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Norm Attaining Operators Versus Bilinear Forms *

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The well known Bishop-Phelps Theorem asserts that the set of norm attaining linear forms on a Banach space is dense in the dual space [3]. This note is an outline of recent results by Y.S. Choi [5] and C. Finet and the author [7], which clarify the relation between two different ways of extending this theorem.

In the best known approach one considers *norm attaining operators.* A bounded linear operator between Banach spaces X and Y attains its norm if there is $x_0 \in B_X$ (the closed unit ball of X) such that

$$\|T(x_0)\| = \|T\| := \sup\{\|T(x)\| : x \in B_X\}.$$

In their seminal paper [3], E. Bishop and R. Phelps already addressed the general question if the set $NA(X,Y)$ of norm attaining operators is dense in the space $L(X,Y)$ of all bounded linear operators. We refer the reader to [10], [4], and [8] for some of the main results on this question.

Alternatively, one may stay in the context of scalar-valued functions and investigate the denseness of *norm attaining multilinear forms or polynomials*, a line of research recently initiated by R. Aron, C. Finet, and E. Werner [2]. For simplicity, only the bilinear case will be discussed here. We denote by $\mathcal{B}(X)$ the space of all continuous bilinear forms on a Banach space X, we say that $\varphi \in \mathcal{B}(X)$ attains its norm if there are $x_0, y_0 \in B_X$ such that

$$|\varphi(x_0, y_0)| = \|\varphi\| := \sup\{|\varphi(x,y)| : x, y \in B_X\},$$

and we denote by $\mathcal{B}_{na}(X)$ the set of norm attaining bilinear forms. The first example of a Banach space X such that $\mathcal{B}_{na}(X)$ is not dense in $\mathcal{B}(X)$, a predual of a Lorentz sequence space, was found in [1]. Sufficient conditions

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for the denseness of norm attaining multilinear forms and polynomials can be found in [2], [6], and [9].

To see the connection between the two approaches, just consider the canonical identification of $\mathcal{B}(X)$ with $L(X, X^*)$ where X^* is the dual space of X. The operator T corresponding to a bilinear form φ is given by

$$[T(x)](y) = \varphi(x, y) \qquad (x, y \in X).$$

We see that the bilinear form φ attains its norm if and only if the operator T attains its norm at a point $x \in B_X$ such that $T(x)$ also attains its norm as a functional on X. Therefore, $T \in NA(X, X^*)$ whenever $\varphi \in \mathcal{B}_{na}(X)$, but we will see that the converse is not true, even in a very simple case. For fixed $x^*, y^* \in X^*$ consider the bilinear form

$$\varphi(x, y) = x^*(x)y^*(y) \qquad (x, y \in X),$$

which corresponds to the operator T given by

$$T(x) = x^*(x)\, y^* \qquad (x \in X).$$

Then T attains its norm whenever x^* does, while φ attains its norm only when both x^* and y^* attain their norms. However, the Bishop-Phelps Theorem allows replacing y^* with a norm attaining functional to get a norm attaining bilinear form which is arbitrarily close to φ.

To summarize the above comments, when identifying bilinear forms with operators, $\mathcal{B}_{na}(X)$ becomes a subset of $NA(X, X^*)$, in general a proper subset, but there are some reasons to believe that $\mathcal{B}_{na}(X)$ might be dense as soon as $NA(X, X^*)$ is dense, and the "new" question on bilinear forms would be nothing but a very special case of the "old" problem on operators. Actually, in all papers dealing with norm attaining bilinear forms the following question appears in a more or less explicit way (see [2], [1], [5].)

QUESTION. Is there a Banach space X such that $NA(X, X^*)$ is dense in $L(X, X^*)$ while $\mathcal{B}_{na}(X)$ is not dense in $\mathcal{B}(X)$?

The following results give an affirmative answer.

THEOREM 1. [5, Theorem 3] *The set of norm attaining bilinear forms is not dense in the space of all continuous bilinear forms on* $L_1[0, 1]$.

THEOREM 2. [7, Theorem 3] *The set of norm attaining operators is dense in the space of all bounded linear operators from* $L_1[0, 1]$ *into* $L_\infty[0, 1]$.

In what follows we will try to point out the main ideas in the proofs of the above results. In both cases one starts with a useful representation of the Banach space under consideration, which reduces the problem to nice questions on Lebesgue measure in the plane. Then, some classical results in Measure Theory come into play.

By quite standard arguments, the spaces $\mathcal{B}(L_1(I))$ and $L(L_1(I), L_\infty(I))$ (from now on we write $I = [0,1]$) can be identified with $L_\infty(I \times I)$. More concretely, the bilinear form φ and the operator T corresponding to a function $h \in L_\infty(I \times I)$ are given by

$$\varphi(f,g) = \int_{I\times I} h(s,t)f(s)g(t)\,dsdt \quad (f,g \in L_1(I)),$$

$$[T(f)](t) = \int_I h(s,t)f(s)\,ds \quad (a.e.\ t \in I, f \in L_1(I)).$$

Thus, we are actually dealing with (equivalence classes of) essentially bounded measurable functions on the unit square. The next step is trying to characterize those functions h which correspond to norm attaining bilinear forms or operators. Actually it suffices to consider the special case of h being the characteristic function χ_S of a measurable set $S \subseteq I \times I$ with positive measure.

Choi proves that the bilinear form corresponding to χ_S can be approximated by norm attaining bilinear forms if and only if there are measurable sets $A, B \subseteq I$, with positive measure, such that the measurable rectangle $A \times B$ is essentially contained in S, in the sense that

$$m((A \times B) \cap S) = m(A \times B), \tag{1}$$

where we use m to denote Lebesgue measure in the unit square. With similar arguments, it is shown in [7, Corollary 2] that the operator corresponding to χ_S attains its norm if and only if there are measurable sets $A, B_n \subseteq I$, also with positive measure, such that

$$\frac{m((A \times B_n) \cap S)}{m(A \times B_n)} \to 1. \tag{2}$$

In fact, if this property is satisfied by any measurable set S with positive measure, then the operator corresponding to any simple function will attain its norm and we can use that simple functions are dense in $L_\infty(I \times I)$ to get the denseness of norm attaining operators. Therefore, Theorems 1 and 2 are consequences of the following properties of Lebesgue measure.

CLAIM 1. There is a measurable set $S \subset I \times I$, with positive measure, such that (1) fails for every measurable rectangle $A \times B$ with positive measure.

CLAIM 2. For any measurable set $S \subseteq I \times I$ with positive measure, there are measurable sets $A, B_n \subseteq I$, with positive measure, satisfying (2).

To find the set S required in Claim 1, Choi uses a Cantor-type set $C \subset I$ with positive measure and takes $S = \{(s,t) \in I \times I : |s-t| \in C\}$. The proof is based on a classical theorem due to Steinhaus: *If A, B are measurable sets in the real line, with positive measure, then the difference set $A - B = \{a - b : a \in A, b \in B\}$ has nonempty interior.* Now, if $A \times B \subseteq S$ we have $A - B \subseteq C \cup -C$, but $C \cup -C$ has empty interior and we get a contradiction with Steinhaus Theorem, unless $m(A \times B) = 0$. Unfortunately, if (1) is satisfied, we only have $A \times B \subseteq S \cup N$ where $m(N) = 0$, so one has to be more careful. Actually, the proof of Steinhaus Theorem gives a somewhat stronger statement which overcomes this difficulty. We refer to [5] for the rather technical details.

The proof of Claim 2 given in [7] is also based on a classical result, namely Lebesgue Density Theorem: *If E is a measurable set in the real line, then E has density 1 at almost all its points.* More precisely, $\lambda(E \backslash \delta(E)) = \lambda(\delta(E) \backslash E) = 0$, where λ denotes Lebesgue measure in the real line and

$$\delta(E) := \{y \in \mathbb{R} : \lim_{h \to 0} \frac{\lambda(E \cap [y-h, y+h])}{2h} = 1\}.$$

Given a measurable set $S \subseteq I \times I$ with $m(S) > 0$ we may fix $x \in I$ and consider the vertical section $S_x := \{t \in I : (x,t) \in S\}$. It is clear that there must be some y with $0 < y < 1$ such that $y \in S_x$ for every x in a set $A \subseteq I$ with positive measure. Lebesgue Density Theorem allows replacing S_x with $\delta(S_x)$ in this assertion, so we actually have $y \in \delta(S_x)$ for every $x \in A$ (see [7] for the details). Then, we may take $B_n = [y - h_n, y + h_n]$ where (h_n) is a sequence of positive numbers tending to zero, and we have

$$\frac{\lambda(S_x \cap B_n)}{\lambda(B_n)} \to 1.$$

for every $x \in A$. Integration over the set A and the Dominated Convergence Theorem yield (2).

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