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The Norm Extension Problem: Positive Results and Limits *

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1. Introduction

If A is an associative algebra with product xy, then we can consider the so called Jordan product of A, namely the one $x \circ y$ defined by

$$x \circ y := \frac{1}{2}(xy + yx) .$$

The algebra obtained by replacing the associative product of A by its Jordan product is of course commutative and satisfies the Jordan identity:

$$x \circ (y \circ x^2) = (x \circ y) \circ x^2 .$$

In other words, from the associative algebra A we have obtained in a natural way a Jordan algebra. All subspaces of the associative algebra A which are closed under the Jordan product are also examples of Jordan algebras (although not every Jordan algebra can be obtained in this way). In particular, if $*$ is an involution (i.e., an involutive linear anti-automorphism) on the associative algebra A, then the set $H(A, *)$ of all $*$-invariant elements of A is a Jordan algebra.

The celebrated Zel'manov prime theorem for Jordan algebras [14] has the nice consequence that, if J is a simple Jordan algebra, and if J is not of the form $H(A, *)$ for some associative algebra A, then J belongs to a small class of simple Jordan algebras the members of which are perfectly well-understood.

Therefore, when one is trying to obtain reasonable normed versions of the Zel'manov prime theorem, the following "norm extension problem" naturally

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arises: if A is a (real or complex) associative algebra with involution $*$, and if $\|.\|$ is an algebra norm on the Jordan algebra $H(A,*)$, is there an algebra norm on A whose restriction to $H(A,*)$ is equivalent to $\|.\|$?

In fact, if a Jordan algebra is of the form $H(A,*)$ for some associative algebra A with involution $*$, then the parameter $(A,*)$ is not unique and can be chosen in such a way that A becomes a "$*$-tight envelope" of the given Jordan algebra $H(A,*)$. This means that A is generated by $H(A,*)$ and every non-zero $*$-invariant ideal of A has a non-zero intersection with $H(A,*)$. In view of this fact, the additional requirement that A is a $*$-tight envelope of $H(A,*)$ can and will be assumed in the formulation of the norm extension problem. To conclude the phrasing of that problem, note that, if it has an affirmative answer, then the algebra norm on A extending the topology of the norm $\|.\|$ on $H(A,*)$ can be chosen in such a way that $*$ becomes continuous.

As a general intrinsic criterion, we have that the norm extension problem has an affirmative answer if (and only if) the "tetrad mapping"

$$(x,y,z,t) \ \rightarrow \ xyzt + tzyx$$

from $H(A,*)\times H(A,*)\times H(A,*)\times H(A,*)$ to $H(A,*)$ is $\|.\|$-continuous [13]. It is also known (and was talked by the first author in the First Badajoz Meeting on Examples and counterexamples in Banach spaces) that the answer to the norm extension problem is affirmative whenever $H(A,*)$ is simple and has a unit, whereas we can have negative answers in the case that $H(A,*)$ is simple but has not a unit [2].

2. Two theorems

In this section we collect two recent positive results concerning the norm extension problem. Recall that an algebra is said to be semiprime if, for every non-zero ideal I in the algebra, there exist x,y in I satisfying $xy \neq 0$.

THEOREM 1. [13] *The norm extension problem has an affirmative answer whenever $H(A,*)$ is semiprime and the norm $\|.\|$ on $H(A,*)$ is complete.*

We do not know if, under the requirements in the above theorem, the algebra norm $|\!|.|\!|$ on A extending the topology of the norm $\|.\|$ on $H(A,*)$ can be chosen complete. In any case, it is proved in [13] that the norm $|\!|.|\!|$ on A can be chosen in such a way that $(A,|\!|.|\!|)$ becomes a $*$-invariant dense subalgebra of a Banach $*$-algebra whose non-zero $*$-invariant ideals have a non-zero intersection with $H(A,*)$. Observe that such a Banach algebra (say

C) is generated by $H(A,*)$ in a algebraic and topological sense, and satisfies $H(C,*) = H(A,*)$.

Now note that, if B is an algebra with involution and n is a natural number, then the algebra $M_n(B)$ (of all $n \times n$ matrices over B) has a natural involution (called the standard involution) consisting in transposing a given matrix and applying the involution of B to each entry.

THEOREM 2. [10] *The norm extension problem has an affirmative answer whenever $A = M_n(B)$, for some natural number n with $n \geq 3$ and some algebra B with a unit and an involution, and the involution $*$ of A is the standard one relative to that of B.*

According to our raising of the norm extension problem, the algebra A in Theorems 1 and 2 above is assumed to be associative, and consequently the algebra B in Theorem 2 must be associative too. Nevertheless, Theorem 2 is proved in [10] with B (and hence A) not necessarily associative, thus beginning the non-associative treatment of the norm extension problem. It is also worth to mention that, for the algebra with involution $(A,*)$ in Theorem 2, we enjoy the "uniqueness of the extended norm topology" (i.e., two algebra norms on A are equivalent whenever they make $*$ continuous and their restrictions to $H(A,*)$ are equivalent), and that, if $\|.\|$ is a complete algebra norm on $H(A,*)$, then the unique algebra norm on A making $*$ continuous and generating on $H(A,*)$ the topology of $\|.\|$ is complete too (see [10] again). Such a nice situation for the norm extension problem was completely unknown up to date in any context.

3. AN ANTI-THEOREM

In this section we exhibit an example showing simultaneously that neither the assumption in Theorem 1 that $H(A,*)$ is semiprime can be dropped, nor the requirement in Theorem 2 that $n \geq 3$ may be relaxed. In fact the example goes further away, and shows in particular that, in the setting of the norm extension problem, no normed space condition on $(H(A,*), \|.\|)$, other than the finite dimensionality, can be sufficient to ensure an affirmative answer to that problem. (By the way, if $H(A,*)$ is finite-dimensional, then it is not difficult to see that A is finite-dimensional too, and therefore the norm extension problem answers affirmatively.)

Our construction begins with the following lemma.

LEMMA. [11] *On every infinite-dimensional normed space there is a dis-*

continuous anti-commutative associative product.

Proof. Let Y be an infinite-dimensional normed space. Let $\mathcal{B}$ be a Hamel basis of Y consisting of norm-one elements of Y, take a countably infinite subset $\mathcal{C}$ of $\mathcal{B}$ such that $\mathcal{B}\backslash\mathcal{C}$ is non-empty, and choose b in $\mathcal{B}\backslash\mathcal{C}$. Then, if $n \to c_n$ is a bijection from $\mathbb{N}$ to $\mathcal{C}$, the product on Y determined by the rules

$$c_n c_m = (n - m)b \text{ for all } n, m \text{ in } \mathbb{N}$$

$$xy = 0 \text{ whenever both } x, y \text{ are in } \mathcal{B} \text{ and some of them is in } \mathcal{B}\backslash\mathcal{C}$$

is a discontinuous anti-commutative associative product on Y. ∎

Now the anti-theorem reads as follows.

ANTI-THEOREM. [10] Let X be an arbitrary infinite-dimensional normed space. Then there exists an associative algebra A with involution $*$, satisfying:

i) $X = H(A, *)$, as vector spaces.

ii) Up to the multiplication by a suitable positive number if necessary, the norm of X becomes an algebra norm on $H(A, *)$.

iii) There is no algebra norm on A whose restriction to $H(A, *)$ is equivalent to the norm of X.

iv) A is a $*$-tight envelope of $H(A, *)$. Moreover, A is of the form $M_2(B)$, where B is an algebra with a unit and an involution, and the involution $*$ of A is the standard one relative to that of B.

Proof. Take a 3-codimensional subspace (say Y) of X. By the lemma above Y can be converted into an associative anticommutative algebra whose product is discontinuous. If B denotes the unital hull of Y, then B is an associative algebra with involution $*$ given by $(\beta + y)^* := \beta - y$ for all $\beta + y$ in $B = \mathbb{K} \oplus Y$ (where $\mathbb{K}$ is the field of real or complex numbers). Now, consider $A = M_2(B)$ with the standard involution, and let u, v, w be norm-one elements in X such that $\mathbb{K}u + \mathbb{K}v + \mathbb{K}w$ is a complement of Y in X. Then the mapping

$$x = \lambda u + \alpha v + \beta w + y \;\to\; [x] := \begin{pmatrix} \lambda + \alpha & \beta + y \\ \beta - y & \lambda - \alpha \end{pmatrix}$$

identifies X with $H(A,*)$ as vector spaces and a straightforward verification shows

$$[\lambda_1 u + \alpha_1 v + \beta_1 w + y_1] \circ [\lambda_2 u + \alpha_2 v + \beta_2 w + y_2]$$
$$= [(\lambda_1\lambda_2 + \alpha_1\alpha_2 + \beta_1\beta_2)u + (\lambda_1\alpha_2 + \lambda_2\alpha_1)v + (\lambda_1\beta_2 + \lambda_2\beta_1)w + \lambda_1 y_2 + \lambda_2 y_1]$$

for all $\lambda_1, \alpha_1, \beta_1, \lambda_2, \alpha_2, \beta_2$ in $\mathbb{K}$ and y_1, y_2 in Y. On the other hand, since the direct sum $X = \mathbb{K}u \oplus \mathbb{K}v \oplus \mathbb{K}w \oplus Y$ is topological, there exists $M > 0$ such that

$$|\lambda| + |\alpha| + |\beta| + \|y\| \ \leq \ M \ \|\lambda u + \alpha v + \beta w + y\|$$

for all λ, α, β in $\mathbb{K}$ and y in Y. Hence, translating the norm of X to $H(A,*)$ via the equality $\|[x]\| := \|x\|$, it follows

$$\|[\lambda_1 u + \alpha_1 v + \beta_1 w + y_1] \circ [\lambda_2 u + \alpha_2 v + \beta_2 w + y_2]\| \ \leq$$
$$M^2 \|[\lambda_1 u + \alpha_1 v + \beta_1 w + y_1]\| \ \|[\lambda_2 u + \alpha_2 v + \beta_2 w + y_2]\|$$

for all $\lambda_1, \alpha_1, \beta_1, \lambda_2, \alpha_2, \beta_2$ in $\mathbb{K}$ and y_1, y_2 in Y. Therefore, up to the multiplication by a suitable positive number if necessary, the norm of X becomes an algebra norm on $H(A,*)$. In this way we have proved (i) and (ii). Assume (iii) does not hold, so that we may assume that the norm of X is the restriction to X of an algebra norm $\|.\|$ on A. Then, for all y_1, y_2 in Y, we have

$$2[y_1 y_2] = [y_1][y_2][w][u+v] + [u+v][w][y_2][y_1] \ ,$$

and therefore

$$\|y_1 y_2\| \ \leq \ \|w\| \ \|u+v\| \ \|y_1\| \ \|y_2\| \ ,$$

contradicting the discontinuity of the chosen product on Y. An elemental proof of (iv) can be also provided (cf. the conclusion of the proof of [10; Theorem 4.3]), but we prefer to say here that (iv) is a consequence of the general fact that, if B is any (possibly non associative) algebra with a unit and an involution, if n is a natural number with $n \geq 2$, and if $*$ denotes the standard involution on $M_n(B)$, then $M_n(B)$ is a $*$-tight envelope of $H(M_n(B),*)$ [10; Corollary 5.2]. ∎

4. Concluding remark

Our goal in this note has been only to present some of the most recent results on the norm extension problem. Since (as we said in the introduction) that problem is closely related to the analytic treatment of the Zel'manov prime theorem, we would not feel reassured without referring the reader to the papers [8], [6], [7], [3], and [4] (see also Section F of [12]), where this topic is widely developed. Also we would like to remark that, sometimes, negative answers to the norm extension problem can be converted into positive results in other fields. For example, the monster built in [2] in relation to the norm extension problem (when $H(A, *)$ is simple but has not a unit) has been exploited in [5] and [9] to analytically distinguishing Jordan polynomials among all associative polynomials (a question raised and germinally discussed in [1]).

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