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Complementation in Spaces of Symmetric Tensor Products and Polynomials

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Our aim here is to announce some properties of complementation for spaces of symmetric tensor products and homogeneous continuous polynomials on a locally convex space E that have, in particular, consequences in the study of the property $(BB)_{n,s}$ recently introduced by Dineen [8].

As it was pointed out by Ryan in his thesis [10], the completion of the space of n-symmetric tensors on the locally convex space E, endowed with the projective topology (denoted by $\hat{\otimes}_{n,s,\pi}E$) is a predual for the space $\mathcal{P}(^nE)$ of all n-homogeneous continuous polynomials on E. Using this fact we get our results for spaces of polynomials from results we obtain about symmetric tensors.

We prove that, for every n, $\hat{\otimes}_{n,s,\pi}E$ is a complemented subspace of $\hat{\otimes}_{n+1,s,\pi}E$ and from that we obtain that for locally convex spaces E and G, the space $\mathcal{P}(^nE;G)$, of all n-homogeneous continuous polynomials from E into G is a complemented subspace of $\mathcal{P}(^{n+1}E;G)$ when we endow these spaces with the strong topology as dual spaces. Moreover the complementation of $\mathcal{P}(^nE;G)$ in $\mathcal{P}(^{n+1}E;G)$ for all the natural topologies on these spaces is also obtained. From this it follows that the property $(BB)_{n,s}$ on a locally convex space implies the property $(BB)_{m,s}$ for $m = 2, \ldots, n$.

We require the following two lemmata in the proof of our theorems.

LEMMA 1. *Let x, y be linearly independent vectors in a vector space E. Then, given $n = 1, 2, \ldots$, there exist $\lambda_1, \ldots, \lambda_{n+1} \in \mathbb{K}$ such that*

$$\otimes_n x = \sum_{k=1}^{n+1} \lambda_k \otimes_n (x + ky).$$

LEMMA 2. *Let E be a locally convex space, $\theta \in \otimes_{n,s} E$ and $\varphi \in E'$, $\varphi \neq 0$. Then there exists a representation $\theta = \sum_{i=1}^{N} \varepsilon_i \otimes_n x_i$, with $\varphi(x_i) \neq 0$ for each $i \in \{1, \ldots, N\}$ (when $\mathbb{K} = \mathbb{R}$ we take $\varepsilon_i \in \{-1, 1\}$ and, when $\mathbb{K} = \mathbb{C}$ it is possible to take $\varepsilon_i = 1$ for every i).*

The next theorem is the main result in the paper. It gives some information in order to understand the structure of the spaces of symmetric tensor products and polynomials. It has some other applications, that can be seen in forthcoming papers.

THEOREM 3. *For any locally convex space E, $\hat{\otimes}_{n,s,\pi} E$ is a complemented subspace of $\hat{\otimes}_{n+1,s,\pi} E$, for each positive integer n.*

From this we obtain immediatly the following

COROLLARY 4. *Let E be a locally convex space, then for $n = 2, 3, \ldots$ and $k \in \mathbb{N}$, $1 \leq k \leq n$ we have that $\hat{\otimes}_{k,s,\pi} E$ is a complemented subspace of $\hat{\otimes}_{n,s,\pi} E$.*

For locally convex spaces E and G let us denote by β the topology, recently introduced in [8], induced on $\mathcal{P}(^n E; G) \simeq \mathcal{L}(\hat{\otimes}_{n,s,\pi} E; G)$ by the strong topology β on $\mathcal{L}(\hat{\otimes}_{n,s,\pi} E; G)$ of uniform convergence on the bounded subsets of $\hat{\otimes}_{n,s,\pi} E$.

The main theorem yields in a natural way to the following corollary that can be considered as a stronger version of a well-known result obtained by Aron and Schottenloher in [3] for E a Banach space and $G = \mathbb{C}$, using a different technique.

COROLLARY 5. *If E and G are locally convex spaces and $n, m \in \mathbb{N}$, with $n \geq m$, then $(\mathcal{P}(^m E; G), \beta)$ is a complemented subspace of $(\mathcal{P}(^n E; G), \beta)$.*

In $\mathcal{P}(^n E; G)$ we can consider, apart from β, the following three usual locally convex topologies: the compact-open topology τ_0, generated by the seminorms $\{||\cdot||_{K,\gamma} : K$ is a compact subset of E, $\gamma \in cs(G)\}$, where $||P||_{K,\gamma} = \sup\{\gamma(P(x)) : x \in K\}$, the topology of uniform convergence on bounded subsets τ_b, generated by the seminorms $\{||\cdot||_{B,\gamma} : B$ is a bounded subset of E, $\gamma \in cs(G)\}$ and the Nachbin ported topology τ_ω defined, when G is normed, by

$$(\mathcal{P}(^n E; G), \tau_\omega) = \varinjlim_{\alpha \in cs(E)} (\mathcal{P}(^n E_\alpha; G), \tau_b)$$

(E_α denotes the normed space associated with the continuous seminorm α on E). In this case, τ_ω is generated by the seminorms p that are *ported* by the origin; i. e. for every neighbourhood V of 0, $V \subset E$, there exists $c(V) > 0$ such that $p(P) \leq c(V) \sup_{x \in V} ||P(x)||$ for every $P \in \mathcal{P}(^nE; G)$. For general locally convex spaces G, τ_ω is defined by

$$(\mathcal{P}(^nE; G), \tau_\omega) = \varprojlim_{\gamma \in cs(G)} (\mathcal{P}(^nE; G_\gamma), \tau_\omega).$$

It is well-known that $\tau_0 \leq \tau_b \leq \beta \leq \tau_\omega$ on $\mathcal{P}(^nE; G)$ and that for Fréchet spaces E, $\tau_0 = \tau_b$ on $\mathcal{P}(^nE; G)$ if and only if E is a Montel space ([6]). Hence, for infinite dimensional Banach spaces E, $\tau_0 < \tau_b$ on $\mathcal{P}(^nE; G)$. On the other hand, if E is a Banach space, $\tau_b = \beta = \tau_\omega$ on $\mathcal{P}(^nE; G)$. The equality $\tau_0 = \tau_\omega$ on $\mathcal{P}(^nE)$ over all n for E Fréchet-Montel has been recently studied because of its equivalence with the corresponding holomorphic problem.

For a Fréchet-Montel space E, $\tau_0 = \tau_\omega$ on $\mathcal{P}(^nE)$ if and only if E has the property $(BB)_{n,s}$ (see below). This property has been introduced by Dineen ([8]) as an $n-$fold version of Taskinen's (BB) property, ([11]) connected with the "Problème des topologies" of Grothendieck.

A locally convex space E has property $(BB)_{n,s}$ for $n = 2, 3, \ldots$ if for every bounded subset B in $\hat{\otimes}_{n,s,\pi}E$ there is a bounded subset C in E such that B is contained in the closed convex hull of $\otimes_{n,s}C = \{\otimes_n x : x \in C\}$. For several classes of Fréchet Montel spaces E, $\tau_0 = \tau_\omega$ on $\mathcal{P}(^nE)$, i.e. E has $(BB)_{n,s}$ property, ([1, 7, 9, 5, 8]) but Ansemil and Taskinen ([2]) gave an example of a Fréchet Montel space E such that $\tau_0 \neq \tau_\omega$ on $\mathcal{P}(^2E)$.

From the next proposition we shall get some information about property $(BB)_{n,s}$, and our main theorem is the key in its proof.

PROPOSITION 6. *Let E and G be locally convex spaces and let $m \in \mathbb{N}$. Then $(\mathcal{P}(^mE; G), \tau)$ is a complemented subspace of $(\mathcal{P}(^nE; G), \tau)$ for $\tau = \tau_0, \tau_b, \beta$ or τ_ω and $n \geq m$.*

COROLLARY 7. *If for a given $n \in \mathbb{N}$, $\tau_b = \beta$ in $\mathcal{P}(^nE; G)$, then $\tau_b = \beta$ on $\mathcal{P}(^mE; G)$ for every m with $1 \leq m \leq n$.*

COROLLARY 8. *If given $n \in \mathbb{N}$, $n \geq 2$, E has the $(BB)_{n,s}$ property, then E has the $(BB)_{m,s}$ property for each positive integer m, $2 \leq m \leq n$.*

This Corollary simplifies the hypothesis in some theorems; see [7], for instance.

The proofs of the mentioned results can be seen in [4].

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