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Some Results on Norm Attaining Bilinear Forms on $L^1[0,1]$

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We characterize the norm attaining bilinear forms on $L^1[0,1]$, and show that the set of norm attaining ones is not dense in the space of continuous bilinear forms on $L^1[0,1]$.

For a real Banach space X we let $\mathcal{B}(X)$ denote the Banach space of continuous bilinear forms on X, endowed with the norm $\|\varphi\| = \sup\{|\varphi(x,y)| : \|x\| \leq 1, \|y\| \leq 1\}$. We say that $\varphi \in \mathcal{B}(X)$ attains its norm if there are $x, y \in X$, $\|x\| = \|y\| = 1$ such that $\|\varphi\| = |\varphi(x,y)|$. We let $\mathcal{B}_{na}(X)$ denote the set of all norm attaining ones.

Motivated by the Bishop-Phelps theorem [3] that the set of norm attaining functionals is dense in X^*, we consider a natural question about the denseness of $\mathcal{B}_{na}(X)$. M. D. Acosta, F. J. Aguirre, and R. Payá [1] first answered this question in the negative. They proved that $\mathcal{B}_{na}(G)$ is not dense in $\mathcal{B}(G)$, where G is the Banach space used by W. T. Gowers [6] to show that ℓ_p $(1 < p < \infty)$ fails Lindenstrauss' property B [7]. On the other hand, it was shown that $\mathcal{B}_{na}(X)$ is dense in $\mathcal{B}(X)$ for a Banach space X with the Radon-Nikodym property or property (α) [8] by R. Aron, C. Finet, and E. Werner [2], and a Banach space X with a monotone shrinking basis and the Dunford-Pettis property like c_0 by Y. S. Choi and S. G. Kim [5]. This note is an announcement of the main results in [4], where we characterize the norm attaining bilinear forms on $L^1[0,1]$ through an integral representation of a bilinear form on $L^1[0,1]$, and show that $\mathcal{B}_{na}(L^1[0,1])$ is not dense in $\mathcal{B}(L^1[0,1])$. Let I stand for the unit interval $[0,1]$.

THEOREM 1. *To each* $\varphi \in \mathcal{B}(L^1(I))$ *there corresponds a unique* f *in* $L^\infty(I \times I)$ *such that* $\|\varphi\| = \|f\|_\infty$ *and*

$$\varphi(g,h) = \int_{I\times I} f(x,y)g(x)h(y)\ dxdy$$

for all $g, h \in L^1(I)$.

For $f \in L^\infty(I \times I)$ we set

$$S_f^+ = \{(x,y) \in I \times I : f(x,y) = \|f\|_\infty\},$$

and

$$S_f^- = \{(x,y) \in I \times I : f(x,y) = -\|f\|_\infty\}.$$

LEMMA 2. *Suppose that* $f \in L^\infty(I \times I)$ *with* $\|f\|_\infty \neq 0$ *and that for any measurable rectangle* $A \times B$ *in* $I \times I$ *with positive measure,* $|(A \times B) \cap S_f^+| < |A \times B|$ *and* $|(A \times B) \cap S_f^-| < |A \times B|$. *Then*

$$\left|\int_{I\times I} f(x,y)g(x)h(y)\ dxdy\right| < \|f\|_\infty$$

for all $g, h \in L^1(I)$, $\|g\|_1 = \|h\|_1 = 1$.

THEOREM 3. $\varphi \in \mathcal{B}(L^1(I))$ *is norm attaining if and only if for its corresponding* $f \in L^\infty(I \times I)$ *as in Theorem 1 there is a measurable rectangle* $A \times B$ *in* $I \times I$ *with positive measure such that either* $|A \times B| = |(A \times B) \cap S_f^+|$ *or* $|A \times B| = |(A \times B) \cap S_f^-|$.

LEMMA 4. *There is a measurable subset* S *of* $I \times I$ *with positive measure such that for any measurable rectangle* $A \times B$ *in* $I \times I$ *with positive measure,* $|(A \times B) \cap S| < |A \times B|$.

THEOREM 5. $\mathcal{B}_{na}(L^1(I))$ *is not dense in* $\mathcal{B}(L^1(I))$.

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