short admissible complexes

$$0 \longleftarrow K_{i-1} \longleftarrow X_i \longleftarrow K_i \longleftarrow 0 \qquad (i = 0, 1, \ldots, n-1),$$

where $K_{-1} = A_+$ and $K_{n-1} = Y$. Each of these short complexes defines, for any $X \in A$-**mod**-A, the exact sequence of groups

$$\mathrm{Ext}^m_{A-A}(X_i, X^*) \to \mathrm{Ext}^m_{A-A}(K_i, X^*) \to \mathrm{Ext}^{m+1}_{A-A}(K_{i-1}, X^*) \to \mathrm{Ext}^{m+1}_{A-A}(X_i, X^*)$$

(see [7, Theorem III.4.4]). Since all the $X_i \in A$-**mod**-A $(i = 0.\ldots, n-1)$ are flat, we have

$$\mathrm{Ext}^m_{A-A}(X_i, X^*) = \mathrm{Ext}^{m+1}_{A-A}(X_i, X^*) = 0 \quad \text{for } m > 0$$

(see [8, Theorem VII.3.25]). Thus we obtain a chain of algebraic isomorphisms

$$\mathrm{Ext}^1_{A-A}(Y, X^*) \cong \mathrm{Ext}^2_{A-A}(K_{n-2}, X^*) \cong \ldots$$
$$\cdots \cong \mathrm{Ext}^n_{A-A}(K_0, X^*) \cong \mathrm{Ext}^{n+1}_{A-A}(A_+, X^*).$$

Since in view of (1), $\mathrm{Ext}^{n+1}_{A-A}(A_+, X^*) = \mathcal{H}^{n+1}(A, X^*)$, (ii) implies that $\mathrm{Ext}^1_{A-A}(Y, X^*) = 0$ for all $X \in A$-**mod**-A. By [8, Theorem VII.3.25], the A-bimodule Y is flat, and so (iii) holds.

Since, A-bimodule A_+ has at least one projective (and consequently flat) resolution in A-**mod**-A (see, for example, [8, Ch. VII, §3.1]), (iii) easily implies (iv). And if there is a resolution of the form (iv), then, for any $X \in A$-**mod**-A, the groups $\mathcal{H}_m(A, X) = \mathrm{Tor}^{A-A}_m(X, A_+)$ computed by using this resolution (see [7, Theorem III.3.15 and Proposition VII.1.2]) vanish for $m > n$ and are Hausdorff for $m = n$. By Proposition 1, the latter implies (i). ∎

Thus the weak bidimension, $\mathrm{w.\,db}\,A$, of A can be defined as the least length of a flat resolution of $A_+ \in A$-**mod**-A. For example, the A-bimodule A_+ is flat if and only if $\mathrm{w.\,db}\,A = 0$, that is, A is amenable (see [9, Theorem 1]).

We recall now that the (*homological*) *bidimension*, $\mathrm{db}\,A$, of A is the homological dimension of $A_+ \in A$-**mod**-A, that is, the least n such that the A-bimodule A_+ has a projective resolution of length n. The same number is the smallest integer n such that $\mathcal{H}^m(A, X) = 0$ for all $X \in A$-**mod**-A and $m > n$. From here we see that $\mathrm{w.\,db}\,A \leq \mathrm{db}\,A$ for every Banach algebra A.

It is known that $\mathrm{db}\,A \leq 2$ for any biprojective A (see [7, Theorem V. 2.28]). On the contrary, the bound $\mathrm{db}\,A \geq 2$ holds for all commutative Banach algebras with infinite spectrum (see [5, §9]) and also for a wide class of noncommutative biprojective Banach algebras (see [13, Theorem 5]). For example, $\mathrm{db}\,C_0(\Omega) \geq 2$ for any infinite, locally compact space Ω, and $\mathrm{db}\,C_0(\Omega) = 2$ with Ω discrete. We recall that

$$\mathrm{db}\,L^1(G) = \begin{cases} 2 & \text{if } G \text{ is compact and infinite (see [4, Theorem 9]),} \\ \infty & \text{if } G \text{ is non-compact and amenable (see [17]),} \end{cases}$$

and that (see [15, Corollary 5])

$$\mathrm{db}\,\mathcal{N}(E) = \begin{cases} 2 & \text{if } E \text{ has the approximation property and} \\ & \text{is infinite-dimensional,} \\ \infty & \text{if } E \text{ does not have the approximation property.} \end{cases}$$

It is not known whether the condition $\mathrm{db}\,A = 1$ holds for some semisimple Banach algebra A. However (we shall show in Theorem 7), this condition is valid for every non-unital biprojective Banach algebra with a "one-sided" (left or right) identity, which is certainly nonsemisimple. In the same theorem, we shall compute the bidimension of any biprojective Banach algebra without a b.a.i.

3. The Tensor Product of Banach Bimodules; Biflat Banach Algebras

Let A and B be Banach algebras, and let B_+^{op} be the Banach algebra opposite to B_+, the unitization of B. We recall that the projective tensor product $C = A_+ \hat{\otimes} B_+^{\mathrm{op}}$ is a Banach algebra, in which the multiplication is well-defined by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_2 b_1 \quad (a_1, a_2 \in A_+, b_1, b_2 \in B_+).$$

Each $X \in B$-**mod**-A (that is, a Banach B-A-bimodule) can be identified with an object in **mod**-C ($x\cdot(a\otimes b) = b\cdot x\cdot a$), and each $Y \in A$-**mod**-B can be identified with an object in C-**mod** ($(a\otimes b)\cdot y = a\cdot y\cdot b$). This identification evidently extends to morphisms (in particular, ${}_B\mathbf{h}_A(X,X) = \mathbf{h}_C(X,X)$ and ${}_A\mathbf{h}_B(Y,Y) = {}_C\mathbf{h}(Y,Y)$), and we can define the tensor product of $X \in B$-**mod**-A and $Y \in A$-**mod**-B by using the explicit construction for the tensor product of the given modules over the algebra C. Namely, the *bimodule tensor product*

$$X \underset{A-B}{\hat{\otimes}} Y = X \underset{C}{\hat{\otimes}} Y$$

is the quotient (Banach) space of the projective tensor product $X \hat{\otimes} Y$ by the closed linear span of the set of elements of the form

$$b\cdot x\cdot a \otimes y - x \otimes a\cdot y\cdot b \quad (a \in A_+, b \in B_+, x \in X, y \in Y). \tag{2}$$

It is easily seen that $X \underset{A-B}{\hat{\otimes}} Y$ has the universal property with respect to continuous bilinear operators R from $X \times Y$, such that

$$R(x\cdot a, y) = R(x, a\cdot y) \qquad (\text{"inner } A\text{-associativity"})$$

and

$$R(b\cdot x, y) = R(x, y\cdot b) \qquad (\text{"outer B-associativity"}),$$

for all $a \in A$, $b \in B$, $x \in X$, $y \in Y$.

Using the universal property just mentioned, it is easy to prove that, for bimodules $X, X' \in B$-**mod**-A, $Y, Y' \in A$-**mod**-B and morphisms $\varphi \in {}_B\mathbf{h}_A(X,X')$ and $\psi \in {}_A\mathbf{h}_B(Y,Y')$, there exists a unique continuous linear operator, $\varphi \underset{A-B}{\hat{\otimes}} \psi$, from $X \underset{A-B}{\hat{\otimes}} Y$ to $X' \underset{A-B}{\hat{\otimes}} Y'$ such that

$$(\varphi \underset{A-B}{\hat{\otimes}} \psi)(x \underset{A-B}{\otimes} y) = \varphi(x) \underset{A-B}{\otimes} \psi(y) \quad (x \in X,\ y \in Y).$$

It also follows from the universal property that there is a natural isometric isomorphism

$$ {}_B\mathbf{h}_A(X, Y^*) \cong (X \underset{A-B}{\hat{\otimes}} Y)^* \tag{3}$$

(cf. [7, Theorem II.5.21]), under which the continuous linear functional t on $X \underset{A-B}{\hat{\otimes}} Y$ corresponds to the morphism $T \in {}_B\mathbf{h}_A(X, Y^*)$ such that

$$(Tx)(y) = t(x \underset{A-B}{\otimes} y) \qquad (x \in X,\ y \in Y).$$

Furthermore, there is a natural isometric isomorphism

$$X \underset{A-B}{\hat{\otimes}} Y \cong Y \underset{B-A}{\hat{\otimes}} X \tag{4}$$

which carries $x \underset{A-B}{\otimes} y$ to $y \underset{B-A}{\otimes} x$.

We now consider an important example.

PROPOSITION 2. *Let $U \in A$-**mod** and $V \in$ **mod**-B, and let $Y = U \hat{\otimes} V$ be the A-B-bimodule obtained from U and V by the tensor product bifunctor (see [7, Ch. II, §5.3]). Then for any $X \in B$-**mod**-A*

$$X \underset{A-B}{\hat{\otimes}} Y \cong V \underset{B}{\hat{\otimes}} X \underset{A}{\hat{\otimes}} U,$$

the isomorphism being natural and isometric.

Proof. For $x \in X$, $u \in U$ and $v \in V$ we put $R(x,u,v) = v \underset{B}{\otimes} x \underset{A}{\otimes} u$. Obviously the operator

$$R\colon X \times U \times V \longrightarrow V \underset{B}{\hat{\otimes}} X \underset{A}{\hat{\otimes}} U$$

is trilinear and $\|R\| \leq 1$. If we apply the universal property of the tensor product $X \hat{\otimes} U \hat{\otimes} V$ with respect to continuous trilinear operators (see [7, Ch. II, §2.3]) we get a continuous linear operator

$$\varphi\colon X \hat{\otimes} Y = X \hat{\otimes} U \hat{\otimes} V \longrightarrow V \underset{B}{\hat{\otimes}} X \underset{A}{\hat{\otimes}} U$$

with $\|\varphi\| \leq 1$. It is easily seen that the kernel of φ contains the set of elements of the form (2). Consequently, φ generates a continuous linear operator

$$\lambda\colon X \underset{A-B}{\hat{\otimes}} Y \longrightarrow V \underset{B}{\hat{\otimes}} X \underset{A}{\hat{\otimes}} U$$

with $\|\lambda\| \leq 1$.

On the other hand, let

$$S\colon V \times X \times U \longrightarrow X \underset{A-B}{\hat{\otimes}} Y$$

be the trilinear operator given by $S(v,x,u) = x \underset{A-B}{\otimes} (u \otimes v)$, where $v \in V$, $x \in X$, $u \in U$; it is easily verified that S is balanced (that is, $S(v \cdot b, x, u) = S(v, b \cdot x, u)$ and $S(v, x \cdot a, u) = S(v, x, a \cdot u)$ for any $b \in B$, $a \in A$). The operator from $V \underset{B}{\hat{\otimes}} X \underset{A}{\hat{\otimes}} U$ into $X \underset{A-B}{\hat{\otimes}} Y$ associated with S is denoted by μ. It is obvious that $\|\mu\| \leq 1$, and that μ is the inverse operator to λ. Thus λ is an isometric isomorphism of Banach spaces. ∎

If $Y \in A$-**mod**-B, then $X \longmapsto X \mathbin{\hat{\otimes}}_{A-B} Y$ is a functor from B-**mod**-A to **Ban**, the category of Banach spaces (the action of this functor on morphisms is given by $\varphi \longmapsto \varphi \mathbin{\hat{\otimes}}_{A-B} 1_Y$). The A-B-bimodule Y is said to be *flat* if the left $A_+ \mathbin{\hat{\otimes}} B_+^{\mathrm{op}}$-module identified with it (see above) is also flat or, equivalently, if the functor

$$? \mathbin{\hat{\otimes}}_{A-B} Y\colon B\text{-}\mathbf{mod}\text{-}A \longrightarrow \mathbf{Ban}\colon X \longmapsto X \mathbin{\hat{\otimes}}_{A-B} Y$$

is exact (see [7, Definition III.3.6]).

PROPOSITION 3. (cf. [8, Proposition VII.1.57]) *If Y is flat in A-**mod**-B, then, for any $Z \in B$-**mod**, $Y \mathbin{\hat{\otimes}}_B Z$ is flat in A-**mod**.*

Proof. Let $X \in$ **mod**-A. Proposition 2 gives us the isometric isomorphism

$$X \mathbin{\hat{\otimes}}_A (Y \mathbin{\hat{\otimes}}_B Z) \cong Y \mathbin{\hat{\otimes}}_{B-A} (Z \mathbin{\hat{\otimes}} X) \cong (Z \mathbin{\hat{\otimes}} X) \mathbin{\hat{\otimes}}_{A-B} Y$$

(the last two spaces are isomorphic in view of (4)). Considering this isomorphism for all X, we can establish the isomorphism between the functors

$$? \mathbin{\hat{\otimes}}_A (Y \mathbin{\hat{\otimes}}_B Z)\colon \mathbf{mod}\text{-}A \longrightarrow \mathbf{Ban} \quad \text{and } (? \mathbin{\hat{\otimes}}_{A-B} Y) \circ (Z \mathbin{\hat{\otimes}} ?),$$

where $Z \mathbin{\hat{\otimes}} ?$ maps **mod**-A into B-**mod**-A, and $? \mathbin{\hat{\otimes}}_{A-B} Y$ maps B-**mod**-A into **Ban**. Since obviously the functor $Z \mathbin{\hat{\otimes}} ?$ preserves the admissibility of short right Banach A-module complexes, the exactness of the functor $? \mathbin{\hat{\otimes}}_{A-B} Y$ implies the exactness of the functor $? \mathbin{\hat{\otimes}}_A (Y \mathbin{\hat{\otimes}}_B Z)$. ∎

In the case $A = B$, every Banach A-bimodule may be considered as a left as well as a right module over (the same) $A^{\mathrm{env}} = A_+ \mathbin{\hat{\otimes}} A_+^{\mathrm{op}}$. We recall (see [6] or [7, Ch. VII]) that the Banach algebra A is said to be biflat, if the A-bimodule A is flat, that is, if the functor

$$? \mathbin{\hat{\otimes}}_{A-A} A\colon A\text{-}\mathbf{mod}\text{-}A \longrightarrow \mathbf{Ban}\colon X \longmapsto X \mathbin{\hat{\otimes}}_{A-A} A$$

is exact.

Note that every biprojective Banach algebra (see [14]) is biflat. The converse is false: for example, the algebra $C_0(\Omega)$ is biflat for any locally compact space Ω, but it is not biprojective if Ω is nondiscrete (see [8, Ch. VII, §1.5]).

We now recall that the *canonical morphism* for $X \in A$-**mod** means the morphism $\pi \in {}_A\mathbf{h}(A \hat{\otimes} X, X)$ defined by $\pi(a \otimes x) = a\cdot x$, where the module operation $A \hat{\otimes} X \in A$-**mod** is well-defined by $a\cdot(b \otimes x) = ab \otimes x$ $(a, b \in A, x \in X)$. The closure of the image of the morphism π is denoted by $A\cdot X$. In particular, for any closed left ideal I in A_+, we denote by $A\cdot I$ the closure of the linear span of elements of the form ab $(a \in A,\ b \in I)$. A is said to be *idempotent* if $A^2 = A$, where A^2 denotes $A\cdot A$.

The next proposition is similar to [14, Lemma 1.1].

PROPOSITION 4. *Let A be a biflat Banach algebra. Then, for each closed left ideal I in A_+, the left Banach A-module $A/A{\cdot}I$ is flat, and the right Banach A-module $(A/A\cdot I)^*$ is injective.*

Proof. The left A-module $A/A\cdot I$ is topologically isomorphic to the A-module $A \hat{\otimes}_A (A_+/I)$ (see [7, Theorem II.3.17]), which is flat, by Proposition 3. Now [7, Theorem VII.1.14] ensures that the right A-module $(A/A\cdot I)^*$ is injective. ∎

COROLLARY 1. *A biflat Banach algebra is idempotent.*

Proof. Taking $A = I$, we deduce from Proposition 4 that the annihilator right A-module $(A/A^2)^*$ is injective. If A is not idempotent then $(A/A^2)^* \neq 0$, and consequently the one-dimensional annihilator right A-module $\mathbb{C}$ can be represented as a modular direct summand $(A/A^2)^*$, and is therefore injective (see [7, Proposition III.1.16]). This implies that the short exact sequence of right A-modules

$$0 \longrightarrow \mathbb{C} \longrightarrow (A_+)^* \longrightarrow A^* \longrightarrow 0$$

splits, and hence that A has a right b.a.i. (see [7, Theorem VII.1.20]). But the latter is impossible provided that A is not idempotent, and so we have a contradiction. ∎

The latter result was established earlier by another method by Helemskii (see [7, Proposition VII.2.6]). In addition, he gave the following characterization of biflatness (see [7, Theorem VII.2.7]): *a Banach algebra A is biflat if and only if $\pi^*\colon A^* \longrightarrow (A \hat{\otimes} A)^*$ has a left inverse morphism of A-bimodules, where $\pi\colon A\hat{\otimes}A \longrightarrow A$ is the canonical morphism.*

The next proposition is similar to [14, Lemma 1.3].

PROPOSITION 5. *Let A be a biflat Banach algebra, and I a closed bi-ideal of A. Then the Banach algebra $A/A\cdot I$ is biflat.*

Proof. Let $B = A/A{\cdot}I$, and let $\sigma\colon A \longrightarrow B$ be quotient map. If $\rho \in {}_A\mathbf{h}_A((A \hat{\otimes} A)^*, A^*)$ is a left inverse to π^*, then we put $\rho_1 = \rho(\sigma\hat{\otimes}\sigma)^*$. It is clear that $\rho_1\colon (B \hat{\otimes} B)^* \longrightarrow A^*$ is a morphism of A-bimodules while for any $a \in A$, $d \in I$ and $f \in (B \hat{\otimes} B)^*$ we have

$$\rho_1(f)(ad) = (d{\cdot}\rho_1(f))(a) = \rho_1(d{\cdot}f)(a) = 0,$$

since $(d{\cdot}f)(b \otimes c) = f(b \otimes cd) = 0$ for all $b, c \in B$. This implies that $\rho_1(f)$ is zero on $A{\cdot}I$ for any $f \in (B \hat{\otimes} B)^*$. The latter means that there exists a unique morphism $\rho_2 \in {}_A\mathbf{h}_A((B \hat{\otimes} B)^*, B^*) = {}_B\mathbf{h}_B((B \hat{\otimes} B)^*, B^*)$ defined by the formula

$$\rho_2(f)(\sigma(a)) = \rho_1(f)(a) \quad (f \in (B \hat{\otimes} B)^*,\ a \in A).$$

Let $\pi_B\colon B \hat{\otimes} B \longrightarrow B$ be the canonical morphism for the B-module B. Obviously $\pi_B(\sigma \hat{\otimes} \sigma) = \sigma\pi$ and hence $(\sigma \hat{\otimes} \sigma)^*(\pi_B)^* = \pi^*\sigma^*$. Then

$$\rho_1(\pi_B)^* = \rho(\sigma \hat{\otimes} \sigma)^*(\pi_B)^* = \rho\pi^*\sigma^* = \sigma^*,$$

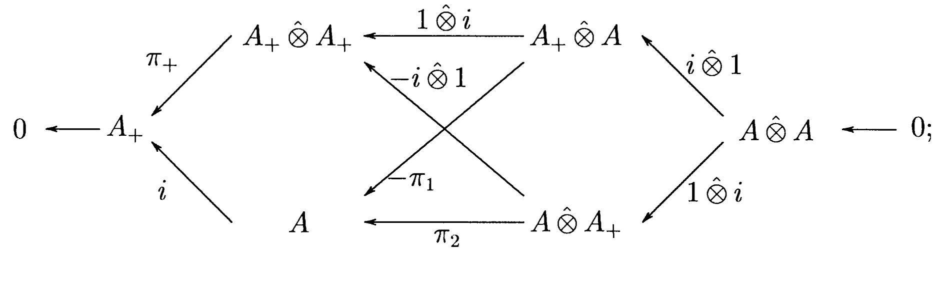
since ρ is a left inverse to π^*. But $\rho_1 = \sigma^*\rho_2$, and $\operatorname{Ker}\sigma^* = 0$. Hence ρ_2 is a left inverse to $(\pi_B)^*$, and therefore (see the result cited above) B is biflat. ∎

4. The Weak Bidimension of Biflat Algebras and the Bidimension of Biprojective Algebras

It is well known (see [7, Theorem VII.2.20]) that a biflat Banach algebra A is amenable (that is, $\mathrm{w.db}\,A = 0$) if and only if it has a b.a.i. The next theorem is similar to [7, Theorem V.2.28].

Theorem 2. *Let A be a biflat Banach algebra. Then* $\mathrm{w.db}\,A \leq 2$.

Proof. Consider the following diagram of A-bimodules and morphisms:



here i is the natural embedding of A in A_+, and π_+, π_1, π_2 are given by $a \otimes b \mapsto ab$. We put $P_0 = (A_+ \hat{\otimes} A_+) \oplus A$, $P_1 = (A_+ \hat{\otimes} A) \oplus (A \hat{\otimes} A_+)$ and we introduce the morphisms $d_{-1}\colon P_0 \longrightarrow A_+$, $d_0\colon P_1 \longrightarrow P_0$, and $d_1\colon A \hat{\otimes} A \longrightarrow P_1$, given "componentwise" with the help of the morphisms indicated on the diagram. The complex

$$0 \longleftarrow A_+ \xleftarrow{d_{-1}} P_0 \xleftarrow{d_0} P_1 \xleftarrow{d_1} A \hat{\otimes} A \longleftarrow 0 \tag{5}$$

is the so-called *entwining resolution* (see [7, Theorem V.2.1]) associated with the left A-module A_+. Obviously, all the morphisms of this resolution are morphisms of A-bimodules. Furthermore, the A-bimodules P_0, P_1 and $A \hat{\otimes} A$ are flat, by [7, Propositions VII.2.2 and VII.2.4]. Consequently, this complex is a flat resolution of length 2 for A-bimodule A_+. By Theorem 1, we have $\operatorname{w.db} A \le 2$. ∎

COROLLARY 2. *Let A be a biflat Banach algebra, and let $X \in A$-**mod**-A. Then:*

(i) *$\mathcal{H}^n(A, X^*) = 0$ for all $n \ge 3$;*

(ii) *$\mathcal{H}_n(A, X) = 0$ for all $n \ge 3$ and $\mathcal{H}_2(A, X)$ is Hausdorff.*

PROPOSITION 6. *Let A be a biflat Banach algebra, and let $X \in A$-**mod**-A. Consider the operator*

$$\Delta_X\colon A \mathop{\hat{\otimes}}_A X \mathop{\hat{\otimes}}_A A \longrightarrow (A \mathop{\hat{\otimes}}_A X) \oplus (X \mathop{\hat{\otimes}}_A A)$$

defined by

$$\Delta_X(a \mathop{\otimes}_A x \mathop{\otimes}_A b) = (a \mathop{\otimes}_A x\cdot b, a\cdot x \mathop{\otimes}_A b) \qquad (a, b \in A,\ x \in X).$$

Then the following conditions are equivalent:

(i) *$\mathcal{H}^2(A, X^*) = 0$;*

(ii) *$\mathcal{H}_2(A, X) = 0$ and $\mathcal{H}_1(A, X)$ is Hausdorff;*

(iii) *the operator Δ_X is topologically injective.*

Proof. Since A is biflat, Corollary 2 and [10, Corollary 1.3] imply that conditions (i) and (ii) are equivalent. It remains for us to prove that (ii) is equivalent to (iii). By [7, Theorem III.3.15 and Proposition VII.1.2], the spaces $\mathcal{H}_n(A,X) = \operatorname{Tor}_n^{A-A}(X, A_+)$ can be computed by using the resolution (5); consequently, they are the homology groups of the complex

$$0 \longleftarrow X \underset{A-A}{\hat{\otimes}} P_0 \xleftarrow{\delta_0} X \underset{A-A}{\hat{\otimes}} P_1 \xleftarrow{\delta_1} X \underset{A-A}{\hat{\otimes}} (A \hat{\otimes} A) \longleftarrow 0,$$

where $\delta_0 = 1_X \underset{A-A}{\hat{\otimes}} d_0$ and $\delta_1 = 1_X \underset{A-A}{\hat{\otimes}} d_1$. In particular, we have $\mathcal{H}_2(A,X) = \operatorname{Ker}\delta_1$, and $\mathcal{H}_1(A,X) = \operatorname{Ker}\delta_0/\operatorname{Im}\delta_1$. Thus the condition (ii) is equivalent to the operator δ_1 being topologically injective. It remains only to note that, by Proposition 2,

$$X \underset{A-A}{\hat{\otimes}} (A \hat{\otimes} A) = A \underset{A}{\hat{\otimes}} X \underset{A}{\hat{\otimes}} A, \qquad X \underset{A-A}{\hat{\otimes}} P_1 = (A \underset{A}{\hat{\otimes}} X) \oplus (X \underset{A}{\hat{\otimes}} A)$$

and $\delta_1 = \Delta_X$, up to an isometric isomorphism of Banach spaces. ∎

Before giving the next result about the weak bidimension of biflat algebras, we touch on the question of the structure of the space $I \underset{A}{\hat{\otimes}} X$, where X is a left Banach A-module, and I is a closed right ideal in the algebra A_+. We consider the operator $\alpha_{I,X} \colon I \underset{A}{\hat{\otimes}} X \longrightarrow X$ defined by $\alpha_{I,X}(a \otimes x) = a \cdot x$ $(a \in I,\ x \in X)$. Obviously, $\operatorname{Im}\alpha_{I,X}$ is dense in $I{\cdot}X$, where $I{\cdot}X$ is the closure of the linear span of elements of the form $a{\cdot}x$ $(a \in I,\ x \in X)$, and if $\alpha_{I,X}$ is a topologically injective operator (that is, an injective operator with closed image), then $I \underset{A}{\hat{\otimes}} X \cong I{\cdot}X$. We recall the well-known result (cf. [8, Theorem VI.3.24]): *if I has, as a Banach algebra, a left b.a.i., then the operator $\alpha_{I,X}$ is topologically injective.* As it is mentioned in [6], this theorem was proved, in essence, by Rieffel [11, Theorem 4.4], who considered the case $I = A$. The following result shows that the converse theorem is true.

THEOREM 3. *Let A be a Banach algebra, and let I be a closed right ideal in A_+. Then the following conditions are equivalent:*

(i) *for each left Banach A-module X, the operator $\alpha_{I,X} \colon I \underset{A}{\hat{\otimes}} X \longrightarrow X$ is topologically injective;*

(ii) *for the left Banach A-module I^* dual to I, the operator $\alpha_{I,I^*} \colon I \underset{A}{\hat{\otimes}} I^* \longrightarrow I^*$ is topologically injective;*

(iii) *I has a left b.a.i.*

Proof. We consider the short exact complex

$$0 \longleftarrow A_+/I \xleftarrow{j} A_+ \xleftarrow{i} I \longleftarrow 0 \tag{6}$$

in **mod**-A (i and j are the natural embedding and the natural projection respectively), and its dual complex

$$0 \longrightarrow (A_+/I)^* \xrightarrow{j^*} (A_+)^* \xrightarrow{i^*} I^* \longrightarrow 0, \tag{7}$$

which is also exact (see [7, Theorem 0.5.2]). Let X be an arbitrary left Banach A-module. Using the isomorphism between X and $A_+ \underset{A}{\hat{\otimes}} X$ (see [8, Corollary VI.3.25]), we see that the operator $\alpha_{I,X}\colon I \underset{A}{\hat{\otimes}} X \longrightarrow X \cong A_+ \underset{A}{\hat{\otimes}} X$ can be obtained by applying the functor $? \underset{A}{\hat{\otimes}} X$ to the morphism $i\colon I \longrightarrow A_+$. Therefore, in view of (3), the operator $\alpha^*_{I,X}\colon X^* \longrightarrow (I \underset{A}{\hat{\otimes}} X)^*$ dual to $\alpha_{I,X}$ may be identified with the operator

$$\lambda = {}_A\mathbf{h}(X, i^*)\colon {}_A\mathbf{h}(X, (A_+)^*) \longrightarrow {}_A\mathbf{h}(X, I^*).$$

If the operator $\alpha_{I,X}$ is topologically injective, then $\alpha^*_{I,X}$ and λ are surjective (see [7, Theorem 0.5.2 or Exercise 0.5.3]).

Now set $X = I^*$ and consider the identity morphism on I^* as an element of the space ${}_A\mathbf{h}(I^*, I^*)$. If the operator λ is surjective, then this element belongs to $\operatorname{Im}\lambda$, and so there exists a morphism of left Banach modules $\rho\colon I^* \longrightarrow (A_+)^*$ such that $\lambda(\rho) = i^*\rho = 1_{I^*}$. In this case, the complex (7) splits in A-**mod**, and hence I has a left b.a.i. (cf. [7, Theorem VII.1.20]). In this way, (i) implies (ii), and (ii) implies (iii). The deduction of (i) from (iii) is obviously provided by [8, Theorem VI.3.24]. ∎

We single out particularly the case $I = A$.

COROLLARY 3. *Let A be a Banach algebra. Then the following conditions are equivalent:*

(i) *for each left Banach A-module X, the operator $æ_X\colon A \underset{A}{\hat{\otimes}} X \longrightarrow X$ given by $a \underset{A}{\otimes} x \mapsto a \cdot x$ is topologically injective;*

(ii) *for the left Banach A-module A^* dual to A, the operator $æ_{A^*}\colon A \underset{A}{\hat{\otimes}} A^* \longrightarrow A^*$ is topologically injective;*

(iii) *A has a left b.a.i.*

Remark 1. For right modules we have of course similar theorem and corollary, which give us criteria for a closed left ideal in A_+ or for A to have a right b.a.i.

LEMMA 1. *Let E_0, E, F_0, and F be Banach spaces and let $\tau\colon E_0 \longrightarrow E$ and $\nu\colon F_0 \longrightarrow F$ be continuous linear operators that are not topologically injective. Then the operator $\theta\colon E_0 \hat{\otimes} F_0 \longrightarrow (E_0 \hat{\otimes} F) \oplus (E \hat{\otimes} F_0$, given by $x \otimes y \mapsto (x \otimes \nu(y), \tau(x) \otimes y)$ is not topologically injective either.*

Proof. Since the operator τ (respectively, ν) is not topologically injective, there exists a sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \in E_0$ (respectively, $\{y_n\}_{n=1}^{\infty}$, $y_n \in F_0$), such that for all n $\|x_n\| = 1$, and $\|\tau(x_n)\| = \alpha_n$, where $\lim_{n\to\infty} \alpha_n = 0$ (respectively, $\|y_n\| = 1$, and $\|\nu(y_n)\| = \beta_n$, where $\lim_{n\to\infty} \beta_n = 0$).

For each n we put $z_n = x_n \otimes y_n$; then $z_n \in E_0 \hat{\otimes} F_0$, and $\|z_n\| = 1$. At the same time, for each n

$$\|\theta(z_n)\| \le \|x_n\|\,\|\nu(y_n)\| + \|\tau(x_n)\|\,\|y_n\| \le \beta_n + \alpha_n,$$

and so $\lim_{n\to\infty} \|\theta(z_n)\| = 0$. It follows that the operator θ is not topologically injective. ∎

THEOREM 4. *Let A be a Banach algebra. Then the following conditions are equivalent:*

(i) *for each $X \in A$-**mod**-A, the operator $\Delta_X\colon A \hat{\otimes}_A X \hat{\otimes}_A A \longrightarrow (A \hat{\otimes}_A X) \oplus (X \hat{\otimes}_A A)$ given by $a \otimes_A x \otimes_A b \mapsto (a \otimes_A x\cdot b, a\cdot x \otimes_A b)$ is topologically injective;*

(ii) *for $X = A^* \hat{\otimes} A^* \in A$-**mod**-$A$, the operator Δ_X is topologically injective;*

(iii) *A has a left or right b.a.i.*

Proof. The implication (i) $\Rightarrow$ (ii) is trivial. To prove the implication (ii) $\Rightarrow$ (iii), suppose (ii) holds. This means that the operator

$$\theta = \Delta_{A^* \hat{\otimes} A^*}\colon (A \hat{\otimes}_A A^*) \hat{\otimes} (A^* \hat{\otimes}_A A) \longrightarrow ((A \hat{\otimes}_A A^*) \hat{\otimes} A^*) \oplus (A^* \hat{\otimes} (A^* \hat{\otimes}_A A))$$

given by $x \otimes y \mapsto (x \otimes \nu(y), \tau(x) \otimes y)$ $(x \in A \hat{\otimes}_A A^*,\ y \in A^* \hat{\otimes}_A A)$ is topologically injective; here $\tau\colon A \hat{\otimes}_A A^* \longrightarrow A^*$ (respectively, $\nu\colon A^* \hat{\otimes}_A A \longrightarrow A^*$) is the operator defined by $\tau(a \otimes_A u) = a\cdot u$ (respectively, $\nu(u \otimes_A a) = u\cdot a$). Then, by Lemma 1, at least one of the operators τ and ν is topologically injective. In view of Corollary 3 and Remark 1, (iii) holds.

We now assume that (iii) holds. Let, for example, A have a left b.a.i., and let $X \in A$-**mod**-A. Then, by [8, Theorem VI.3.24],

$$A \hat{\otimes}_A X \hat{\otimes}_A A = A \hat{\otimes}_A (X \hat{\otimes}_A A) \cong A\cdot(X \hat{\otimes}_A A).$$

Since the latter space is a closed subspace of $X \hat{\otimes}_A A$, we obtain that the operator Δ_X is topologically injective, that is, (i) holds. ∎

By combining Proposition 6 with Theorem 4 we get the following theorem.

THEOREM 5. *Let A be a biflat Banach algebra. Then the following conditions are equivalent:*

(i) $\mathcal{H}^2(A, X^*) = 0$ *for all* $X \in A$-**mod**-A;

(i') $\mathcal{H}^2(A, (A^* \hat{\otimes} A^*)^*) = 0$;

(ii) $\mathcal{H}_2(A, X) = 0$ *and* $\mathcal{H}_1(A, X)$ *is Hausdorff for all* $X \in A$-**mod**-A;

(ii') $\mathcal{H}_2(A, A^* \hat{\otimes} A^*) = 0$ *and* $\mathcal{H}_1(A, A^* \hat{\otimes} A^*)$ *is Hausdorff;*

(iii) *A has a left or right b.a.i.*

Corollary 2 and Theorem 5 yield the following corollary.

COROLLARY 4. *Let A be a biflat Banach algebra. Then* $\operatorname{w.db} A \leq 1$ *if and only if A has a left or right b.a.i.*

The following theorem is the main result of the paper. It is a consequence of Theorem 2, Corollary 4 and [7, Theorem VII.2.20].

THEOREM 6. *Let A be a biflat Banach algebra. Then*

$$\operatorname{w.db} A = \begin{cases} 0 & \text{if } A \text{ has a b.a.i.,} \\ 1 & \text{if } A \text{ has a left or right, but no two-sided, b.a.i.,} \\ 2 & \text{if } A \text{ has neither a left nor a right b.a.i.} \end{cases}$$

COROLLARY 5. *Let E be an infinite-dimensional Banach space with the approximation property, and set $A = \mathcal{N}(E)$. Then* $\operatorname{w.db} A = 2$ *(and so* $\operatorname{w.db} A = \operatorname{db} A$, see [15, Corollary 5]*).*

Proof. Since our algebra is biflat (see [15, Corollary 2]), it is sufficient to note that it has neither a left nor a right b.a.i. (cf. [15, Corollary 6]). ∎

Remark 2. By analogy with the proof of [15, Theorem 1 and Corollary 5], one can prove the following: *if a Banach space E does not have the approximation property, then* $\operatorname{w.db}\mathcal{N}(E) = \infty$. Thus $\operatorname{w.db}\mathcal{N}(E) = \operatorname{db}\mathcal{N}(E)$ for any Banach space E.

COROLLARY 6. *Let A be a commutative biflat Banach algebra. Then*

$$\mathrm{w.\,db}\, A = \begin{cases} 0 & \textit{if } A \textit{ has a b.a.i.,} \\ 2 & \textit{if } A \textit{ does not have a b.a.i.} \end{cases}$$

For example, if $A = \ell_1$ with coordinatewise multiplication, then $\mathrm{w.\,db}\, A = 2$.

One of the applications of Theorem 6 is that it enables one to establish a lower bound for the bidimension of certain biflat algebras.

COROLLARY 7. *Let A be a biflat Banach algebra and suppose that A has neither a left nor a right b.a.i. Then* $\mathrm{db}\, A \geq 2$.

Concerning biprojective algebras, the following turns out to be true.

THEOREM 7. *Let A be a biprojective Banach algebra. Then*

$$\mathrm{db}\, A = \begin{cases} 0 & \textit{if } A \textit{ has an identity,} \\ 1 & \textit{if } A \textit{ has a left or right, but no two-sided, identity,} \\ 2 & \textit{if } A \textit{ has neither a left nor a right identity} \\ & \textit{and does not have a b.a.i.} \end{cases}$$

Proof. The case "A has an identity" is trivial. If A is non-unital, but has a "one-sided" identity, then the result follows from the proof of [12, Theorem 1].

Let us now prove the non-trivial part. We assume that A has neither a left nor a right identity and does not have a b.a.i. either. The bound $\mathrm{db}\, A \leq 2$ follows from [7, Theorem V.2.28]. To obtain a contradiction, suppose that $\mathrm{db}\, A < 2$. Then, for each $X \in A$-**mod**, the *homological dimension*, $\mathrm{dh}_A\, X$, of X is less than 2 (see [7, Corollary III.5.16]). Since A does not have a right identity, [16, Corollary 1] and Corollary 3 imply that A has a left b.a.i. Since A does not have a left identity, we obtain, by the concepts of symmetry, that A has a right b.a.i. Consequently, the Banach algebra A has a left as well as a right b.a.i. and hence (see [1]) also a two-sided b.a.i. But we have assumed that A is an algebra without a b.a.i., and so we have a contradiction. The theorem is proved. ∎

COROLLARY 8. *Let A be a biprojective, non-amenable Banach algebra without "one-sided" identities. Then* $\mathcal{H}^2(A, A \hat{\otimes} A) \neq 0$.

Proof. If the groups $\mathcal{H}^2(A, A \hat{\otimes} A) = \operatorname{Ext}^2_{A-A}(A_+, A \hat{\otimes} A)$, computed by using the (projective) resolution (5),vanish, then obviously there exists a morphism of A-bimodules $\xi \colon P_1 \longrightarrow A \hat{\otimes} A$ that is left inverse to d_1. But then the homological dimension of $A_+ \in A$-**mod**-A (that is, $\operatorname{db} A$) is less than 2. We now only have to apply Theorem 7. ∎

Remark 3. Most likely, $\mathcal{H}^2(A, A \hat{\otimes} A) \neq 0$ and $\operatorname{db} A = 2$ for any biprojective A without "one-sided" identities; however, as yet, there is no complete proof of this conjecture. (See [13, Theorem 5] or [7, Assertion V.2.30(II)] for the special case in which A is semisimple and has the approximation property.)

Remark 4. Recall the known question (see [7, Postscript, §7]), which was also posed by A.L.T. Patersen in his preprint "Virtual diagonals and n-amenability for Banach algebras", kindly sent to us. For which classes of (unital) Banach algebras is it true that $\operatorname{db} A \hat{\otimes} B = \operatorname{db} A + \operatorname{db} B$ or is the analogous "additivity formula" true when db is replaced by a certain other homological characteristic of Banach algebras (for example, by w. db)? Using the results of the present paper, we shall prove in subsequent article that the formula

$$\operatorname{w.db} A \hat{\otimes} B = \operatorname{w.db} A + \operatorname{w.db} B$$

is valid for any Banach algebras A and B, having b.a.i.

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