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The Powers of the Bivariate Bernstein Operators

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1. Introduction

Kelisky and Rivlin [8] were the first to investigate the powers of the well–known Bernstein polynomials

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right) \qquad (n \in \mathbb{N})$$

which are defined recursively: $B_n^k(f;x) = B_n(B_n^{k-1}(f;x))$. They studied the convergence of $B_n^k(f;x)$ as $k \to \infty$, both in the case that k is independent on n and, for polynomial f, when k is a function of n. Their results were used and extended in further research papers ([3, 7, 10, 11, 12]).

In this note we generalize the work of Kelisky and Rivlin on powers of the Bernstein operators to the two–dimensional case.

The bivariate Bernstein operators on the simplex

$$S = \{(x,y) \mid x+y \le 1, x \ge 0, y \ge 0\}$$

are given by

$$(1) \quad B_n(f;x,y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{n!}{i!j!(n-i-j)!} x^i y^j (1-x-y)^{n-i-j} f\left(\frac{i}{n}, \frac{j}{n}\right).$$

Multidimensional Bernstein operators were early introduced by Stancu [14] and Lorentz [9]. In 1986 Ditzian [5] proved an inverse result. The weighted approximation problem was discussed in [4]. Very recently, lower and upper bounds for the multivariate Bernstein operators on a simplex or a cube are achieved in [15].

In this note we first show that for every fixed $n = 1, 2, 3, \ldots$ $B_n^k(f; x, y)$ converges as $k \to \infty$ to the linear function interpolating to the function f at each vertex of the simplex S. This was proved in the one–dimensional case in [8, 13], and the result is true even in the higher dimensional case (see [3]).

Furthermore, we study the behaviour of $B_n^{g(n)}(f; x, y)$, where $g(n)$ is a positive integer, as $n \to \infty$. As main result we determine, for polynomial f, the limit $\lim_{n\to\infty} B_n^{g(n)}(f; x, y)$ provided $\lim_{n\to\infty}(g(n)/n)$ exists.

Finally, we remark that our results imply the above mentioned results in the one–dimensional case if we put $y = 0$.

2. Preliminary Results

For the study of the powers of the Bernstein operators there is no loss of generality replacing f by a polynomial. In particular, for the special functions $f(x, y) = x^p y^q$ $(p, q \in \mathbb{N}_0)$ we calculate the associated Bernstein polynomial $B_n(x^p y^q; x, y)$.

Lemma 1. *For* $p, q = 0, 1, 2, 3, \ldots$ *with* $p + q = s \le n$ *we have*

$$B_n(x^p y^q; x, y) = \frac{1}{n^{p+q}} \sum_{r=0}^{n} \pi_r n^r \sum_{i+j=r} \sigma_p^i \sigma_q^j x^i y^j, \tag{2}$$

where π_r *is defined as*

$$\pi_r = \begin{cases} (1 - \frac{1}{n})(1 - \frac{2}{n}) \cdots (1 - \frac{r-1}{n}) & \text{for} \quad r = 2, 3, \ldots, s, \\ 1 & \text{for} \quad r = 0, 1. \end{cases} \tag{3}$$

The quantities σ_n^k denote the Stirling numbers of the second kind, defined by

$$x^n = \sum_{k=0}^{n} \sigma_n^k x^{(k)} \qquad (n \in \mathbb{N}_0), \tag{4}$$

where $x^{(k)} = x(x-1)\cdots(x-k+1)$ is the falling factorial (see, e.g., [6]). For the one–dimensional version of Lemma 1 see [8] (cf. [1, Proof of Lemma 1]).

Proof of Lemma 1. Following an argument of Ditzian [5] we set

$$F(x,y,z)=\sum_{k=0}^{n}\sum_{m=0}^{n-k}\binom{n}{k}\binom{n-k}{m}x^k y^m z^{n-k-m}=(x+y+z)^n.$$

Note that $B_n(\mathbf{1};x,y)=F(x,y,1-x-y)=\mathbf{1}\quad(n=1,2,\dots)$.
For $\frac{\partial^i}{\partial x^i}x^k=k^{(i)}x^{k-i}$, we have, by (4),

$$k^p x^k=\sum_{i=0}^{p}\sigma_p^i k^{(i)}x^k=\sum_{i=0}^{p}x^i\sigma_p^i\frac{\partial^i}{\partial x^i}x^k.$$

Therefore, we get with $z=1-x-y$

$$B_n(x^py^q;x,y)=\sum_{k=0}^{n}\sum_{m=0}^{n-k}\binom{n}{k}\binom{n-k}{m}\left(\frac{k}{n}\right)^p\left(\frac{m}{n}\right)^q x^k y^m z^{n-k-m}$$

$$=\frac{1}{n^{p+q}}\sum_{k=0}^{n}\sum_{m=0}^{n-k}\binom{n}{k}\binom{n-k}{m}\sum_{i=0}^{p}x^i\sigma_p^i\left(\frac{\partial^i}{\partial x^i}x^k\right)\sum_{j=0}^{q}y^j\sigma_q^j\left(\frac{\partial^j}{\partial y^j}y^m\right)z^{n-k-m}$$

$$=\frac{1}{n^{p+q}}\sum_{i=0}^{p}\sum_{j=0}^{q}\sigma_p^i\sigma_q^j x^i y^j\frac{\partial^{i+j}}{\partial x^i\partial y^j}\sum_{k=0}^{n}\sum_{m=0}^{n-k}\binom{n}{k}\binom{n-k}{m}x^k y^m z^{n-k-m}$$

$$=\frac{1}{n^{p+q}}\sum_{i=0}^{p}\sum_{j=0}^{q}\sigma_p^i\sigma_q^j x^i y^j n^{(i+j)}(x+y+z)^{n-i-j}$$

$$=\frac{1}{n^{p+q}}\sum_{i=0}^{p}\sum_{j=0}^{q}n^{i+j}\pi_{i+j}\sigma_p^i\sigma_q^j x^i y^j.\qquad\blacksquare$$

Following Kelisky and Rivlin [8], we use the language of linear algebra. We consider for $p,q\in\mathbb{N}_0$ the functions $f(x,y)=x^py^q$, where $p+q=s$ is a fixed integer satisfying $s\le n$. Let $A=(A_{ij})$ denote the block–matrix consisting of the submatrices

$$(5)\qquad A_{ij}=\pi_i n^{i-j}\begin{pmatrix}\sigma_j^i\sigma_0^0 & \cdots & \sigma_0^i\sigma_j^0\\ \vdots & & \vdots\\ \sigma_j^0\sigma_0^i & \cdots & \sigma_0^0\sigma_j^i\end{pmatrix}\qquad(i,j=1,\dots,s).$$

Because of $\sigma_n^k=0\ (k>n)$, A is upper triangular with $A_{ii}=E_{i+1}\ (i=1,\dots,s)$, where E_m denotes the $m\times m$ unit matrix.

Then A is the corresponding matrix of the operator B_n. By Lemma 1, the coefficient of x^iy^j in $B_n(x^py^q;x,y)$ is the element in the $(j+1)$-th row and the $(q+1)$-th column of the submatrix $A_{i+j,p+q}$.

LEMMA 2. *The matrix A is similar to a diagonal matrix.*

Proof. It is sufficient to prove that A possesses $m = \sum_{k=2}^{s+1} k$ linearly independent eigenvectors. We show that for every eigenvalue π_i ($i = 1, \dots, s$) there exist $i+1$ linearly independent eigenvectors.

The eigenvectors v corresponding to the eigenvalue π_i are the nontrivial solutions of the system of linear equations

$$(A - \pi_i E_m)v = o. \tag{6}$$

If we divide the matrix $(A - \pi_i E_m)$, the vector v and the null vector in suited submatrices the system becomes

$$(A - \pi_i E_m)v = \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ O & O & D_{23} \\ O & O & D_{33} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} o \\ o \\ o \end{pmatrix},$$

i.e. the single equations

$$D_{11}v_1 + D_{12}v_2 + D_{13}v_3 = o, \quad D_{23}v_3 = o \quad \text{and} \quad D_{33}v_3 = o.$$

Here, O and o denote a suited null matrix resp. null vector.

The square submatrix D_{33} is upper triangular with $\det D_{33} = \pi_{i+1} \cdots \pi_s \neq 0$ and hence $v_3 = o$. From this we get

$$D_{11}v_1 = -D_{12}v_2. \tag{7}$$

By the same argument, there holds $\det D_{11} \neq 0$. If we choose $v_2 = e_j$ ($j = 1, \dots, i+1$), where e_j is the unit vector of $i+1$ components and the 1 at position j, Equation (7) has an unique solution $v_1 = v_1^{(j)}$. The vectors

$$v = v^{(j)} = \begin{pmatrix} v_1^{(j)} \\ e_j \\ o \end{pmatrix} \qquad (j = 1, \dots, i+1)$$

are solutions of (6) which obviously are linearly independent. This proves Lemma 2. ∎

Let V be the matrix consisting of the m eigenvectors from Lemma 2 as columns. Since these eigenvectors are linearly independent the matrix V is invertible and we have $V^{-1}AV = \Lambda$ with a diagonal matrix Λ of the form

$$\Lambda = \begin{pmatrix} \pi_1 E_2 & O & O & \cdots & O \\ O & \pi_2 E_3 & O & \cdots & O \\ O & O & \pi_3 E_4 & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & \pi_s E_{s+1} \end{pmatrix}.$$

Analogous to A we divide the matrix $V = (V_{ij})$ in submatrices V_{ij} of type $(i+1, j+1)$ $(i, j = 1, \dots, s)$.

LEMMA 3. *The matrix $V = (V_{ij})$ is upper triangular and the submatrices V_{ii} $(i = 1, \dots, s)$ are unit matrices.*

Proof. In Lemma 2 we have seen that there are $i+1$ linearly independent eigenvectors to each eigenvalue π_i $(i = 1, \dots, s)$. Let $v_i^{(j)}$ $(j = 1, \dots, i+1)$ be eigenvectors belonging to the eigenvalue π_i $(i = 1, \dots, s)$. Lemma 3 follows by the special form of the matrix V which consists of the eigenvectors as columns. ∎

Recall that V consists of the eigenvectors $v_i^{(j)}$ $(j = 1, \dots, i+1)$ of A belonging to the eigenvalue π_i $(i = 1, \dots, s)$. Let the matrix V^{-1} be build up by the $(i+1) \times (j+1)$–submatrices W_{ij} $(i, j = 1, \dots, s)$.

LEMMA 4. *The matrix $V^{-1} = (W_{ij})$ is upper triangular and $W_{kk} = E_{k+1}$ for $k = 1, \dots, s$.*

Proof. Since V is upper triangular, it follows the first part of Lemma 4. In order to show $W_{kk} = E_{k+1}$ $(k = 1, \dots, s)$, we represent V^{-1} by its adjoint

$$V^{-1} = \frac{1}{\det V} \widetilde{V}^T \tag{8}$$

with $\widetilde{v}_{ij} = (-1)^{i+j} \det \widetilde{V}_{ij}$, where $\widetilde{V}_{ij}$ denotes the minor of V, i.e. the matrix obtained from V by crossing out the i-th row and the j-th column. Note that $\det V = 1$.

Now let k be fixed and $W_{kk} = (w_{ij})$. Then $w_{ij} = \widetilde{v}_{ji}$.

First we study the case $i = j$. We arrange V^{-1} to be a $m \times m$–matrix.

For $i = j$ we have

$$\widetilde{V}_{ii} = \begin{pmatrix} 1 & \widetilde{v}_{12} & \cdots & \widetilde{v}_{1m} \\ 0 & 1 & \cdots & \widetilde{v}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

and therefore $\widetilde{v}_{ii} = (-1)^{2i} \det \widetilde{V}_{ii} = 1$ which gives, by (8), $w_{ii} = 1$.

In the case $i < j$ the matrix $\widetilde{V}_{ij}$ is upper triangular and the main diagonal contains at least one zero. Therefore

$$\widetilde{v}_{ij} = (-1)^{i+j} \det \widetilde{V}_{ij} = 0 \qquad \text{and thus} \qquad w_{ji} = \frac{1}{\det V} \widetilde{v}_{ij} = 0.$$

It remains the case $i > j$. Let v_{ij} be an element of V_{kk} with $i > j$. Furthermore, let V_{kk}^* be the submatrix which results from V_{kk} by crossing out the i-th row and the j-th column. Because of $i > j$ the matrix V_{kk}^* contains a row consisting of all entries 0.

Next we consider the matrix

$$D = \begin{pmatrix} V_{11} & V_{12} & \cdots & V_{1k}^* \\ O & V_{22} & \cdots & V_{2k}^* \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & V_{kk}^* \end{pmatrix},$$

where V_{mk}^* represent the submatrices V_{mk} $(m = 1, \ldots, k-1)$ without the j-th column. We see that D has one row consisting of all entries 0. Thus $\det D = 0$ and it follows

$$\widetilde{v}_{ij} = (-1)^{i+j} \det \widetilde{V}_{ij} = 0 \qquad \text{which implies} \qquad w_{ji} = \frac{1}{\det V} \widetilde{v}_{ij} = 0.$$

Hence W_{kk} is an unit matrix for $k = 1, \ldots, s$. ∎

For abbreviation, let C_{ij} denote the matrix of type $(i+1, j+1)$ with

$$A_{ij} = \frac{\pi_i}{n^{j-i}} C_{ij}. \tag{9}$$

LEMMA 5. *For the matrix* $V^{-1} = (W_{ij})$ *we have*

$$W_{1j} = C_{1j}. \tag{10}$$

Proof. On account of the properties of the Stirling numbers of the second kind C_{1j} is of the form

$$C_{1j} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Because of $W_{11} = E_2$ the assertion is obvious for $j = 1$. We proceed with mathematical induction on j. Let now $j > 1$. If we calculate the submatrix at position $(1, j)$ on both sides of the equation $V^{-1}A = \Lambda V^{-1}$, i.e.

$$\begin{pmatrix} W_{11} & W_{12} & W_{13} & \cdots & W_{1s} \\ O & W_{22} & W_{23} & \cdots & W_{2s} \\ O & O & W_{33} & \cdots & W_{3s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & W_{ss} \end{pmatrix} \begin{pmatrix} A_{11} & \cdots & A_{1j} & \cdots & A_{1s} \\ \vdots & \ddots & \vdots & & \vdots \\ O & \cdots & A_{jj} & \cdots & A_{js} \\ \vdots & & \vdots & \ddots & \vdots \\ O & \cdots & O & \cdots & A_{ss} \end{pmatrix}$$

$$= \begin{pmatrix} \pi_1 E_2 & O & O & \cdots & O \\ O & \pi_2 E_3 & O & \cdots & O \\ O & O & \pi_3 E_4 & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & \pi_s E_{s+1} \end{pmatrix} \begin{pmatrix} W_{11} & \cdots & W_{1j} & \cdots & W_{1s} \\ \vdots & \ddots & \vdots & & \vdots \\ O & \cdots & W_{jj} & \cdots & W_{js} \\ \vdots & & \vdots & \ddots & \vdots \\ O & \cdots & O & \cdots & W_{ss} \end{pmatrix},$$

we receive

$$\sum_{i=1}^{j} W_{1i} A_{ij} = \pi_1 W_{1j}.$$

With $A_{jj} = \pi_j E_{j+1}$ we get

$$\sum_{i=1}^{j-1} W_{1i} A_{ij} = (\pi_1 - \pi_j) W_{1j}$$

and hence

$$W_{1j} = \frac{1}{\pi_1 - \pi_j} \sum_{i=1}^{j-1} W_{1i} A_{ij} = \frac{1}{\pi_1 - \pi_j} \sum_{i=1}^{j-1} \frac{1}{n^{j-i}} \pi_i W_{1i} C_{ij}$$

$$= \frac{1}{\pi_1 - \pi_j} \sum_{i=1}^{j-1} \frac{1}{n^{j-i}} \pi_i C_{1i} C_{ij}.$$

The special form of the submatrix C_{ij} implies

$$W_{1j} = \frac{1}{\pi_1 - \pi_j} \sum_{i=1}^{j-1} \frac{1}{n^{j-i}} \pi_i \sigma_j^i C_{1j}.$$

In order to prove (10), we have to show

$$\sum_{i=1}^{j-1} \frac{1}{n^{j-i}} \pi_i \sigma_j^i = \pi_1 - \pi_j \qquad (j = 2, 3, \dots)$$

which easily follows by induction on j. ∎

We defined the matrix A to consist of submatrices A_{ij}. Now we consider the actŭal rows and columns of A. For that reason in the following we denote the number of rows and columns of the matrix A until the end of the submatrices A_{rr} by $m(r)$. It is obvious that for $r = 1, \dots, s$

$$m(r) = \sum_{i=2}^{r+1} i \qquad \text{and we set} \qquad m(s) =: m.$$

Furthermore, for $p + q = r$ let $e_{pq}^{(r)}$ denote the unit vector of m components and the 1 at position $m(r) - p$.

3. The Powers B_n^k as $k \to \infty$

In this section we investigate the limit of B_n^k as $k \to \infty$ for fixed n.

Lemma 6. *Let* $A^k e_{pq}^{(s)} = \left(a_{1,m-p}^{(k)}, \dots, a_{m,m-p}^{(k)}\right)^T$, *then*

$$B_n^k(x^p y^q; x, y) = \sum_{\mu=0}^{s} \sum_{\nu=0}^{s-\mu} a_{m(\mu+\nu)-\mu, m-p}^{(k)} x^\mu y^\nu, \tag{11}$$

where $a_{m(0),m-p}^{(k)}$ *is to be read as* 0.

Proof. Let

$$P(x, y) = \sum_{\mu=0}^{s} \sum_{\nu=0}^{s-\mu} \alpha_{\mu\nu} x^\mu y^\nu$$

be an arbitrary polynomial of degree s with $\alpha_{0,0} = 0$ (for example $B_n^j(x^p y^q; x, y)$ with $p + q = s$). Then we have, by Lemma 1,

$$B_n(P(x,y); x, y) = \sum_{\mu=0}^{s}\sum_{\nu=0}^{s-\mu} \beta_{\mu\nu} x^\mu y^\nu = \sum_{\mu=0}^{s}\sum_{\nu=0}^{s-\mu} \alpha_{\mu\nu} B_n(x^\mu y^\nu; x, y)$$

$$= \sum_{\mu=0}^{s}\sum_{\nu=0}^{s-\mu} \alpha_{\mu\nu} \frac{1}{n^{\mu+\nu}} \sum_{r=0}^{s} \pi_r n^r \sum_{i+j=r} c_{ij}^{(\mu,\nu)} x^i y^j$$

$$= \sum_{\mu=0}^{s}\sum_{\nu=0}^{s-\mu}\sum_{r=0}^{s}\sum_{i+j=r} \alpha_{\mu\nu} \frac{1}{n^{\mu+\nu}} \pi_r n^r c_{ij}^{(\mu,\nu)} x^i y^j$$

$$= \sum_{\mu=0}^{s}\sum_{\nu=0}^{s-\mu}\sum_{r=0}^{s}\sum_{i+j=r} \alpha_{\mu\nu} a_{m(r)-i, m(\mu+\nu)-\mu} x^i y^j.$$

This obviously means that

$$(\beta_{1,0}, \beta_{0,1}, \dots, \beta_{s,0}, \dots, \beta_{0,s})^T = A(\alpha_{1,0}, \alpha_{0,1}, \dots, \alpha_{s,0}, \dots, \alpha_{0,s})^T.$$

The lemma now follows by mathematical induction on k. ∎

LEMMA 7. *Let n be a fixed positive integer, then*

$$(12) \qquad \lim_{k\to\infty} A^k e_{pq}^{(s)} = \begin{cases} (1,0,0,\dots,0)^T & \text{for } p = s, \\ (0,1,0,\dots,0)^T & \text{for } q = s, \\ (0,0,0,\dots,0)^T & \text{for } p, q < s. \end{cases}$$

Proof. The equation $A = V\Lambda V^{-1}$ gives $\lim_{k\to\infty} A^k = V\left(\lim_{k\to\infty}\Lambda^k\right)V^{-1}$. Because of $\lim_{k\to\infty}\pi_1^k = 1$ and $\lim_{k\to\infty}\pi_j^k = 0$ $(j = 2, \dots, s)$ there holds

$$\lim_{k\to\infty}\Lambda^k = \lim_{k\to\infty}\begin{pmatrix} \pi_1^k E_2 & O & \cdots & O \\ O & \pi_2^k E_3 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \pi_s^k E_{s+1} \end{pmatrix} = \begin{pmatrix} E_2 & O & \cdots & O \\ O & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & O \end{pmatrix}.$$

By Lemma 3, V_{11} is an unit matrix and we get

$$\lim_{k\to\infty} A^k = V\left(\lim_{k\to\infty}\Lambda^k\right)V^{-1} = \begin{pmatrix} W_{11} & \cdots & W_{1s} \\ O & \cdots & O \\ \vdots & & \vdots \\ O & \cdots & O \end{pmatrix}.$$

Applying Lemma 5, it follows (12). ∎

Combining the preliminary results we receive as our first main result

THEOREM 1. *For each function f defined on*

$$S = \{(x,y) \mid x+y \leq 1, x \geq 0, y \geq 0\}$$

and every fixed positive integer n there holds

$$\begin{aligned} &\lim_{k\to\infty} B_n^k(f;x,y) \\ (13) \qquad &= f(0,0) + (f(1,0) - f(0,0))\,x + (f(0,1) - f(0,0))\,y. \end{aligned}$$

Proof. Let $B_n(f;x,y) = \sum_{i=0}^{s}\sum_{j=0}^{s-i} \alpha_{ij}x^iy^j$, then

$$B_n^k(f;x,y) = B_n^{k-1}(B_n(f;x,y)) =$$

$$B_n^{k-1}\left(\sum_{i=0}^{s}\sum_{j=0}^{s-i}\alpha_{ij}x^iy^j\right) = \sum_{i=0}^{s}\sum_{j=0}^{s-i}\alpha_{ij}B_n^{k-1}(x^iy^j;x,y).$$

Applying Lemma 6 and 7, we obtain

$$\lim_{k\to\infty} B_n^k(f;x,y) = \alpha_{0,0} + \sum_{i=1}^{s}\alpha_{i,0}x + \sum_{j=1}^{s}\alpha_{0,j}y.$$

By considering the definition of the bivariate Bernstein operators it follows (13). ∎

4. THE POWERS $B_n^{g(n)}$ AS $n \to \infty$

After investigating the iterates of the Bernstein operators for fixed n we now study the iterates $B_n^{g(n)}$, where $g(n)$ represents a positive integer for every $n \in \mathbb{N}$. For that reason we study the behaviour of the eigenmatrix $V = V(n)$ and its inverse $V^{-1} = V^{-1}(n)$ as $n \to \infty$.

From the equation $AV = V\Lambda$ we obtain for the submatrices V_{ij} of the matrix V

$$(14) \qquad V_{ij} = \frac{\pi_i}{\pi_j - \pi_i}\sum_{k=i+1}^{j}\frac{1}{n^{k-i}}C_{ik}V_{kj}$$

with C_{ik} as defined in (9).

We remark that for every $i, j \in \mathbb{N}$ with $i > j$

$$\pi_i - \pi_j = -\pi_j\left[n^{-1}\left(j + (j+1) + \cdots + (i-1)\right) + O(n^{-2})\right] \qquad (n \to \infty).$$

LEMMA 8. *For the submatrices* $V_{ij} = V_{ij}(n)$ $(i, j = 1, \dots, s)$ *of* $V = V(n)$ *we have*

$$V_{ij}(n) = \begin{cases} \frac{1}{n^{j-i}} \prod_{k=i}^{j-1} \frac{\pi_k}{\pi_j - \pi_k} \prod_{r=i}^{j-1} C_{r,r+1} + D_{ij}(n) & \text{for } i < j, \\ E_{i+1} & \text{for } i = j, \\ O & \text{for } i > j, \end{cases} \tag{15}$$

where $D_{ij}(n)$ *is a certain* $(i+1) \times (j+1)$*–matrix whose entries are all* $O(n^{-1})$ *as* $n \to \infty$.

Proof. In the cases $i = j$ and $i > j$ Lemma 8 follows by Lemma 3. In the case $i < j$ we start with the equations

$$V_{ij} = \frac{\pi_i}{\pi_j - \pi_i} \sum_{k=i+1}^{j} \frac{1}{n^{k-i}} C_{ik} V_{kj} \qquad (i < j) \tag{16}$$

which follow from $AV = V\Lambda$ and Lemma 3. Note that, by (3), $\pi_i \to 1$ and $\pi_i - \pi_j = O(n^{-1})$ as $n \to \infty$. Formula (16) implies (15) for the special case $i = j - 1$. Then Eq. (15) follows also for $i = j - 2, \dots, 1$ by induction on i. ▮

By (9), (5) and $\sigma_{i+1}^{i} = \binom{i+1}{2}$ for all $i \in \mathbb{N}$ it can be easily shown that the element of $\prod_{r=i}^{j-1} C_{r,r+1}$ at position (α, β) is

$$\binom{j-i}{\beta-\alpha} \prod_{\mu=i+2-\alpha}^{j-\beta+1} \binom{\mu}{2} \prod_{\nu=\alpha}^{\beta-1} \binom{\nu}{2}, \qquad \text{for} \quad 2 \le \alpha \le i, \quad \alpha \le \beta \le \alpha + j - i,$$

$$\prod_{\mu=i+1}^{j} \binom{\mu}{2}, \qquad \text{for} \quad (\alpha, \beta) = (1, 1) \quad \text{and for} \quad (\alpha, \beta) = (i+1, j+1)$$

and 0 else.

In order to handle the factor $\frac{1}{n^{j-i}} \prod_{k=i}^{j-1} \frac{\pi_k}{\pi_j - \pi_k}$ in (15) note that for $k < j$

$$\begin{aligned} \pi_j - \pi_k &= -\pi_k \left[n^{-1} \left(k + (k+1) + \dots + (j-1) \right) + O(n^{-2}) \right] \\ &= -\pi_k \frac{(j-k)(j+k-1)}{2n} + O(n^{-2}) \qquad (n \to \infty). \end{aligned}$$

After some calculations we get

LEMMA 9. *For $i,j=1,\dots,s$ we have* $\lim_{n\to\infty} V_{ij}(n) = V_{ij}^* = \left(v_{\alpha\beta}^{(i,j)^*}\right)$ *with*

$$
(17)\quad v_{\alpha\beta}^{(i,j)^*} = \begin{cases} (-1)^{j-i}\dfrac{\binom{j-1}{i-1}^2}{\binom{2j-2}{j-i}}\dbinom{j-i}{\beta-\alpha}\dfrac{\binom{i-1}{\alpha-1}\binom{i-1}{\alpha-2}}{\binom{j-1}{\beta-1}\binom{j-1}{\beta-2}} & \text{for } 2\le\alpha\le i,\\ & \alpha\le\beta\le\alpha+j-i,\\ (-1)^{j-i}\dfrac{\binom{j}{i}\binom{j-1}{i-1}}{\binom{2j-2}{j-i}} & \text{for } (\alpha,\beta)=(1,1) \text{ and}\\ & (\alpha,\beta)=(i+1,j+1),\\ 0 & \text{else.} \end{cases}
$$

In a completely analogous fashion the limit of V^{-1} as $n\to\infty$ may be calculated. We suppress the details and give only the result.

LEMMA 10. *For the submatrices W_{ij} $(i,j=1,\dots,s)$ of $V^{-1}=V^{-1}(n)$ we have*

$$
(18)\qquad W_{ij}(n) = \begin{cases} \dfrac{1}{n^{j-i}}\displaystyle\prod_{k=i+1}^{j}\frac{\pi_{k-1}}{\pi_i-\pi_k}\prod_{r=i}^{j-1}C_{r,r+1}+D_{ij}(n) & \text{for}\quad i<j,\\ E_{i+1} & \text{for}\quad i=j,\\ O & \text{for}\quad i>j, \end{cases}
$$

where $D_{ij}(n)$ is a certain $(i+1)\times(j+1)$–matrix whose entries are all $O(n^{-1})$ as $n\to\infty$.

Proceeding as above we get

LEMMA 11. *For $i,j=1,\dots,s$ we have* $\lim_{n\to\infty} W_{ij}(n) = W_{ij}^* = \left(w_{\alpha\beta}^{(i,j)^*}\right)$

with

$$(19)\quad w_{\alpha\beta}^{(i,j)^*} = \begin{cases} \dfrac{\binom{j-1}{i-1}^2}{\binom{j+i-1}{j-i}} \dbinom{j-i}{\beta-\alpha} \dfrac{\binom{i-1}{\alpha-1}\binom{i-1}{\alpha-2}}{\binom{j-1}{\beta-1}\binom{j-1}{\beta-2}} & \text{for } 2 \le \alpha \le i, \\ & \alpha \le \beta \le \alpha + j - i, \\ \dfrac{\binom{j}{i}\binom{j-1}{i-1}}{\binom{j+i-1}{j-i}} & \text{for } (\alpha, \beta) = (1,1) \\ & \text{and } (\alpha,\beta) = (i+1, j+1), \\ 0 & \text{else.} \end{cases}$$

In order to formulate our second main result we put $\Pi_j = \lim_{n\to\infty} \pi_j^n \quad (j \in \mathbb{N})$ which is easily determined, by (3), to be $\Pi_j = \exp\left(-\binom{j}{2}\right)$.

THEOREM 2. *Suppose $g(n)$ is a positive integer for every $n \in \mathbb{N}$ and*

$$(20)\qquad \lim_{n\to\infty} \frac{g(n)}{n} =: \gamma$$

exists, where $\gamma = \infty$ should be allowed. Then for $p + q = s \in \mathbb{N}$ we have

$$(21)\qquad \lim_{n\to\infty} B_n^{g(n)}(x^p y^q; x, y) = \sum_{k=1}^{s} \sum_{i+j=k} b_{ij} x^i y^j,$$

where the coefficients $b_{ij} = b_{ij}(p,q)$ are given by

$$(22)\quad b_{ij} = \begin{cases} c_{ks}^{(\gamma)} \dbinom{s-1}{k-1}^2 \dbinom{s-k}{p-i} \dfrac{\binom{k-1}{i-1}\binom{k-1}{i}}{\binom{s-1}{p-1}\binom{s-1}{p}} & \text{for } 1 \le i \le p \le s-1, \\ c_{ks}^{(\gamma)} \dbinom{s}{k}\dbinom{s-1}{k-1} & \text{for } i = p = 0 \\ & \text{and } i = k,\ p = s, \\ 0 & \text{else} \end{cases}$$

with

$$(23)\qquad c_{ks}^{(\gamma)} = \sum_{r=k}^{s} (-1)^{r-k}\, \Pi_r^{\gamma} \frac{\binom{s-k}{r-k}^2}{\binom{2r-2}{r-k}\binom{s+r-1}{s-r}} \qquad (k \in \mathbb{N}).$$

In the case $\gamma = \infty$ the quantity Π_j^{γ} in (22) means 1 for $j = 1$ resp. 0 for $j \ge 2$.

Proof. By the equation $A^{g(n)} = V\Lambda^{g(n)}V^{-1}$ we conclude

$$\lim_{n\to\infty} A^{g(n)} = V^*\Lambda^*\left(V^*\right)^{-1},$$

where

$$\Lambda^* = \lim_{n\to\infty} \Lambda^{g(n)}$$

is a diagonal matrix with entries $\Pi_j^\gamma = \lim\limits_{n\to\infty} \Pi_j^{g(n)/n}$ $(j = 1,\ldots,s)$ on its main diagonal. The matrices

$$V^* = \lim_{n\to\infty} V \qquad \text{and} \qquad \left(V^{-1}\right)^* = \left(V^*\right)^{-1} = \lim_{n\to\infty} V^{-1}$$

are completely determined in (17) and (19). Thus the convergence in (21) is proved.

We set $\lim\limits_{n\to\infty} A^{g(n)} = A^*$ with $A^* = \left(A^*_{\mu\nu}\right)$ $(\mu,\nu = 1,\ldots,s)$ with submatrices $A^*_{\mu\nu}$ of type $(\mu+1,\nu+1)$. Note that $A^*_{\mu\nu}$ is a null matrix for $\mu > \nu$.

We consider the case $\mu \le \nu$. In this case we have

$$A^*_{\mu\nu} = \sum_{r=\mu}^{\nu} V^*_{\mu r}\, W^*_{r\nu}\, \Pi_r^\gamma,$$

where $V^*_{\mu r}$ and $W^*_{r\nu}$ represent the submatrices of V^* and $\left(V^{-1}\right)^*$ respectively.

The entries of the submatrices $A^*_{\mu\nu}$ are given by

$$\text{(24)}\quad a_{\alpha\beta}^{(\mu,\nu)^*} = \sum_{r=\mu}^{\nu} \Pi_r^\gamma \sum_{t=1}^{r+1} v_{\alpha t}^{(\mu,r)^*}\, w_{t\beta}^{(r,\nu)^*} \quad (1 \le \alpha \le \mu+1,\ 1 \le \beta \le \nu+1).$$

Now we insert (17) and (19) in (24). For $(\alpha,\beta) = (1,1)$ we get

$$a_{11}^{(\mu,\nu)^*} = \sum_{r=\mu}^{\nu} \Pi_r^\gamma\, v_{11}^{(\mu,r)^*}\, w_{11}^{(r,\nu)^*} = \sum_{r=\mu}^{\nu} \Pi_r^\gamma\, (-1)^{r-\mu}\, \frac{\binom{r}{\mu}\binom{r-1}{\mu-1}}{\binom{2r-2}{r-\mu}} \cdot \frac{\binom{\nu}{r}\binom{\nu-1}{r-1}}{\binom{\nu+r-1}{\nu-r}}.$$

Using the identity $\displaystyle\binom{r}{\mu}\binom{\nu}{r} = \binom{\nu-\mu}{r-\mu}\binom{\nu}{\mu}$ for $\mu \le r \le \nu$ we have

$$\text{(25)}\qquad a_{11}^{(\mu,\nu)^*} = \binom{\nu}{\mu}\binom{\nu-1}{\mu-1} \sum_{r=\mu}^{\nu} \Pi_r^\gamma\, (-1)^{r-\mu}\, \frac{\binom{\nu-\mu}{r-\mu}^2}{\binom{2r-2}{r-\mu}\binom{\nu+r-1}{\nu-r}}.$$

Furthermore, we get $a_{1\beta}^{(\mu,\nu)^*} = 0$ $(\beta = 2,\ldots,\nu+1)$ and for $\alpha = \mu+1$

$$\text{(26)}\qquad a_{\mu+1,\beta}^{(\mu,\nu)^*} = a_{1,\nu+2-\beta}^{(\mu,\nu)^*}.$$

Now we deal with the general case $2 \le \alpha \le \mu$, $\alpha \le \beta \le \alpha + \nu - \mu$:

$$\sum_{t=1}^{r+1} v_{\alpha t}^{(\mu,r)^*} \; w_{t\beta}^{(r,\nu)^*}$$

$$= \sum_{t=\alpha}^{\beta} (-1)^{r-\mu} \frac{\binom{r-1}{\mu-1}^2}{\binom{2r-2}{r-\mu}} \binom{r-\mu}{t-\alpha} \frac{\binom{\nu-1}{r-1}^2}{\binom{\nu+r-1}{\nu-r}} \binom{\nu-r}{\beta-t} \frac{\binom{\mu-1}{\alpha-1}\binom{\mu-1}{\alpha-2}}{\binom{\nu-1}{\beta-1}\binom{\nu-1}{\beta-2}}$$

$$= (-1)^{r-\mu} \frac{\binom{r-1}{\mu-1}^2 \binom{\nu-1}{r-1}^2}{\binom{2r-2}{r-\mu}\binom{\nu+r-1}{\nu-r}} \cdot \frac{\binom{\mu-1}{\alpha-1}\binom{\mu-1}{\alpha-2}}{\binom{\nu-1}{\beta-1}\binom{\nu-1}{\beta-2}} \sum_{t=0}^{\beta-\alpha} \binom{r-\mu}{t} \binom{\nu-r}{(\beta-\alpha)-t}$$

$$= (-1)^{r-\mu} \frac{\binom{\nu-\mu}{r-\mu}^2 \binom{\nu-1}{\mu-1}^2}{\binom{2r-2}{r-\mu}\binom{\nu+r-1}{\nu-r}} \cdot \frac{\binom{\mu-1}{\alpha-1}\binom{\mu-1}{\alpha-2}}{\binom{\nu-1}{\beta-1}\binom{\nu-1}{\beta-2}} \binom{\nu-\mu}{\beta-\alpha},$$

so that, by (24),

$$a_{\alpha\beta}^{(\mu,\nu)^*} = \binom{\nu-1}{\mu-1}^2 \binom{\nu-\mu}{\beta-\alpha} \frac{\binom{\mu-1}{\alpha-1}\binom{\mu-1}{\alpha-2}}{\binom{\nu-1}{\beta-1}\binom{\nu-1}{\beta-2}} \sum_{r=\mu}^{\nu} \Pi_r^{\gamma} (-1)^{r-\mu} \frac{\binom{\nu-\mu}{r-\mu}^2}{\binom{2r-2}{r-\mu}\binom{\nu+r-1}{\nu-r}}.$$

The coefficient b_{ij} in (22) is the entry of $A^*_{i+j,p+q}$ at position $(j+1, q+1)$, i.e.

$$b_{ij} = a_{j+1,q+1}^{(i+j,p+q)^*} = a_{k+1-i,s+1-p}^{(k,s)^*}$$

$$= \binom{s-1}{k-1}^2 \binom{s-k}{p-i} \frac{\binom{k-1}{i-1}\binom{k-1}{i}}{\binom{s-1}{p-1}\binom{s-1}{p}} c_{ks}^{(\gamma)} \qquad (1 \le i < k),$$

$$b_{0k} = a_{k+1,s+1-p}^{(k,s)^*} = \begin{cases} a_{k+1,s+1}^{(k,s)^*} = \binom{s}{k}\binom{s-1}{k-1} c_{ks}^{(\gamma)} & \text{for } p = 0, \\ 0 & \text{else,} \end{cases}$$

and

$$b_{k0} = a_{1,s+1-p}^{(k,s)^*} = \begin{cases} 0 & \text{for } 0 \le p \le s-1, \\ a_{11}^{(k,s)^*} = \binom{s}{k}\binom{s-1}{k-1} c_{ks}^{(\gamma)} & \text{for } p = s. \end{cases}$$

Hence, the proof of Theorem 2 is completed. ∎

Remark. In order to see that Theorem 2 also contains the one–dimensional case we put $p = s$ ($s \in \mathbb{N}$) and $y = 0$. Then we have

$$\lim_{n\to\infty} B_n^{g(n)}(x^s; x, y = 0) = \sum_{k=1}^{s} b_{k0}\, x^k$$

with

$$b_{k0} = \binom{s}{k}\binom{s-1}{k-1} c_{ks}^{(\gamma)}$$

which is the classical result of Kelisky and Rivlin [8].

References

[1] Abel, U., On the asymptotic approximation with operators of Bleimann, Butzer and Hahn, *Indag. Mathem., New Ser.,* **7** (1996), 1–9.

[2] Arends, G., "Untersuchungen über die Potenzen der zweidimensionalen Bernstein–Operatoren", Diplomarbeit, Fachhochschule Giessen–Friedberg, 1992.

[3] Chang Gengzhe and Feng Yuyu, Limit of iterates for Bernstein polynomials defined on higher dimensional domains, *Kexue Tongbao, Foreign Lang. Ed.,* **31** (1986), 157–160.

[4] Ding–Xuan Zhou, Weighted approximation by multidimensional Bernstein operators, *J. Approx. Theory,* **76** (1994), 403–422,

[5] Ditzian, Z., Inverse theorems for multidimensional Bernstein operators, *Pacific J. Math.,* **121** (1986), 293–319.

[6] Jordan, C., "Calculus of finite differences", Chelsea, New York, 1965.

[7] Karlin, S. and Ziegler, Z., Iteration of positive approximation operators, *J. Approx. Theory,* **3** (1970), 310–339.

[8] Kelisky, R. P. and Rivlin, T. J., Iterates of Bernstein polynomials, *Pacific J. Math.,* **21** (1967), 511–520.

[9] Lorentz, G. G., "Bernstein polynomials", University of Toronto Press, Toronto 1953.

[10] Micchelli, C., The saturation class and iterates of the Bernstein polynomials, *J. Approx. Theory,* **8** (1973), 1–18.

[11] Nagel, J., Asymptotic properties of powers of Bernstein operators, *J. Approx. Theory,* **29** (1980), 323–335.

[12] Nagel, J., Asymptotic properties of powers of Kantorovic operators, *J. Approx. Theory,* **36** (1982), 268–275.

[13] Sikkema, P. C., Über Potenzen von verallgemeinerten Bernstein–Operatoren, *Mathematica (Cluj),* **31** (1966), 173–180.

[14] Stancu, D. D., Some Bernstein polynomials in two variables and their applications, *Soviet Math.,* **1** (1960), 1025–1028.

[15] Xin–Long Zhou, Approximation by multivariate Bernstein operators, *Resultate Math.,* **25** (1994), 166–191.