EXTRACTA MATHEMATICAE Vol. 11, Núm. 3, 493 – 498 (1996)

A Characterization of Value Efficiency

Alfonso Mateos and Sixto Ríos-Insua *

Dept. of Artificial Intelligence, Technical Univ. of Madrid, 28660 *Madrid, Spain*

(Research announcement presented by M. Molina)

AMS *Subject Class.* (1991): 92B5, 90A10 *Received February* 27, 1996

1. Introduction

An important issue in multi-attribute decision making consists of identifying the set of efficient solutions. The importance of this set is that the decision maker (DM) can restrict his attention to it, discarding all other solutions, because a nonefficient solution can never be optimal. Several methods have been developed to aid a DM in generating all or representative subsets of efficient solutions, [1] and [4], or to approximate it [7]. However, most of these methods may be hard to apply to nonlinear problems being very difficult in that case to generate the efficient set.

In this paper, we propose a characterization of the efficient set for imprecise vector value functions, which intends to facilitate the checking of the efficiency. Our framework will be the multi-attribute decision making problem under certainty with a set of alternatives structured as vectors of specific levels of achievement against a number of factors $z = (z_1, \ldots, z_n) \in Z \subseteq \mathbb{R}^n$. We assume that we have to maximize each z_i, and there is partial information on DM's preferences in the sense that we are not able to assess a scalar value function, [2] and [3], but a *vector value function* [8] $v = (v_1, \ldots, v_p) : Z \to \mathbb{R}^p$. which represents a strict partial order $\succ$ (irreflexive, transitive and asymmetric) in Z, such that for any $z, z' \in Z$

$$z' \succ z \Leftrightarrow v(z') \geq v(z)$$

where $v(z') \geq v(z)$ if and only if $v_i(z') \geqq v_i(z)$ for all $i = 1, \ldots, p$ and there is $j \in \{1, \ldots, p\}$ such that $v_j(z') > v_j(z)$.

*Supported by the CICYT project TIC95-0028

This vector function may be seen as a proxy for an underlying (scalar) value function. It seems appropriate in the hierarchical structures which often exhibits well defined multi-attribute problems, where there is a natural grouping of attributes in the lowest-level of the hierarchy.

Given a vector value function v, we state the vector optimization problem as

$$\max_{z \in Z} \; v(z)$$

and we have that a point $z \in Z$ is a *vector value efficient solution* if there exist no $z' \in Z$ such that $z' \succ z$.

This set of solutions, called *vector value efficient set* and denoted $\mathcal{E}(Z, v)$, is where the DM should make his choice. Now, we may state our multi-attribute problem as: "Given Z and v, find $\mathcal{E}(Z, v)$".

If $\mathcal{E}(Z, v) = \{z\}$, i.e., the vector value efficient set has a unique solution, then it can be considered as a solution of the decision problem. However, this is not the case in most real problems, as $\mathcal{E}(Z, v)$ has many points and thus, the generation of such set cannot be considered as the solution of a decision problem. Then, based on the characterization of the value efficient set we shall provide a method, valid for nonlinear problems, to interactively reduce the value efficient set and alleviate the solution process.

The paper includes four more sections. In the second section we provide the relation of the value efficiency with an extended vector value function concept and its characterization. In the third section we study the reduction of the value efficient set from increasingly more precise vector value functions. The four section presents the outline of an algorithm for solving discrete problems which shows an interactive method based on the above reduction. Finally, some conclusions are considered in the last section.

2. Value Efficiency and its Characterization

Given our partial information problem by a vector value function $v : Z \to \mathbb{R}^p$, it leads us to consider $\mathcal{E}(Z, v)$. Consider an additive aggregation with weights or scaling constants $k = (k_1, \ldots, k_p)$ as a valid approximation [9], which has the form

$$v^k(z) = (kv)(z) = \sum_{i=1}^{p} k_i v_i(z)\,.$$

Suppose there is partial information about the scaling constants k in the form of constraints set K_*, which is next introduced. Let $K^0 \equiv \mathbb{R}^p_+$, which is a constant and convex cone.

DEFINITION 1. Let $K \supseteq K^0$ be a constant, convex, closed and acute cone which we call information cone, and K^P its positive polar. The set $K_* = K^P \cap S_p$ is called *information set associated to* K, where S_p is the simplex on $\mathbb{R}^p$.

We shall assume that K^P is defined by a polyhedral cone, and it will be possible to determine its set of generators [10], which normalized in S_p will be denoted $K_* = G\{k^1, \ldots, k^q\}$. Given a vector value function v and an information set K_*, the class of value functions consistent with such information is denoted by $\mathcal{V} = \mathcal{V}(K_*)$, and defined as

$$\mathcal{V}(K_*) = \{v^k : v^k = kv,\ k \in K_*\}.$$

Now, we introduce the ordering: "Given $z, z' \in Z$, $z' \succ_* z$ if and only if $v^k(z') \geqq v^k(z)$ for any $k \in K_*$ and $v^{k'}(z') > v^{k'}(z)$ for at least one $k' \in K_*$". It is clear that $\succ_*$ is a strict partial order from which we can define the notion of *value efficient solution* and the corresponding *value efficient set* will be

$$\mathcal{E}(Z, \mathcal{V}) = \{z \in Z : \nexists\, z' \in Z \text{ such that } z' \succ_* z\}.$$

Next, based on the information set, we extend the concept of vector value function

DEFINITION 2. For a vector value function v and an information set $K_* = G\{k^1, \ldots, k^q\}$, the function $v^K = (k^1 v, \ldots, k^q v)$ with domain in Z, is called *vector value function associated to* K_*.

Thus, we shall have

$$z' \succ_K z \Leftrightarrow v^K(z') \geq v^K(z)$$

where $v^K(z') \geq v^K(z)$ if and only if $(k^i v)(z') \geqq (k^i v)(z)$ for all $i = 1, \ldots, q$ and there is $j \in \{1, \ldots, q\}$ such that $(k^j v)(z') > (k^j v)(z)$. In analogous way, we shall have the vector value efficient set for v^K, denoted $\mathcal{E}(Z, v^K)$. Note that, if we have null information over the scaling constants, i.e., $K_*^0 = G\{(1,0,\ldots,0), \ldots, (0,\ldots,0,1)\}$, it will be $v = v^{K^0}$. On other hand, if K is a halfspace, then $K_* = \{k\}$ and $v^K = kv$ will be a scalar value function.

Now, we shall provide a characterization that will give a practical way to obtain $\mathcal{E}(Z, \mathcal{V})$.

THEOREM 1. *Let* $v : Z \to \mathbb{R}^p$ *be a vector value function and* $K_* = G\{k^1, \ldots, k^q\}$ *an information set, then*

$$\mathcal{E}(Z, \mathcal{V}) \equiv \mathcal{E}(Z, v^K).$$

This theorem allow us to use the vector function v^K rather than the infinite class $\mathcal{V}(K_*)$, to show the value efficiency of a solution. Note that in the special case of the null information set about the scaling constants, i.e., $K_* = K_*^0$, we have $\mathcal{E}(Z, \mathcal{V}) \equiv \mathcal{E}(Z, v)$. Next, we consider the solutions which maximizes v^k for some $k \in K_*$.

DEFINITION 3. For a fixed $k \in K_*$, the set

$$\mathcal{O}(Z, v^k) = \left\{ z' \in Z : v^k(z') = \max_{z \in Z} v^k(z) \right\}$$

is called *value optimization set.*

This set will be well defined if, for example, we assume that Z is compact and the functions v_i are continuous. Next result states that the solutions which maximize the function v^k for some k, are vector value efficient solutions (analogous results are in [12] and [13]).

THEOREM 2. *Let* $v : Z \to \mathbb{R}^p$ *be a vector value function and* $K_* = G\{k^1, \ldots, k^q\}$ *an information set, then*

$$\bigcup_{k \in \mathrm{int}(K_*)} \mathcal{O}(Z, v^k) \subseteq \mathcal{E}(Z, v^K).$$

3. INTERACTIVE REDUCTION OF THE SET OF SOLUTIONS

We shall consider the progressive reduction of the value efficient set based on an increasingly more precise vector value function, given through information sets over the DM's preferences.

PROPOSITION 1. *Given a vector value function* v *and two information sets* K_*, K'_* *such that* $K'_* \subseteq \mathrm{int}(K_*)$, *then*

$$\mathcal{E}(Z, v^{K'}) \subseteq \mathcal{E}(Z, v^K).$$

THEOREM 3. *Let* v *a vector value function and* $\{K_*^n\}$ *be a decreasing sequence of information sets such that* $K_*^1 \subseteq \mathrm{int}(K_*^0)$ *and* $K_*^n \downarrow p$ *when* $n \to \infty$, *then* $z \in \mathcal{O}(Z, pv)$.

To obtain the information sets we may generate constraints through pairwise comparisons of solutions. This has been made in [14], [6] and [11]. The procedure will be to compare pairs of solutions z, z' and if the DM states a strict preference between them, we shall have a constraint which will lead to a new information set, i.e., if DM prefers z' to z then, add to the information set $(kv)(z') > (kv)(z)$. We do not write constraints for any of the indifference or I do not know responses.

PROPOSITION 2. *Let $z', z \in Z$, v a vector value function and let $K_* = G\{k^1, \dots, k^q\}$ be an information set. If the DM reveals that z is at most as preferred as z', we generate the information set*

$$K'_* = \left\{ k = \sum_{i=1}^{q} \alpha_i k^i : (kv)(z') \geqq (kv)(z) \ \text{with} \ (\alpha_1, \dots, \alpha_q) \in S_q \right\}$$

where $K' \neq \emptyset$ and if $K'_* \subseteq \text{int}(K_*)$, then

$$\mathcal{E}(Z, v^{K'}) \subseteq \mathcal{E}(Z, v^{K}) .$$

4. An Interactive Algorithm for Discrete Problems

The above results are valid for general problems, however, to show the method, we outline an algorithm to solve discrete problems, whose aim is the interactive reduction of the value efficient set and is based on pairwise comparisons to derive successive information sets. This algorithm is developed in an interactive framework [5] which intends to incorporate the DM as an active element of the decision aid process.

The algorithm can be divided in three phases. The first one, includes the initialization and the generation of vector value efficient set $\mathcal{E}(Z, v^{K^i})$ (which is computationally easy for discrete problems). The second phase, checks for optimality. If the DM is satisfied with a solution $z \in \mathcal{E}(Z, v^{K^i})$, stop. Otherwise, go to the third phase, which includes the generation of a new information set (proposition 2) and the reduction of the set of solutions $\mathcal{E}(Z, v^{K^i})$, to start again the process.

5. Conclusions

We have considered the multi-attribute decision making problem under partial information over the preferences, which are modeled by a vector value function. We extend the notion of efficiency to this problem and we propose its characterization based on more precise vector value functions with scaling constants in a polyhedral set. We provide a procedure to interactively reduce the value efficient set for discrete problems, which is based on revealed preferences over pairwise comparisons expressed as constraints over the scaling constants. This method may be an important aid for a DM in the process of reaching a final solution.

References

[1] CHANKONG, V., HAIMES, Y.Y, "Multiobjective Decision Making, Theory and Methodology", North Holland, Amsterdam, 1983.

[2] DEBREU, G., Representation of a preference ordering by a numerical function, in "Decision Processes" (R.M. Thrall, C.H. Coombs and R.L. Davies, Eds.), Wiley, New York, 1954.

[3] FISHBURN, P.C., "Decision and Value Theory", Wiley, New York, 1964.

[4] GAL, T., "Postoptimal Analyses, Parametric Programming, and Related Topics", W. de Gruyter, Berlin, 1995.

[5] GARDINER, L.R., STEUER, R.E., Unified interactive multiple objective programming, *Eur. J. Opl. Res.* **74** (1994), 391–406.

[6] MALAKOOTI, B., Assessment through strength of preference, *Large Scale Systems* **8** (1985), 169–182.

[7] MATEOS, A., RÍOS-INSUA, S., An approximation to the value efficient set, in "Twenty Years of East-West Cooperation in MCDM" (1996, T. Gal and T. Hanne, eds.), Springer-Verlag (to appear).

[8] ROBERTS, F.S., "Measurement Theory", Addison-Wesley, Reading, Mass., 1979.

[9] RAIFFA, H., "The Art & Science of Negotiation", Harvard University Press, Cambridge, Mass., 1982.

[10] TAMURA, K., A method for constructing the polar cone of a polyhedral cone, with applications to linear multicriteria decision problems, *J. Opt. Th. and App.* **19** (1976), 547–564.

[11] TANER, O.V., KÖKSALAN, M.M., Experiments and an improved method for solving the discrete alternative multiple-criteria problem, *J. Opl. Res. Soc.* **42** (1991), 383–391.

[12] WHITE, D.J., "Optimality and Efficiency", Wiley, Chichester, 1982.

[13] WHITE, D.J., Multiple-objective weighting factor auxiliary optimization problems, *J. of Multi-Criteria Decision Analysis* **4** (2) (1995), 122–132.

[14] ZIONTS, S., WALLENIUS, J., An interactive multiple objective linear programming method for a class of underlying nonlinear utility functions, *Man. Sci.* **29** (1983), 519–529.