EXTRACTA MATHEMATICAE Vol. **12**, Núm. 2, 173 – 178 (1997)

# On Weakly Uniformly Convex Spaces According to Calder

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AMS *Subject Class.* (1991): 46B20

## 1. Weak forms of strict and uniform convexity appearing in the literature

Throughout this paper, $X$ denotes a normed linear space, of dimension at least 2, over the real field $\mathbb{R}$. $U(X)$ denotes the unit (closed) ball of $X$, and $\Sigma(X)$ its unit sphere; we shall also denote these sets simply by $U$ and $\Sigma$ when no confusion can arise.

Given $x$ and $y$ in $X$, we denote by $[x,y]^d$ the "metric segment" determined by $x$ and $y$:

$$[x,y]^d = \{z \in X\colon \|x-z\| + \|z-y\| = \|x-y\|\}. \tag{1.1}$$

We define, for $X$ given, the following functions: for $x, y \in U$, we set

$$\delta^*(x,y) = \sup\{1-\|w\|\colon w \in [x,y]^d\} = 1 - \inf\{\|w\|\colon w \in [x,y]^d\}. \tag{1.2}$$

Moreover, for $\epsilon \in [0,2]$, we set:

$$\delta^*(\epsilon) = \inf\{\delta^*(x,y)\colon x,y \in \Sigma;\ \|x-y\| \geq \epsilon\}; \tag{1.3}$$

$$\delta_1^*(\epsilon) = \inf\{\delta^*(x,y)\colon x,y \in U;\ \|x-y\| \geq \epsilon\}; \tag{1.4}$$

$$\delta_2^*(\epsilon) = \inf\{\delta^*(x,y)\colon x,y \in \Sigma;\ \|x-y\| = \epsilon\}; \tag{1.5}$$

$$\delta_3^*(\epsilon) = \inf\{\delta^*(x,y)\colon x,y \in U;\ \|x-y\| = \epsilon\}. \tag{1.6}$$

Note that for any pair $x, y \in U$:

$$0 \leq \delta^*(x,y) \leq 1$$

and that

$$0 \leq \delta^*(\epsilon) = 1 - \sup\{\{\inf \|w\| \colon w \in [x,y]^d\} \colon x, y \in \Sigma;\ \|x-y\| \geq \epsilon\} \leq 1.$$

Similar relations hold for the other three functions now defined.

The following definition was introduced in [1]. Say that $X$ is "weakly uniformly convex" if the following property holds:

(C) $\delta^*(\epsilon) > 0$ for all $\epsilon \in (0,2]$.

We shall say that $X$ is weakly d-uniformly convex, ($W_d$UC) for short, when it satisfies (C).

Also, it was shown in [1], Theorem 3.2, that $X$ satisfies (C) if and only if

($C^{\#}$) for $\epsilon > 0$ there exists $\delta > 0$ such that if $\|x\| = \|y\| = 1+\delta$ and $\|x-y\| \geq \epsilon$, then there exists $w \in [x,y]^d$ such that $\|w\| < 1$.

Successively, in [2], the same property was considered but with a slightly different definition, obtained by changing (C) into

(C') $\delta_1^*(\epsilon) > 0$ for all $\epsilon \in (0,2]$.

In [3] similar notions were introduced in the more general context of metric spaces, together with the similar "non-uniform" definitions (these properties were called there "uniform M-convexity" and "strict M-convexity"; the prefix M stands for Menger, who first considered, in 1928, the property $d(x,z) + d(z,y) = d(x,y)$). Such definitions were made precise in [6], where it was indicated that most "results" in [3] were not clear or correct. We note in passing that also a proposition indicated in [6] (Theorem 2) is wrong, namely the proposition saying that any closed set $A$ is "d-convex"; i.e., $x, y \in A$ implies $[x,y]^d \in A$: in fact, we see that this is not true by simply considering as $A$ a set like $\{x,y\}$ with $x \neq y$.

The definitions of [3] are the following.

The space $X$ is "strictly M-convex" if the following condition holds:

(Ch*) for every triplet $x, y, t$ in $X$ all different, $r > 0$, $\|x-t\| \leq r$ and $\|y-t\| \leq r$, there exists $z \in [x,y]^d$, $z$ different from $x, y, t$, such that $\|z-t\| < r$.

The space $X$ is "uniformly M-convex" if the following condition holds:

(Ch) for every pair of positive numbers $\epsilon, r$, there corresponds a positive number $\delta$ such that for every triplet $x, y, t$ in $X$ all different and satisfying $\|x-y\| \geq \epsilon$, $\|x-t\| < r+\delta$, $\|y-t\| < r+\delta$, there exists $z \in [x,y]^d$ such that $\|z-t\| < r$.

After a translation and a change of scale (with a proof similar to that of the equivalence between (C) and (C$^{\#}$)), it is possible to see that the conditions (Ch) and (C') are equivalent.

The above notions have been quoted very seldom in the literature: we only recall that in [5], 5.15, it was noted that (C) does not imply "weakly normal structure" for the space.

We recall that the term weakly uniformly convex space appears in general in the literature to denote a different notion, obtained by considering functionals in $X^*$ (see e.g. [4]).

## 2. Some equivalences

We shall see that the above "different" definitions (and some related ones) agree. We start with a simple lemma.

Lemma 2.1. *If $[x', y'] \subset [x, y]$, then $[x', y']^d \subset [x, y]^d$.*

*Proof.* Let $z \in [x', y']^d$, so $\|z - x'\| + \|z - y'\| = \|x' - y'\|$; then

$$\begin{aligned}\|z - x\| + \|z - y\| &\leq \|z - x'\| + \|x - x'\| + \|z - y'\| + \|y - y'\| \\ &= \|x - x'\| + \|x' - y'\| + \|y' - y\| = \|x - y\|,\end{aligned}$$

so $z \in [x, y]^d$. ∎

Note that $\delta^*(0) = \delta_1^*(0) = \delta_2^*(0) = \delta_3^*(0) = 0$; $\delta^*(2) = \delta_1^*(2) = \delta_2^*(2) = \delta_3^*(2) = 1$. The following inequalities are trivially true, in any space, for every $\epsilon \in [0, 2]$:

$$\delta_1^*(\epsilon) \leq \delta^*(\epsilon)$$
$$\delta_3^*(\epsilon) \leq \delta_2^*(\epsilon)$$

We shall now prove the following.

Theorem 2.2. *In any space $X$, we have, for every $\epsilon \in [0, 2]$:*

$$\delta^*(\epsilon) = \delta_2^*(\epsilon); \tag{2.1}$$

$$\delta_1^*(\epsilon) = \delta_3^*(\epsilon). \tag{2.2}$$

*Proof.* The proof will be achieved by showing that, for every $\epsilon \in [0, 2]$, we have:

a) $\delta_2^*(\epsilon) \le \delta^*(\epsilon)$;

b) $\delta_3^*(\epsilon) \le \delta_1^*(\epsilon)$.

We prove a). Take $x, y \in \Sigma$; $\|x-y\| \ge \epsilon$, $0 < \epsilon < 2$. Let $Y$ be the two-dimensional subspace of $X$ generated by $x$ and $y$. Take a functional $f \in Y^*$ such that $\|f\| = 1$ and $f(x-y) = 0$: let $f(x) = f(y) = a > 0$; $f(z) = 1$ for some $z \in \Sigma(Y)$. We can find $x', y'$ in $\Sigma(Y)$ such that: $f(x'-y') = 0$; $y' - x' = \lambda(y-x)$ for some $\lambda > 0$; $\|x'-y'\| = \epsilon$: in fact, this is trivially true if $a = 1$, otherwise it is enough to move the line passing through $x$ and $y$ along the direction of $z$. Let $f(x') = f(y') = b$ $(a \le b \le 1)$. Now take $x'' = (a/b)y'$, $y'' = (a/b)y'$: we have $f(x'') = f(y'') = a$. Moreover $\|x''\| \le 1$, $\|y''\| \le 1$ imply $[x'', y''] \subset [x, y]$.

Now, for any $\eta > 0$, we can choose $u'$ in $[x', y']^d$ (but not necessarily in $Y$) such that $\|x' - u'\| + \|u' - y'\| = \|x' - y'\|$ and $\|u'\| < 1 - \delta^*(x', y') + \eta \le 1 - \delta_2^*(\epsilon) + \eta$.

But then, for $u = (a/b)u'$ we have:

$$\|x''-u\|+\|u-y''\| = (a/b)\|x'-u'\|+(a/b)\|u'-y'\| = (a/b)\|x'-y'\| = \|x''-y''\|,$$

so $u \in [x'', y'']^d$, and then also $u \in [x, y]^d$ (see Lemma 2.1). Therefore $\delta^*(x, y) \ge 1 - \|u\| \ge 1 - \|u'\| > \delta_2^*(\epsilon) - \eta$. Since $\eta > 0$ is arbitrary, this implies $\delta^*(x, y) \ge \delta_2^*(\epsilon)$: since this is true for all $x, y \in \Sigma$, $\|x - y\| \ge \epsilon$, this proves a).

b) Let $x, y \in U$; $\|x - y\| \ge \epsilon$, $0 < \epsilon < 2$. We can take $x'$ on the segment joining $x$ and $y$ so that $\|x' - y\| = \epsilon$. For any $\eta > 0$, there exists $u \in [x', y]^d$ such that $\|u\| < 1 - \delta^*(x', y) + \eta$. But then (by Lemma 2.1) also $u \in [x, y]^d$, so $\delta^*(x, y) = 1 - \inf\{\|v\|; v \in [x, y]^d\} \ge 1 - \|u\| > \delta^*(x', y) - \eta \ge \delta_3^*(\epsilon) - \eta$. Since $\eta > 0$ is arbitrary, this implies $\delta^*(x, y) \ge \delta_3^*(\epsilon)$ for all $x, y \in U$; $\|x - y\| \ge \epsilon$: thus b) is proved. ∎

THEOREM 2.3. *For any space $X$, we have, for every $\epsilon \in [0, 2]$:*

$$\delta^*(\epsilon) \le \frac{2}{\epsilon}\delta_1^*(\epsilon) \tag{2.3}$$

*Proof.* Take $x, y \in U$; $\|x - y\| \ge \epsilon > 0$. We want to show that $\delta^*(x, y) \ge \frac{\epsilon}{2}\delta^*(\epsilon)$. Assume that $0 < \|x\| \le \|y\| \le 1$; $\|x\| \cdot \|y\| < 1$ (otherwise there is nothing to prove). Let $\|x - y\| \ge \epsilon > 0$, then take $w$ on the line joining 0 and $y$, such that $\|w - x\| = \|w - y\| = s$ (of course $\epsilon/2 \le s < 1$). Then set $x' = \frac{x-w}{s}$, $y' = \frac{y-w}{s}$; we have $\|x'\| = \|y'\| = 1$; $\|x' - y'\| = \frac{\|x-y\|}{s} > \epsilon$; so, given $\eta > 0$, there exists $k' \in [x', y']^d$ such that $\|k'\| < 1 - \delta^*(\epsilon) + \eta$. If we set

$k = w + sk'$, then we obtain: $\|k-x\|+\|k-y\| = \|w+sk'-x\|+\|w+sk'-y\| = \|sk'-sx'\|+\|sk'-sy'\| = s(\|k'-x'\|+\|k'-y'\|) = s\|x'-y'\| = \|x-y\|$, so $k \in [x,y]^d$;

moreover $\|k\| \le \|w\| + \|sk'\| < 1 - s + s(1-\delta^*(\epsilon)+\eta) = 1 - s\delta^*(\epsilon) + s\eta$, so ($s \ge \epsilon/2$) we obtain $\delta^*(x,y) \ge 1 - \|k\| > \frac{\epsilon}{2}(\delta^*(\epsilon) - \eta)$. Since $\eta > 0$ is arbitrary, this concludes the proof. ∎

COROLLARY 2.4. *A space $X$ is* ($\mathrm{W}_d$UC) *if and only if one of the following four properties is satisfied:*

(C) $\delta^*(\epsilon) > 0$ *for all* $\epsilon \in (0,2]$;

(C′) $\delta_1^*(\epsilon) > 0$ *for all* $\epsilon \in (0,2]$;

(C″) $\delta_2^*(\epsilon) > 0$ *for all* $\epsilon \in (0,2]$;

(C‴) $\delta_3^*(\epsilon) > 0$ *for all* $\epsilon \in (0,2]$.

Consider now the following definition. We say that $X$ is weakly d-strictly convex, ($\mathrm{W}_d$SC) for short, when

(C*) $\delta^*(x,y) > 0$ for every pair $x, y$ in $\Sigma$, $x \neq y$.

Concerning this property the following simple remark applies. It is possible to see that (Ch*) is equivalent to

(C′*) $\delta^*(x,y) > 0$ for every pair $x, y$ in $U$, $x \neq y$.

It is also clear that (C′*) implies (C*), but clearly also the converse is true. In fact, let (C*) hold, and take $x, y$ in $U$, $x \neq y$: if $\|x\| = \|y\| = 1$, then $\delta^*(x,y) > 0$ because of (C*); otherwise, all points in the open segment (x,y) have a norm smaller than 1. So we have:

COROLLARY 2.5. *Property* ($\mathrm{W}_d$SC) *can be defined by any of the (equivalent) properties* (C*), (C′*), (Ch*).

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