EXTRACTA MATHEMATICAE Vol. **11**, Núm. 3, 485 – 488 (1996)

Suitable Norms for Simultaneous Approximation *

Carlos Benítez, Manuel Fernández and María L. Soriano

Dpto. de Matemáticas, Univ. de Extremadura, 06071-*Badajoz, Spain*

(Research announcement)

AMS *Subject Class.* (1991): 41A65, 41A28, 46B20 *Received November* 28, 1996

Let E be a normed linear space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and M be a subset of E. A point $u_0 \in M$ is a *best approximation* (BA) to $x \in E$ from M (in short, $u_0 \in P_M(x)$), when $\|x - u_0\| \leq \|x - u\|$, for every $u \in M$. Assume now that $E \times E$ is endowed with a norm, a point $u_0 \in M$ is a *best simultaneous approximation* (BSA) to $x, y \in E$ from M (in short, $u_0 \in Z_M(x,y)$), when $(u_0, u_0) \in P_{d(M\times M)}(x,y)$, where $d(M \times M) = \{(u,u) : u \in M\}$.

We discuss those conditions that must be required to a norm in $E \times E$ so that the set $Z_M(x,y)$ agrees with our primary idea about simultaneity. (A similar study, but for mixed approximation problems, has been done in [3]). With this in mind, a tentative list of conditions to be analysed should include:

(i) There is no any priority between points to be approximated simultaneously,

$$Z_M(x,y) = Z_M(y,x), \text{ for every } x, y \in E \text{ and } M \subset E.$$

(ii) Each BA of a point is a BSA of two copies of the same point,

$$P_M(x) \subset Z_M(x,x), \text{ for every } x \in E \text{ and } M \subset E.$$

(iii) Each BA of two points is a BSA of both points,

$$P_M(x) \cap P_M(y) \subset Z_M(x,y), \text{ for every } x, y \in E \text{ and } M \subset E.$$

(iv) If two points have common BA, each BSA is BA to some of the two points,

$$Z_M(x,y) \subset P_M(x) \cup P_M(y), \text{ when } P_M(x) \cap P_M(y) \neq \emptyset.$$

(v) If two points have common BA, they are, exactly, the BSA of both points,

$$P_M(x) \cap P_M(y) = Z_M(x,y), \text{ when } P_M(x) \cap P_M(y) \neq \emptyset.$$

Through very simple propositions we characterize the norms in $E \times E$ for

*A full version will appear in *Acta Math. Hung.*

which the above conditions are fulfiled.

PROPOSITION 1. *A norm in* $E \times E$ *fulfils* (i) *if and only if it is commutative (or symmetric), that is,*

$$\|(x,y)\| = \|(y,x)\|, \quad \text{for every } x, y \in E. \tag{1}$$

PROPOSITION 2. *A norm in* $E \times E$ *fulfils* (ii) *if and only if any of the following equivalent statements holds,*

(2.1) $\|x\| = \|y\| \Rightarrow \|(x,x)\| = \|(y,y)\|$.

(2.2) $\|x\| < \|y\| \Rightarrow \|(x,x)\| < \|(y,y)\|$.

(2.3) $\|x\| \leq \|y\| \Rightarrow \|(x,x)\| \leq \|(y,y)\|$.

(2.4) *There is* $C > 0$ *such that* $\|(x,x)\| = C\|x\|$, *for every* $x \in E$.

(2.5) $P_M(x) = Z_M(x,x)$, *for every* $x \in E$ *and* $M \subset E$.

These norms give a complete answer to the trivial problem of BSA of two copies of the same point. However, the next example shows that this condition is too weak to give a good answer to the problem of BSA of two different points.

EXAMPLE 1. If E is $\mathbb{R}^3$ with the norm $\|(r,s,t)\| = \sup(|r|,|s|,|t|)$ and M is the linear subspace spanned by $(1,1,1)$, then $P_M(0,4,0) = P_M(0,0,4) = \{(2,2,2)\}$, but, if we consider the (i) and (ii)-norm $\|(x,y)\| = \|x+y\| + \|x-y\|$, then we have the undesirable fact that $Z_M((0,4,0),(0,0,4)) = \{(1,1,1)\}$.

PROPOSITION 3. *A norm in* $E \times E$ *fulfils* (iii) *if and only if*

$$\|x\| \leq \|x+z\|,\ \|y\| \leq \|y+z\| \ \Rightarrow\ \|(x,y)\| \leq \|(x+z, y+z)\|. \tag{3}$$

The next example shows that even these norms are rather weak.

EXAMPLE 2. If E is $\mathbb{R}^2$ endowed with the euclidean norm and M be the linear subspace spanned by $(0,1)$, then $P_M(1,0) = P_M(-1,0) = \{(0,0)\}$, but if we consider the (i) and (iii)-norm $\|(x,y)\| = \sup(\|x\|, \|y\|, \|x-y\|)$, then, $Z_M((1,0),(-1,0)) = \{(0,r) : |r| \leq \sqrt{3}\}$. That is, there are BSA that are really bad approximations to both points.

PROPOSITION 4. *A norm in* $E \times E$ *fulfils* (iv) *if and only if*

$$\|x\| < \|x+z\|,\ \|y\| < \|y+z\| \ \Rightarrow\ \|(x,y)\| < \|(x+z, y+z)\|. \tag{4}$$

Remark 1. It is obvious that (iv) implies (ii) and we can prove that (iv) implies (iii) when the spheres of E have not bits of real hyperplanes, but we do not know, in the general case, if (iv) implies (iii).

Pursuing this gradual analysis of conditions in order that a norm in $E \times E$ is suitable for simultaneous approximation, note that the stronger condition

$$Z_M(x,y) \subset P_M(x) \cup P_M(y), \qquad \text{for every } x, y \in E \text{ and } M \subset E$$

is very far from being good in simultaneous approximation. (If we consider the norm $\|(x,y)\| = \|x\| + \|x-y\|$, that fulfils the above condition, we have the undesirable fact $Z_M(x,y) = P_M(x)$, for every $x, y \in E$ and $M \subset E$).

Also note that condition

$$P_M(x) \cup P_M(y) \subset Z_M(x,y), \text{ when } P_M(x) \cap P_M(y) \neq \emptyset,$$

cannot be satisfied by a norm since it implies that $2n\|(x,x)\| \leq \|(x,-x)\|$, for every $n \in \mathbb{N}$, which is impossible.

This trip across the norms in $E \times E$ that are suitable for BSA from subsets of E, ends in a result that, like their similar, is easy to prove.

PROPOSITION 5. *A norm in $E \times E$ fulfils* (v) *if and only if it fulfils (3) and*

$$(5) \qquad \|x\| < \|x+z\|, \ \|y\| \leq \|y+z\| \ \Rightarrow \ \|(x,y)\| < \|(x+z, y+z)\|,$$

(or $\leq$ and $<$ in the left hand).

Remark 2. It is obvious that (v) implies (iii) and (iv), but, by the same reason that we do not know if (iv) implies (iii), we also do not know if (5) implies (iii) when the spheres of E have bits of real hyperplanes.

Note finally that the classical norms

$$\|(x,y)\| = (\|x\|^p + \|y\|^p)^{1/p} \quad (1 \leq p < \infty)$$

satisfy (v), but $\|(x,y)\| = \sup(\|x\|, \|y\|)$ (the most frequently used norm in the literature of simultaneous approximation) only fulfils (iii) and (iv).

An analogous list of conditions can be obtained by changing "M subset of E" to "M linear subspace of E" in (i), ... , (v). For example, Proposition 2 is the same for M a linear subspace, but in Proposition 3 condition (3) must be changed into

$$x \perp z, y \perp z \Rightarrow \|(x,y)\| \leq \|(x+z, y+z)\|,$$

where $x \perp z$ means that x is orthogonal to z in the Birkhoff sense, i.e., $\|x\| \leq \|x + \lambda z\|$, for every $\lambda \in \mathbb{K}$.

In Example 1 it has been showed that the norm in $E \times E$ defined by $\|(x,y)\| = \|x+y\| + \|x-y\|$ is a (ii)-norm, but not (iii)-norm. The last result studies the behaviour of the more general norm $\|(x,y)\| = \|x+y\| + \|x-y\|_\star$, where $\| \ \|_\star$ is an arbitrary norm in E , with respect to the conditions (i), ... , (v). Let denote $S = \{x \in E : \|x\| = 1\}$ as the unit sphere of E with respect to the main norm $\| \ \|$. The notations (3S), (4S), and (5S) will be used to indicate that the known properties (3), (4), and (5) hold for every $x, y \in S$, and $z \in E$.

PROPOSITION 6. *Let $E \times E$ be endowed with a norm $\|(x,y)\| = \|x+y\| + \|x-y\|_\star$.*

(a) *If E is a real bidimensional linear space, then* (4S) *holds.*

(b) *If E is a real bidimensional linear space, then E is rotund if and only if either* (3S) *or* (5S) *holds.*

(c) *E is an inner product space if and only if either* (3) *or* (5) *holds.*

(d) *When E is rotund, E is an inner product space if and only if* (4) *holds.*

(e) *When* $\dim E \geq 3$, *E is an inner product space if and only if* (3S) *holds.*

(f) *Let E be rotund with real dimension* ≥ 3. *Then E is an inner product space if and only if either* (4S) *or* (5S) *holds.*

REFERENCES

[1] BLASCHKE, W., "Kreis und Kugel", Walter de Gruyter & Co., 2nd. ed., (Berlin, 1956).

[2] DANES, P., On local and global moduli of convexity, *Comm. Math. Univ. Carol.,* **17**(3) (1976), 413–420.

[3] HADJISAVVAS, N., NASSOPOULOS, B., PANTEDELIS, G., On the renorming of a vector space endowed with two norms and the simultaneous best approximation problem, *Math. Ann.,* **280** (1988), 549–557.

[4] KAKUTANI, S., Some characterizations of euclidean spaces, *Jap. J. Math.,* **16** (1939), 93–97.

[5] LORCH, E.R., On some implications which characterize Hilbert spaces, *Ann. of Math.,* **49**(3) (1948), 523–532.

[6] SORIANO, M.L., "Aproximación Simultánea en Espacios Normados", Pub. Dpto. Matem. Univ. Extremadura, **23**, (Badajoz, 1990).