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The Johnson-Lindenstrauss Space

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The Johnson-Lindenstrauss space is a Banach space containing an uncomplemented copy of c_0, the quotient by which is a non-separable Hilbert space. This serves as a counterexample for many questions [3]. We are indebted to the organizers for suggesting that we talk about this space. This has forced us to think about the general problem of whether every copy of c_0 in a given Banach space is complemented. First, a positive result. Say that a topological space has the Veech property if every separable subset is metrizable.

Theorem. (Sobczyk's Theorem) (i) *Let K be a compact Veech space. Then every isometric copy of c_0 in $C(K)$ is complemented.*

(ii) *Let X be a Banach space whose dual ball, equipped with the weak* topology, has the Veech property. Then every isomorphic copy of c_0 in X is complemented.*

The isometry requirement in (i) can be relaxed a little; it suffices to assume that each coordinate functional (on the copy of c_0) is an extreme point of the dual ball. Since the dual ball of a separable space is weak* metrizable,c_0 is complemented in any separable superspace. This is what Sobczyk [20] originally proved, with a simpler proof given later by Veech [21], [15, Proposition 2.4]. Rosenthal observed in [21] that the same proof yields the stronger result (ii), and that weakly compactly generated spaces satisfy this hypothesis. Note that the property in (ii) passes to subspaces, since the Veech property obviously passes to continuous images.

So a Banach space containing an uncomplemented copy of c_0 cannot be separable. Here we study 3 such examples. The first is ℓ_∞, a result first proved explicitly by Phillips [19] and implicitly by Nakamura and Kakutani [16, §7] with a simpler proof being given by Whitley ([22] or [15, Theorem 2.2]). We give another simple proof here, and show how it leads to two further examples: a certain $C(K)$ space, and the Johnson-Lindenstrauss space.

EXAMPLE 1. In ℓ_∞, the subspace c_0 is not complemented.

Proof. Like several known proofs, we begin with the fact that there is an uncountable family $\{N_\gamma : \gamma \in \Gamma\}$ of infinite subsets of the natural numbers such that the intersection of any two of them is finite [16]. Denote by $\mathcal{X}_\gamma$ the characteristic function of N_γ, and let X be the closed linear span of $c_0 \cup \{\mathcal{X}_\gamma : \gamma \in \Gamma\}$. We will show that c_0 is not complemented in X; this clearly implies that c_0 is not complemented in ℓ_∞.

For any finite linear combination $x = \sum_{i=1}^n \lambda_i \mathcal{X}_{\gamma_i}$, it is easy to find some $y \in c_0$ (in fact, with finite support) such that $\|x - y\| = \max_{i=1}^n |\lambda_i|$. This implies that the quotient X/c_0 is isometric to $c_0(\Gamma)$. Now note that X^* is weak* separable (since $X \subset \ell_\infty$), whereas $c_0(\Gamma)^*$ is not (any countable subset of $\ell_1(\Gamma)$ has countable support and so is contained in a weak* closed separable subspace). Thus $c_0(\Gamma)$ cannot be isomorphic to any subspace of X. In particular X is not isomorphic to $c_0(\Gamma) \oplus c_0$ and c_0 is not complemented in X. ∎

Recall that a topological space is said to be scattered if every subset has an isolated point. We write K' for the derived set of K, i.e. the set of non-isolated points, and $K^{(n)}$ for the n^{th} derived set; thus $K^{(2)} = K''$ etc. It is well known (see e.g. the survey [24]) that $C(K)$ is an Asplund space if and only if K is scattered. Throughout, K is compact.

EXAMPLE 2. There is an Asplund $C(K)$ space containing an uncomplemented copy of c_0. Furthermore K is separable and $C(K)$ is Lipschitz homeomorphic to $c_0(\Gamma)$ (for some uncountable set Γ) even though neither of these Banach spaces is isomorphic to any subspace of the other. Also $C(K)$ admits an equivalent Fréchet smooth norm.

Proof. We simply look at the previous example in more detail. Let $A = X + \mathbb{R}1$ be the sum of X and the constant functions. Then A is clearly a subalgebra of ℓ_∞; by the Stone-Weierstraß Theorem it must be isomorphic to some $C(K)$ space. Of course A contains an uncomplemented copy of c_0. This space was first defined by Johnson and Lindenstrauss [13, Example 2], and later studied further by Aharoni and Lindenstrauss [1].

Since being Asplund is a 3-space property, $C(K)$ must be an Asplund space. An easier way to see this is to identify K directly. It is not hard to see that K is homeomorphic to the one-point compactification of the disjoint union $\mathbb{N} \cup \Gamma$, topologized as follows:

- For every $n \in \mathbb{N}$, any set containing n is a neighborhood of n.
- For every $\gamma \in \Gamma$, a set S is a neighborhood of γ if (and only if) $\gamma \in S$ and $N_\gamma \setminus S$ is finite.

It is obvious that K is scattered and separable. Since the subset Γ is not separable, K cannot be metrizable. This is perhaps the simplest known example of a separable compact Hausdorff space which is not metrizable.

There are several ways to see that $C(K)$ is not isomorphic to any subspace of $c_0(\Gamma)$. The following seems to be the most elementary. A simple direct argument shows that $c_0(\Gamma)$ satisfies the hypothesis of Sobczyk's Theorem. In particular, c_0 is complemented in any Banach space isomorphic to a subspace of $c_0(\Gamma)$. However c_0 is not complemented in $C(K)$. So neither of $C(K)$ or $c_0(\Gamma)$ is isomorphic to any subspace of the other.

It is shown in [1] that $C(K)$ is Lipschitz homeomorphic to $c_0(\Gamma)$. A Fréchet smooth norm for $C(K)$ was first given explicitly in [13], with a construction similar to that given below in Example 3. ∎

More generally, if K is any compact Hausdorff space whose n^{th} derived set $K^{(n)}$ is empty for some finite n, it is shown in [8, Theorem 3] that $C(K)$ is Lipschitz homeomorphic to $c_0(\Gamma)$ for some Γ, and in [7] that there is a Fréchet smooth norm for $C(K)$. However there are $C(K)$ Asplund spaces which do not admit even a Gateaux smooth norm [10].

The class just described contains all known examples of Banach spaces which are Lipschitz equivalent but not isomorphic. In particular, there is still no known example of two non-isomorphic separable Banach spaces being Lipschitz equivalent. It is possible to prove that if K is scattered, then $C(K)$ is c_0-saturated, i.e. that every infinite-dimensional subspace contains an isomorphic copy of c_0 [18]. It has been conjectured that whenever two Banach spaces are Lipschitz homeomorphic, then they have the same separable subspaces. This is true in the reflexive case: more generally, if X and Y are Lipschitz homeomorphic, then every separable subspace of X is isomorphic to a subspace of Y^{**} [11].

There is not enough space here to say much about this subject, for which we refer to the ground-breaking paper [11] or the survey [23]. Of course, reflexivity is preserved by Lipschitz homeomorphisms. We mention just one more typical result: for $1 < p < \infty$, any Banach space Lipschitz equivalent to ℓ_p is actually isomorphic to ℓ_p. This has recently been proved also for uniform homeomorphisms [14]. For $p = 1$ the problem remains open.

It is high time to introduce the few elementary facts about WCG spaces which we need. None of them are deep; it is worth emphasizing that we do not need to know anything about projections in WCG spaces, or injections into $c_0(\Gamma)$. First of all, a Banach space is said to be weakly compactly generated (WCG) if it contains a weakly compact set whose linear span is dense. The most natural examples are reflexive spaces, separable spaces, the spaces $c_0(\Gamma)$ and the spaces $L_1(\mu)$ for μ a σ-finite measure. The following well known result is often proved using barycentres. The argument here does not require any knowledge of measure theory.

THEOREM. *If K is a weakly compact subset of the Banach space X, then the restriction map $R : X^* \to C(K)$ is weak* to weak continuous. If K generates X, then R is injective.*

Proof. [12, proof of Theorem 4] Recall that $\mu(X^*, X)$, the Mackey* topology on X^*, is the topology of uniform convergence on weakly compact subsets of X.This is a natural topology to work with in WCG spaces. Since R is Mackey* to norm continuous, it is continuous in the corresponding weak topologies. But the dual of X^* under $\mu(X^*, X)$ is just X, so R is weak* to weak continuous. Clearly R is injective if K generates X. ∎

The bulk of the next result was first proved in [2].

COROLLARY. *For any Banach space X, the following are equivalent:*

(i) *X is weakly compactly generated.*

(ii) *There is a Banach space Y, and an injective weak* to weak continuous linear mapping $T : X^* \to Y$.*

(iii) *The unit ball of X^*, equipped with the weak* topology, is affinely homeomorphic to a weakly compact subset of a Banach space.*

In particular, WCG spaces satisfy the hypotheses of Sobczyk's Theorem.

Proof. The preceding theorem shows that (i) implies (ii). For the converse, note that the adjoint $T^* : Y^* \to X$ is weak* to weak continuous and has dense range. The equivalence of (ii) and (iii) should be clear. The final assertion follows from the well known easy exercise that every weakly compact subset of a separable space is (weakly) metrizable. ∎

For spaces of continuous functions, the word "affinely" in Corollary 1 is redundant.

COROLLARY. [2] *A $C(K)$ space is WCG if and only if K is homeomorphic to a weakly compact subset of a Banach space.*

Proof. (If) [12, Theorem 4] Suppose that K is a weakly compact subset of some Banach space X, and write U for the unit ball of X^*. In the notation of the preceding theorem, $L = R(U)$ is a weakly compact subset of $C(K)$ which separates the points of K.

Routine arguments show that L^n, the set of n-fold products of elements of L, is a weakly compact subset of $C(K)$. (Use Lebesgue's Theorem to show that a bounded pointwise convergent sequence is weakly convergent.) If we assume that L is contained in the unit ball of $C(K)$, then $\{1\} \cup \bigcup_{n=1}^{\infty} \frac{1}{n} L^n$ will be a weakly compact set, whose linear span is a subalgebra of $C(K)$. By the Stone-Weierstraß Theorem, $C(K)$ is WCG.

(Only if) If $C(K)$ is WCG, Corollary 1 tells us that the unit ball of $C(K)^*$, equipped with the weak* topology, is homeomorphic to a weakly compact subset of a Banach space. But K embeds therein. ∎

This can be used to give an alternative proof that Example 2 is not WCG.

Let us say that X is an extension of Y by Z if X contains (a subspace isomorphic to) Y and $X/Y \cong Z$. The extension is said to be non-trivial if Y is not complemented in X. Note that the space $C(K)$ of Example 2 is an extension of the separable space c_0 by the WCG space $c_0(\Gamma)$. Thus being WCG is not a 3-space property. It is proved in [4] that any extension of a separable dual space by a WCG space is already WCG. So it is not surprising that the non-dual space c_0 should appear as the subspace in a counterexample for the 3-space problem.

It is also proved in [4] that if every extension of Y by a reflexive space is WCG, then every extension of Y by a WCG space is WCG. With hindsight, we may confidently predict that there is an extension of c_0 by a reflexive space which is not WCG. In fact, Example 2 can be modified to yield such an example: this is the Johnson-Lindenstrauss space. The first step of their construction is a special case of the result just stated.

LEMMA. *Suppose that there is a non-WCG extension of some Banach space Y by $c_0(\Gamma)$. Then there is a non-WCG extension of Y by $\ell_2(\Gamma)$.*

Proof. Suppose that Y is a subspace of X and that $X/Y \cong c_0(\Gamma)$. Let U denote the $\ell_2(\Gamma)$ unit ball in $c_0(\Gamma)$. Put $V = \{x \in X : \|x\| \leq 1, x + Y \in U\}$, and denote by X_V the linear span of V, equipped with the norm whose unit ball is V. Since the intersection $V \cap Y$ is simply the unit ball of Y, we see that Y is a subspace of X_V. Clearly $X_V/Y \cong \ell_2(\Gamma)$. Since V is bounded, the inclusion mapping $X_V \to X$ is continuous; it obviously has dense range. Thus if X is not WCG, X_V cannot be either. ∎

EXAMPLE 3. ([13]) There is a Banach space $J\!L$ containing an uncomplemented copy of c_0, for which the quotient is a non-separable Hilbert space. It follows that $J\!L^*$ is WCG, although $J\!L$ is not. Furthermore,$J\!L^*$ is weak* separable, although its unit ball is not. Also, $J\!L$ admits an equivalent Fréchet smooth norm.

Proof. We follow the notation of Examples 1 and 2. Applying the Lemma to the latter gives us a non-WCG extension of c_0 by $\ell_2(\Gamma)$, which we denote by $J\!L$. Clearly c_0 is not complemented in $J\!L$ since $J\!L$ is not isomorphic to the WCG space $c_0 \oplus \ell_2(\Gamma)$. It follows from the lifting property of ℓ_1 that $J\!L^* \cong \ell_1 \oplus \ell_2(\Gamma)$. Thus $J\!L^*$ is WCG.

The inclusion mapping $J\!L \to \ell_\infty$ is continuous, so $J\!L^*$ is weak* separable. (A word of caution here: if X is a Banach space, E is a dense subspace, and E^* is weak* separable, we cannot always conclude that X^* is weak* separable [9, Example 1.1].) It can also be shown that no non-separable subspace Y of $J\!L$ is isomorphic to any subspace of ℓ_∞. More precisely [13, p.223], if Y contains c_0, then for any sequence $(f_n) \subset J\!L^*$ and any $\varepsilon > 0$, there is an $x \in Y$ with $\|x\| > 1-\varepsilon$ and $|f_n(x)| < \varepsilon$ for all n. As observed in [6], this implies that the unit ball of $J\!L^*$ is not weak* separable; see the following Proposition.

To continue, let us study the weak* topology more closely. Since the linear span of all the vectors e_n and χ_γ is dense in $J\!L$, a bounded net (y_α, z_α) in $\ell_1 \oplus \ell_2(\Gamma)$ converges weak* to (y, z) if and only if $y_\alpha(n) \to y(n)$ for each n and $\sum_{n\in N_\gamma} y_\alpha(n) + z_\alpha(\gamma) \to \sum_{n\in N_\gamma} y(n) + z(\gamma)$ for each $\gamma \in \Gamma$. In particular the net $(e_n)_{n\in N_\gamma}$ in ℓ_1 converges weak* to e_γ in $\ell_2(\Gamma)$. This proves again that $J\!L^*$ really is weak* separable.

Now we exhibit a Fréchet smooth norm on $J\!L$. If on ℓ_1, we denote by $\|\cdot\|_1$ and $\|\cdot\|_2$ the natural and ℓ_2 norms respectively, then

$$\|(y,z)\| = \|y\|_1 + (\|y\|_1^2 + \|y\|_2^2 + \|z\|^2)^{\frac{1}{2}}$$

defines a locally uniformly convex dual norm on $\ell_1 \oplus \ell_2(\Gamma)$. ∎

We remark that the index 2 in the construction of $J\!L$ is not important. For $1 < p < \infty$, we can construct a non-WCG extension of c_0 by $\ell_p(\Gamma)$. The choice $p = \infty$ returns us to Example 2. The choice $p = 1$ is not interesting because $\ell_1(\Gamma)$ is not WCG when Γ is uncountable.

PROPOSITION. [6] *The unit ball of a dual space X^* is weak* separable if and only if X is isometric to a subspace of ℓ_∞. A dual space X^* is itself weak* separable if and only if there is an injective operator $T : X \to \ell_\infty$.*

Proof. (If) In both cases, consider the transpose of the given mapping $X \to \ell_\infty$.

(Only if) Choose a sequence (f_n) which is weak* dense in the dual ball (respectively, which is bounded and separates points of X). Define $T : X \to \ell_\infty$ by $Tx = (f_n(x))$. ∎

Historically, *JL* was the first example of a non-WCG space with WCG dual. The first counterexample to the 3-space problem for WCG spaces was $D[0,1]$, the space of continuous functions on the "split interval" space [5, Example 2]. In this example, the subspace is $C[0,1]$ and the quotient space is $c_0(\Gamma)$. This Banach space is not weakly Lindelöf (since the uncountable collection of characteristic functions $\chi_{[t,1]}$ is closed and discrete in the weak topology) and hence it is not WCG. In fact the existence of a Banach space with these properties can easily be deduced from (the historically later) Example 2. Let Y be any Banach space containing a complemented copy of c_0, in particular $C[0,1]$. Then $X = (Y/c_0) \oplus C(K)$ is a non-trivial extension of Y by $c_0(\Gamma)$.

It is natural to attempt to apply the Johnson-Lindenstrauss modification to $D[0,1]$, as well to Example 2. This leads to a Banach space X containing an uncomplemented copy of $C = C[0,1]$ such that X/C is a non-separable Hilbert space. This can also be deduced by applying the argument of the previous paragraph to *JL*. In effect, we have a commutative diagram here: starting with Example 2, these two processes can be applied in either order.

There is even a non-trivial extension of $D[0,1]$ by $c_0(\Gamma)$ (or $\ell_p(\Gamma)$ for $1 < p < \infty$) since $D[0,1]$ contains a complemented copy of c_0 [17].

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