EXTRACTA MATHEMATICAE Vol. 11, Núm. 2, 375 – 380 (1996)

Properties and Applications of the Local Functional Calculus

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AMS *Subject Class.* (1991): 47A11, 47A60 *Received March* 4, 1996

1. INTRODUCTION

The "holomorphic functional calculus" was developed by N. Dunford and A.E. Taylor in the forties (see [2], [6]). Given a continuous operator T acting on a complex Banach space X, this calculus associates an operator $f(T) \in L(X)$ to each holomorphic function f, defined on a neighborhood of the spectrum of T. Gindler [4] extended it by associating a closed operator to each meromorphic function of a specific class.

Many efforts have been devoted to extend the holomorphic functional calculus in other directions. In [1], we studied a local functional calculus defined as follows. Given an operator $T \in L(X)$ with the Single-Valued Extension Property (SVEP in short) and an arbitrary analytic function $f : \triangle(f) \subset \mathbb{C} \longrightarrow \mathbb{C}$, we set

$$D(f[T]) := \{x \in X \ : \ \sigma(x,T) \subset \triangle(f)\} \quad \text{and} \quad f[T]x := \frac{1}{2\pi i}\int_\Gamma f(\lambda)\widehat{x}_T(\lambda)d\lambda.$$

We obtained a linear operator $f[T] : D(f[T]) \subset X \longrightarrow X$. In the case $\sigma(T) \subset \triangle(f)$, we have that $f[T]$ coincides with $f(T)$, the operator of the holomorphic functional calculus.

The purpose of this paper is to develop this local functional calculus. The operator $f[T]$ is neither continuous nor closed in general, but we provide sufficient conditions for continuity and closability. We prove a local spectral mapping theorem, which asserts that $\sigma(x, f[T]) = f(\sigma(x,T))$, for all x in the domain of $f[T]$.

Whether the operator $f[T]$ is always closable remains an open problem, but we show this property for functions f of the meromorphic functional calculus, and we give a sufficient condition for the operator $f\{T\}$ of the meromorphic functional calculus to be the minimal closed extension of $f[T]$. Moreover, we describe the local resolvent function in terms of the local functional calculus and, as an application of this result, we derive local resolvent equations and an expression for the derivatives of the local resolvent function, and we describe $f[T+N]$ in terms of $f[T]$ when N is a quasinilpotent operator.

Given an operator $T \in L(X)$, a complex number λ belongs to the *resolvent set* $\rho(T)$ of T if there exists $(\lambda - T)^{-1} =: R(\lambda, T) \in L(X)$. We denote by $\sigma(T) := \mathbb{C}\backslash\rho(T)$ the *spectrum* of T. The *resolvent map* $R(.,T) : \rho(T) \to L(X)$ is analytic.

Moreover, given an arbitrary linear operator $A : D(A) \subset X \longrightarrow X$ and $x \in X$, a complex number λ belongs to the *local resolvent set* of A at x, denoted $\rho(x, A)$, if there exists an analytic function $w : U \subset \mathbb{C} \longrightarrow D(A)$, defined on a neigbourhood U of λ, which satisfies

$$(\mu - A)w(\mu) = x, \tag{1}$$

for every $\mu \in U$. The *local spectrum set* of A at x is $\sigma(x, A) := \mathbb{C} \setminus \rho(x, A)$. Since w is not necessarily unique, a complementary property is needed to prevent ambiguity.

A linear operator A satisfies the *Single Valued Extension Property* (SVEP) if for open subset $U \subset \mathbb{C}$ every analytic function $h : U \longrightarrow D(A)$, the condition $(\lambda - A)h(\lambda) \equiv 0$ implies $h \equiv 0$. If A satisfies the SVEP, then for every $x \in X$ there exists a unique analytic function $\widehat{x}_A$ defined on $\rho(x, A)$ satisfying (1), which is called the *local resolvent function of A at x*. See [3] for further details.

If for all closed set $F \subset \mathbb{C}$, $\{x \in X \ : \ \sigma(x, A) \subset F\}$ is closed, then we say that A has *property (C)*. If $T \in L(X)$ satisfies property (C) then T has the SVEP [5, Theorem 2.13].

2. Properties of the Local Functional Calculus

The operator provided by the local functional calculus is not continuous or closed in general. If T is the operator on $\ell_2(\mathbb{N})$ defined by $Te_n = \frac{1}{n}e_n$, and $f(\lambda) = 1/\lambda$, then $f[T]$ is not continuous, since $\|f[T]e_n\| = n$. Moreover, $f[T]$ is not closed, but it is closable because T^{-1} is a closed extension of $f[T]$.

In the following result we derive a sufficient condition for $f[T]$ to be a continuous operator.

THEOREM 2.1. *Assume $T \in L(X)$ has the SVEP, and let f be an analytic function. If $D(f[T])$ is closed, then $f[T]$ is continuous, and $f[T] = f(T|D(f[T]))$.*

COROLLARY 2.1. *Assume $T \in L(X)$ has property (C), and let f be an analytic function.*

1. *$D(f[T])$ is closed if and only if $\cup\{\sigma(x,T) : x \in D(f[T])\}$ is closed.*
2. *If $f[T]$ is a closed operator, then it is continuous.*

The converses of Theorem 2.1 and part (2) of Corollary 2.1 are not true. If T is again the operator on $\ell_2(\mathbb{N})$ given by $Te_n = \frac{1}{n}e_n$ and $f(\lambda) = \sin\frac{1}{\lambda}$, then $D(f[T])$ is not closed and $f[T]$ is continuous, but it is not closed.

The following result is a local spectral mapping theorem for the local functional calculus.

THEOREM 2.2. *Assume $T \in L(X)$ has property (C), and let f be an analytic function on $\triangle(f)$. Then $f(\sigma(x,T)) = \sigma(x, f[T])$ for every $x \in D(f[T])$.*

3. MEROMORPHIC CALCULUS AND LOCAL FUNCTIONAL CALCULUS

The definition of the holomorphic functional calculus was extended to meromorphic functions by Gindler [4]. Let f be a meromorphic function on an open set $\triangle(f)$ containing $\sigma(T)$, and let $\alpha_1, \ldots, \alpha_k$ be the poles of f in $\sigma(T)$, with multiplicities $n_1, \ldots, n_k$, respectively. We assume that the poles of f are not eigenvalues of T, and consider the polynomial $p(\lambda) = \prod_{i=1}^{k}(\alpha_i - \lambda)^{n_i}$.

The function $g(\lambda) := f(\lambda)p(\lambda)$ is analytic on a neighborhood of $\sigma(T)$. So we can define the function $f\{T\}$ of the *meromorphic functional calculus* by

$$f\{T\} := g(T)p(T)^{-1},$$

obtaining a closed operator $f\{T\}$.

Here we study the relation between the meromorphic calculus and the local functional calculus.

THEOREM 3.1. *Assume $T \in L(X)$ has the SVEP and let f be a function of the meromorphic functional calculus. Then $f\{T\}$ is a closed extension of $f[T]$. Consequently $f[T]$ is closable.*

The closure of the operator $f[T]$ is known in some cases.

Proposition 3.1. *Let T be a normal operator on a Hilbert space $\mathcal{H}$ and let f be an function of the meromorphic calculus. Then $\overline{f[T]} = f\{T\}$.*

Open Problem Is the operator $f[T]$ always closable?

4. Applications and Examples

Given operators $S, T \in L(X)$, for $\lambda, \mu \in \rho(T)$, the equality

$$R(\lambda, T) - R(\mu, T) = (\mu - \lambda)R(\lambda, T)R(\mu, T) \tag{2}$$

is called the *first resolvent equation*, and for $\lambda \in \rho(S) \cap \rho(T)$, the equality

$$R(\lambda, S) - R(\lambda, T) = R(\lambda, S)(S - T)R(\lambda, T); \tag{3}$$

is called the *second resolvent equation.* If we try to establish equalities (2) and (3) for the local resolvent function, we find that the "products of vectors" $\widehat{x}_T(\lambda)\widehat{x}_T(\mu)$ and $\widehat{x}_S(\lambda)(S - T)\widehat{x}_T(\lambda)$ are undefined. This drawback may be solved using the local functional calculus.

For every $\lambda \in \mathbb{C}$, we denote by f_λ the function given by $f_\lambda(\mu) := (\lambda - \mu)^{-1}$.

Lemma 4.1. *Assume $T \in L(X)$ has the SVEP and let $x \in X$. If $\lambda \in \rho(x, T)$, then $\widehat{x}_T(\lambda) = f_\lambda[T]x$.*

This lemma will be the key to establish the local resolvent equations.

Theorem 4.1. *Let $x \in X$ and let $S, T \in L(X)$ satisfying the SVEP.*

(i) *(The first local resolvent equation). If $\lambda, \mu \in \rho(x, T)$, then*

$$f_\lambda[T]x - f_\mu[T]x = (\mu - \lambda)f_\lambda[T]f_\mu[T]x. \tag{4}$$

(ii) *(The second local resolvent equation). If $\lambda \in \rho(x, T) \cap \rho(x, S)$, then*

$$f_\lambda[S]x - f_\lambda[T]x = f_\lambda[S](S - T)f_\lambda[T]x. \tag{5}$$

Next, using the first local resolvent equation we derive some well-known properties of the local resolvent function that will be later needed. Here f_λ^n is the function given by $f_\lambda^n(\mu) := (\lambda - \mu)^{-n}$.

PROPOSITION 4.1. *Assume $T \in L(X)$ has the SVEP and let $x \in X$. For $\lambda \in \rho(x,T)$, we have*

$$\frac{d^n \hat{x}_T(\lambda)}{d\lambda^n} = (-1)^n n! f_\lambda^{n+1}[T]x.$$

Moreover, if $|\lambda - \mu| < dist(\lambda, \sigma(x,T))$, then $\hat{x}_T(\mu) = \sum_{n=0}^{\infty} (\lambda - \mu)^n f_\lambda^{n+1}[T]x$.

Finally, we give a perturbation result similar to [2, Corollary VII.6.12] for the local functional calculus. Recall that $N \in L(X)$ is said to be *quasinilpotent* if $\sigma(N) = \{0\}$.

THEOREM 4.2. *Assume $T \in L(X)$ has the SVEP, and let N be a quasinilpotent operator that commutes with T. If f is an analytic function, then we have $D(f[T+N]) = D(f[T])$ and*

$$f[T+N] = \sum_{n=0}^{\infty} \frac{N^n}{n!} \left(\frac{d^n f}{d\lambda^n}\right)[T].$$

Finally, we illustrate some properties of the local calculus using the multiplication operator in $L_2(\mu)$. Given a finite positive measure μ with bounded support, the multiplication operator on $L_2(\mu)$ is given by $Mx(\lambda) := \lambda x(\lambda)$, $x \in L_2(\mu)$, and the support of the measure μ coincides with the spectrum $\sigma(M)$ of M.

THEOREM 4.3. *Let M be the multiplication operator on $L_2(\mu)$ and let f an analytic function on $\triangle(f)$. Then the following assertions hold.*

(i) $\sigma(x,M) = \{\lambda \in \mathbb{C} \ : \ \forall \varepsilon > 0 \quad \chi_{D(\lambda,\varepsilon)}x \neq 0\}$, *where $\chi_{D(\lambda,\varepsilon)}$ is the characteristic function of the disk of center λ and radius ε.*

(ii) *If $x \in D(f[T])$, then $f[M]x(\lambda) = f(\lambda)x(\lambda)$.*

(iii) *$f[M]$ is injective if and only if f is nonzero on every component of $\triangle(f)$ and no zero of f is an eigenvalue of M.*

(iv) *If $\sigma(M) \setminus \{\alpha_0\} \subset \triangle(f)$, and α_0 is not an isolated point of $\sigma(M)$, then $f[M]$ is closable. Moreover $f[M]$ is continuous if and only if $f \in L_\infty(\mu)$.*

(v) *The domain of $f[M]$ is dense if and only if $\mu(\sigma(M) \setminus \triangle(f)) = 0$.*

References

[1] BERMÚDEZ, T., GONZÁLEZ, M., MARTINÓN, A., Stability of the local spectrum, to appear in *Proc. Amer. Math. Soc.*

[2] DUNFORD, N., SCHWARTZ, J.T., "Linear Operators, Part I: General Theory", Interscience, New York, 1958.

[3] ERDELYI, I., LANGE, R., "Spectral Decompositions on Banach Spaces", Lecture Notes in Math. 623, Springer, 1977.

[4] GINDLER, H., An operational calculus for meromorphic functions, *Nagoya Math. J.,* **26** (1966), 31–38.

[5] RADJABALIPOUR, M., Decomposable Operator, *Bull. Iranian Math. Soc.,* **9** (1978), 1L-49L.

[6] TAYLOR, A.E., LAY, D.C., "Introduction to functional analysis" (2nd. ed.), Wiley, 1980.