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# Asymptotics for Quasilinear Equations with Nearly Critical Growth

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## 1. INTRODUCTION

Let $\Omega$ be a smooth, open, bounded and starshaped domain in $\mathbb{R}^N$, with $N \geq 3$; and $1 < p < N$. We consider the following problem

$$(P_{\lambda\varepsilon}) \quad \begin{cases} -div(|\nabla u|^{p-2}\nabla u) - \lambda u^{p-1} = & u^{p^*-1-\varepsilon} & \text{in } \Omega, \\ u > & 0 & \text{in } \Omega, \\ u = & 0 & \text{on } \partial\Omega. \end{cases}$$

where $p^* = \frac{Np}{N-p}$ is the critical Sobolev exponent, in the sense that the embedding of $W_0^{1,p}(\Omega)$ into $L^q(\Omega)$ is compact for $q < p^*$ and only continuous for $q = p^*$; $\lambda$ and $\varepsilon$ are nonnegative reals.

Let $\lambda_1$ be the first eigenvalue of the problem

$$(P) \quad \begin{cases} -div(|\nabla u|^{p-2}\nabla u) = & \lambda|u|^{p-2}u & \text{in } \Omega, \\ u = & 0 & \text{on } \partial\Omega. \end{cases}$$

The eigenvalue $\lambda_1$ exists, it is simple and isolated (see for instance [1]).

It is well known that the problem $(P_{\lambda 0})$ has a solution only in the two following cases (see for instance [7] and [8]):

i) $1 < p^2 \leq N$ and $0 < \lambda < \lambda_1$,

ii) $p^2 > N$ and $\lambda_* < \lambda < \lambda_1$, where $\lambda_*$ is some positive constant.

In the case $p = 2$, the asymptotic behavior of the solutions of $(P_{\lambda\varepsilon})$ has been intensively studied in the last years. In this paper, we will be concerned with the behavior of the solutions when $\varepsilon$ goes to 0 or $\lambda$ goes to $\lambda_1$ in the case $1 < p < N$.

This paper is organized in the following way: in section 2 we will recall some existence and nonexistence results for $(P_{\lambda\varepsilon})$ and we will establish some a priori bounds. In section 3, we will study the epi-limits of the functionals associated to $(P_{\lambda\varepsilon})$ and in section 4, we describe the behavior of the solutions when $\lambda$ goes to $\lambda_1$ or $\varepsilon$ goes to 0.

## 2. Existence and nonexistence results

Let us first recall the following result (see for instance [7] and [8]).

THEOREM 2.1. (Pohozaev identity) *Let $u \in W_0^{1,p}(\Omega)$ be such that*

$$-div(|\nabla u|^{p-2}\nabla u) = f(u), \quad (u \in \Omega),$$

*where $f$ is some real function. Then for every $x_0 \in \Omega$ we have*

$$N\int_\Omega F(u)dx - \frac{N-p}{p}\int_\Omega uf(u)dx = \frac{p-1}{p}\int_{\partial\Omega}(x-x_0,\nu)\left(\frac{\partial u}{\partial \nu}\right)^p d\sigma,$$

*where $F(x) = \int_0^x f(t)dt$ and $\nu$ is the outward normal on $\partial\Omega$.*

Let $I_{\lambda\varepsilon}$ be the functional defined on $W_0^{1,p}(\Omega)$ by

$$I_{\lambda\varepsilon}(u) = \begin{cases} \int_\Omega |\nabla u|^p dx - \lambda \int_\Omega |u|^p dx & \text{if } \int_\Omega |u|^{p^*-\varepsilon}dx = 1, \\ +\infty & \text{otherwise}. \end{cases}$$

PROPOSITION 2.2. i) *Assume that $\varepsilon > 0$. Then, for all $\lambda < \lambda_1$, $I_{\lambda\varepsilon}$ has a positive minimizer $\bar{u}_{\lambda\varepsilon}$.*

ii) *If $\varepsilon = 0$, there exists a nonnegative constant $\lambda_*$ such that $I_{\lambda 0}$ has a positive minimizer $\bar{u}_{\lambda 0}$ if and only if $\lambda \in ]\lambda_*, \lambda_1[$.*

iii) *For all $\varepsilon \geq 0$, we have $-div(|\nabla \bar{u}_{\lambda\varepsilon}|^{p-2}\nabla \bar{u}_{\lambda\varepsilon}) - \lambda \bar{u}_{\lambda\varepsilon}^{p-1} = I_{\lambda\varepsilon}(\bar{u}_{\lambda\varepsilon})\bar{u}_{\lambda\varepsilon}^{p^*-1-\varepsilon}$.*

iv) *The function $u_{\lambda\varepsilon}$ defined by $u_{\lambda\varepsilon} = (I_{\lambda\varepsilon}(\bar{u}_{\lambda\varepsilon}))^{\frac{1}{p^*-p-\varepsilon}}\bar{u}_{\lambda\varepsilon}$ is a solution of $(P_{\lambda\varepsilon})$.*

*Proof.* i) (a) $I_{\lambda\varepsilon}$ is proper. Indeed, for every $\varphi \in C_0^\infty(\Omega)$, we have

$$I_{\lambda\varepsilon}\left(\frac{\varphi}{\|\varphi\|_{p^*-\varepsilon}}\right) < +\infty.$$

(b) $I_{\lambda\varepsilon}$ is lower semicontinuous for the weak topology of $W_0^{1,p}(\Omega)$ : Let $(u_n)_n$ converge to $u$ in the weak topology of $W_0^{1,p}(\Omega)$ and then, in the strong topologies of $L^p(\Omega)$ and $L^{p^*-\varepsilon}(\Omega)$. We may assume that $\liminf_{n\to+\infty} I_{\lambda\varepsilon}(u_n) < +\infty$, otherwise the conclusion is trivially reached. This implies the existence of a subsequence $(u_{n_k})_k$ of $(u_n)_n$ which belongs to the unit sphere of $L^{p^*-\varepsilon}(\Omega)$. Hence, so does $u$. We then deduce from the expression of $I_{\lambda\varepsilon}(u_n)$ that $\liminf_{n\to+\infty} I_{\lambda\varepsilon}(u_n) \geq I_{\lambda\varepsilon}(u)$.
(c) $I_{\lambda\varepsilon}$ is coercive on $W_0^{1,p}(\Omega)$, since $\lambda < \lambda_1$.

Properties (a)-(c) imply that $I_{\lambda\varepsilon}$ has a minimizer $\bar{u}_{\lambda\varepsilon}$ on $W_0^{1,p}(\Omega)$, which is positive in $\Omega$ by means of classical arguments.

ii) For the existence of $\lambda_*$, one can see [7]. Recall that $\lambda_*$ is equal to 0 if $N \geq p^2$ and $\lambda_*$ is positive if $p^2 > N$.

iii) For every $\varphi \in C_0^\infty(\Omega)$ and every positive $t$, we consider

$$w_{\varepsilon t} = \frac{\bar{u}_{\lambda\varepsilon} + t\varphi}{\|\bar{u}_{\lambda\varepsilon} + t\varphi\|_{p^*-\varepsilon}}.$$

From the definition of $\bar{u}_{\lambda\varepsilon}$, we deduce

$$\begin{aligned} I_{\lambda\varepsilon}(w_{\varepsilon t}) &= \frac{1}{\|\bar{u}_{\lambda\varepsilon} + t\varphi\|_{p^*-\varepsilon}^p}\left(\int_\Omega |\nabla(\bar{u}_{\lambda\varepsilon} + t\varphi)|^p dx - \lambda\int_\Omega |\bar{u}_{\lambda\varepsilon} + t\varphi|^p dx\right) \\ &\geq I_{\lambda\varepsilon}(\bar{u}_{\lambda\varepsilon}) = \int_\Omega |\nabla \bar{u}_{\lambda\varepsilon}|^p dx - \lambda\int_\Omega |\bar{u}_{\lambda\varepsilon}|^p dx. \end{aligned}$$

By easy computations we obtain

$$\begin{aligned} I_{\lambda\varepsilon}(\bar{u}_{\lambda\varepsilon})\frac{\|\bar{u}_{\lambda\varepsilon} + t\varphi\|_{p^*-\varepsilon}^p - 1}{t} \leq{}& \frac{\|\nabla(\bar{u}_{\lambda\varepsilon} + t\varphi)\|_p^p - \|\nabla\bar{u}_{\lambda\varepsilon}\|_p^p}{t} \\ &- \lambda\int_\Omega |\bar{u}_{\lambda\varepsilon}|^{p-1}\varphi\, dx + o(1). \end{aligned}$$

It is easy to see that

$$\lim_{t\to o^+}\frac{\|\bar{u}_{\lambda\varepsilon} + t\varphi\|_{p^*-\varepsilon}^p - 1}{t} = p\int_\Omega |\bar{u}_{\lambda\varepsilon}|^{p^*-1-\varepsilon}\varphi\, dx,$$

and,

$$\lim_{t\to 0^+} \frac{\|\nabla(\bar{u}_{\lambda\varepsilon} + t\varphi)\|_p^p - \|\nabla \bar{u}_{\lambda\varepsilon}\|_p^p}{t} = p\int_\Omega |\nabla \bar{u}_{\lambda\varepsilon}|^{p-2}\nabla \bar{u}_{\lambda\varepsilon}\nabla\varphi\, dx,$$

which gives,

$$I_{\lambda\varepsilon}(\bar{u}_{\lambda\varepsilon})\int_\Omega |\bar{u}_{\lambda\varepsilon}|^{p^*-1-\varepsilon}\varphi\, dx \le \int_\Omega |\nabla \bar{u}_{\lambda\varepsilon}|^{p-2}\nabla \bar{u}_{\lambda\varepsilon}\nabla\varphi\, dx - \lambda\int_\Omega |\bar{u}_{\lambda\varepsilon}|^{p-1}\varphi\, dx.$$

If we replace $\varphi$ by $-\varphi$ we obtain the required result.

iv) Follows immediately from (iii) and the definition of $u_{\lambda\varepsilon}$. ∎

PROPOSITION 2.3. *We have*

$$\|\bar{u}_{\lambda\varepsilon}\|_p \le c, \quad \|\bar{u}_{\lambda\varepsilon}\|_{W_0^{1,p}(\Omega)} \le c, \quad \|u_{\lambda\varepsilon}\|_{W_0^{1,p}(\Omega)} \le c,$$

*for some constant c independant of $\varepsilon$ and $\lambda$.*

*Proof.* Since $\int_\Omega |\bar{u}_{\lambda\varepsilon}|^{p^*-\varepsilon}dx = 1$ for every $\lambda$ and $\varepsilon$, one can prove using Hölder inequality, that $\|\bar{u}_{\lambda\varepsilon}\|_p \le c_1$ for some constant $c_1$ independant of $\lambda$ and $\varepsilon$. Let $v$ be any element of $W_0^{1,p}(\Omega)$. Then,

$$I_{\lambda\varepsilon}(\bar{u}_{\lambda\varepsilon}) \le \int_\Omega \left|\nabla\left(\frac{v}{\|v\|_{p^*-\varepsilon}}\right)\right|^p dx \le c_2,$$

where $c_2$ is independant of $\lambda$ and $\varepsilon$. And then, $\int_\Omega |\nabla \bar{u}_{\lambda\varepsilon}|^p dx \le c_2 + \lambda_1 c_1$. The last assertion is a consequence of the definition of $u_{\lambda\varepsilon}$. ∎

## 3. EPI-CONVERGENCE OF THE FUNCTIONALS $I_{\lambda\varepsilon}$

The purpose of this work is to study the behavior of $\bar{u}_{\lambda\varepsilon}$ and $u_{\lambda\varepsilon}$ when $\lambda$ goes to $\lambda_1$ or $\varepsilon$ goes to 0. This will be done, using epi-convergence arguments. For this we begin by recalling this variational convergence

DEFINITION 3.1. ([2, Def. 1.9]) Let $(X,\tau)$ be a metric space, $(F_\varepsilon)_\varepsilon$ and $F_0$ be functionals defined on $X$ with values in $\mathbb{R}\cup\{+\infty\}$. We say that $(F_\varepsilon)_\varepsilon$ epi-converges to $F_0$ in the topology $\tau$ if and only if the two following assertions are satisfied

(EP1) For every $x \in X$, there exist $(x_\varepsilon^0)_\varepsilon \subset X$, such that $x_\varepsilon^0 \to x$ and $\limsup_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon^0) \le F_0(x)$.

(EP2) For every $x \in X$ and every $(x_\varepsilon^0)_\varepsilon \subset X$ such that $x_\varepsilon^0 \to x$, we have $\liminf_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) \ge F_0(x)$.

THEOREM 3.2. ([2, Th. 1.10]) *Assume that*

i) *$F_\varepsilon$ has a minimizer $\bar{x}_\varepsilon$ on $X$,*

ii) *The sequence $(\bar{x}_\varepsilon)_\varepsilon$ is $\tau$-relatively compact,*

iii) *The sequence $(F_\varepsilon)_\varepsilon$ epi-converges to $F_0$, in $\tau$.*

*Then, every limit point $\bar{x}$ of the sequence $(\bar{x}_\varepsilon)_\varepsilon$ is a minimizer of $F_0$ on $X$. And if $(x_{\varepsilon'})_{\varepsilon'}$ converges to $\bar{x}$, we have $\lim_{\varepsilon'\to 0} F_{\varepsilon'}(\bar{x}_{\varepsilon'}) = F_0(\bar{x})$.*

PROPOSITION 3.3. ([2, p. 40]) *Assume that $(F_\varepsilon)_\varepsilon$ epi-converges to $F_0$ in the topology $\tau$ and that $G$ is $\tau$-continuous. Then, $(F_\varepsilon + G)_\varepsilon$ epi-converges to $F + G$ in the topology $\tau$.*

LEMMA 3.4. (Brezis-Lieb) *Let $q > 1$ and assume that $(f_k)_k$ converges to $f$ in the weak topology of $L^q(\Omega)$ and almost everywhere in $\Omega$. Then, we have*

$$\lim_{k\to+\infty} \left( \int_\Omega |f_k|^q\,dx - \int_\Omega |f_k - f|^q\,dx \right) = \int_\Omega |f|^q\,dx.$$

This result has been established in [4] and can be extended in the following way (see [3]).

LEMMA 3.5. *If $(u_\varepsilon)_\varepsilon$ converges to $u$ in the weak topology of $W_0^{1,p}(\Omega)$, we have*

$$\lim_{\varepsilon\to 0} \left( \int_\Omega |u_\varepsilon|^{p^*-\varepsilon}\,dx - \int_\Omega |u_\varepsilon - u|^{p^*-\varepsilon}\,dx \right) = \int_\Omega |u|^{p^*}\,dx.$$

Let us recall also that the best constant $S_N$ for the Sobolev embedding from $W_0^{1,p}(\Omega)$ into $L^{p^*}(\Omega)$ is defined by

$$S_N = \inf \left\{ \int_\Omega |\nabla u|^p\,dx : \quad u \in W_0^{1,p}(\Omega) \quad \text{and} \quad \int_\Omega |u|^{2^*}\,dx = 1 \right\}.$$

$S_N$ is independant of $\Omega$ and is never achieved when $\Omega$ is bounded (see for instance [17]).

PROPOSITION 3.6. *For every nonnegative and fixed $\varepsilon$, $(I_{\lambda\varepsilon})_\lambda$ epi-converges in the strong topology of $W_0^{1,p}(\Omega)$, when $\lambda$ goes to $\lambda_1$, to the functional $I_{\lambda_1\varepsilon}$.*

*Proof.* *Verification of* (EP1): Let $u \in W_0^{1,p}(\Omega)$ and set $u_\lambda^0 = u$, for every $\lambda < \lambda_1$. If $\int_\Omega |u|^{p^*-\varepsilon}\,dx \neq 1$, we have $I_{\lambda\varepsilon}(u_\lambda^0) = +\infty$, which implies

$$\limsup_{\lambda\to\lambda_1} I_{\lambda\varepsilon}(u_\lambda^0) \le I_{\lambda_1\varepsilon}(u).$$

If $\int_\Omega |u|^{p^*-\varepsilon}\,dx = 1$, we have

$$I_{\lambda\varepsilon}(u_\lambda^0) = \int_\Omega |\nabla u|^p\,dx - \lambda\int_\Omega |u|^{p^*}\,dx.$$

Then,

$$\limsup_{\lambda\to\lambda_1} I_{\lambda\varepsilon}(u_\lambda^0) = I_{\lambda_1\varepsilon}(u).$$

*Verification of* (EP2): Let $(u_\lambda)_\lambda$ be any sequence converging to $u$ in the strong topology of $W_0^{1,p}(\Omega)$. If $\int_\Omega |u|^{p^*-\varepsilon}\,dx = 1$, we have

$$\liminf_{\lambda\to\lambda_1} I_{\lambda\varepsilon}(u_\lambda) \ge \int_\Omega |\nabla u|^p\,dx - \lambda_1\int_\Omega |u|^p\,dx = I_{\lambda_1\varepsilon}(u).$$

If $\int_\Omega |u|^{p^*-\varepsilon}\,dx \neq 1$, $u$ can't be the limit in the strong topology of $W_0^{1,p}(\Omega)$ of a sequence $(u_\lambda)_\lambda$ of the unit sphere of $L^{p^*-\varepsilon}(\Omega)$. Hence,

$$\liminf_{\lambda\to\lambda_1} I_{\lambda\varepsilon}(u_\lambda) = +\infty = I_{\lambda_1\varepsilon}(u),$$

which ends the proof. ∎

Let $v_\varepsilon$ be a solution of the following variational problem

$$S_\varepsilon = \inf\left\{\int_\Omega |\nabla v|^p\,dx \;:\; v\in W_0^{1,p}(\Omega) \text{ and } \int_\Omega |v|^{p^*-\varepsilon}\,dx = 1\right\}.$$

LEMMA 3.7. *$(v_\varepsilon)_\varepsilon$ is a minimizing sequence for $S_N$ in the sense that*

$$\lim_{\varepsilon\to 0}\frac{\|\nabla v_\varepsilon\|_p^p}{\|v_\varepsilon\|_{p^*}^p} = S_N.$$

*Proof.* Let $(w_j)_j$ be a minimizing sequence for $S_N$ such that

$$\int_\Omega |w_j|^{p^*}\,dx = 1,$$

for all $j$. One has

$$\int_\Omega |\nabla w_j|^p\,dx \le S_N + o_j;$$

where $\lim_j o_j = 0$. We have

$$S_\varepsilon \leq \int_\Omega |\nabla \left(\frac{w_j}{\|w_j\|_{p^*-\varepsilon}}\right)|^p\,dx \leq \frac{1}{\|w_j\|^p_{p^*-\varepsilon}}(S_N + o_j),$$

and then,

$$\limsup_{\varepsilon\to 0} S_\varepsilon \leq S_N + o_j.$$

On the other hand we have

$$S_N \leq \frac{1}{\|v_\varepsilon\|^p_{p^*}} \int_\Omega |\nabla v_\varepsilon|^p\,dx$$

and

$$1 = \int_\Omega |v_\varepsilon|^{p^*-\varepsilon}\,dx \leq (\int_\Omega |v_\varepsilon|^{p^*}\,dx)^{\frac{p^*-\varepsilon}{p^*}}(meas(\Omega))^{\frac{\varepsilon}{p^*}};$$

which gives

$$S_N \leq (meas(\Omega))^{\frac{\varepsilon}{p^*(p^*-\varepsilon)}} \int_\Omega |\nabla v_\varepsilon|^p\,dx \leq S_N + o_j.$$

And then,

$$S_N = \lim_{\varepsilon\to 0} \int_\Omega |\nabla v_\varepsilon|^p\,dx.$$

∎

COROLLARY 3.8. *$(v_\varepsilon)_\varepsilon$ converges weakly to 0 in $W_0^{1,p}(\Omega)$.*

*Proof.* It follows immediately from [17].

PROPOSITION 3.9. *For every $\lambda \in ]\lambda_*, \lambda_1[$, the sequence $(I_{\lambda\varepsilon})_\varepsilon$ epi-converges in the weak topology of $W_0^{1,p}(\Omega)$ to the functional $I^0_\lambda$ defined on this space by*

$$I^0_\lambda(u) = \begin{cases} \int_\Omega |\nabla u|^p\,dx - \lambda \int_\Omega |u|^p\,dx + S_N(1 - \int_\Omega |u|^p\,dx)^{\frac{p}{p^*}} & \text{if } \int_\Omega |u|^{p^*}\,dx \leq 1 \\ +\infty & \text{otherwise}. \end{cases}$$

*Proof.* Since the mapping $u \to \lambda \int_\Omega |u|^p\,dx$ is continuous on $W_0^{1,p}(\Omega)$ we can assume, using Proposition 3.3, that $\lambda = 0$.

*Verification of* (EP1): Let $u \in W_0^{1,p}(\Omega)$. If $\int_\Omega |u|^{p^*}\,dx > 1$ we have by setting $u^0_\varepsilon = u$ for all $\varepsilon > 0$,

$$\limsup_{\varepsilon\to 0} I_{0\varepsilon}(u^0_\varepsilon) = +\infty = I^0_0(u).$$

Assume now that $\int_\Omega |u|^{p^*}\, dx \leq 1$ and let

$$u^0_\varepsilon = \frac{u - bv_\varepsilon}{\|u - bv_\varepsilon\|_{p^*-\varepsilon}},$$

where $b = (1 - \int_\Omega |u|^{p^*}\, dx)^{\frac{1}{p^*}}$. $(u - bv_\varepsilon)_\varepsilon$ converges weakly in $W^{1,p}_0(\Omega)$ and a.e. on $\Omega$ to $u$. By Lemma 3.4, we have

$$\int_\Omega |u - bv_\varepsilon|^{p^*-\varepsilon}\, dx - b^{p^*-\varepsilon} \int_\Omega |v_\varepsilon|^{p^*-\varepsilon}\, dx = \int_\Omega |u|^{p^*}\, dx + o(\varepsilon),$$

which gives

$$\lim_{\varepsilon\to 0} \int_\Omega |u - bv_\varepsilon|^{p^*-\varepsilon}\, dx = b^{p^*} + \int_\Omega |u|^{p^*}\, dx = 1.$$

The sequence $(|\nabla v_\varepsilon|\, dx)_\varepsilon$ converges weakly to 0 in $L^p(\Omega)$ and a.e. on $\Omega$. By Lemma 3.4, we obtain

$$\lim_{\varepsilon\to 0} \left( \int_\Omega |\nabla u^0_\varepsilon|^p\, dx - \int_\Omega |\nabla (u^0_\varepsilon - u)|^p\, dx \right) = \int_\Omega |\nabla u|^p\, dx,$$

and then,

$$\lim_{\varepsilon\to 0} \int_\Omega |\nabla u^0_\varepsilon|^p\, dx = \int_\Omega |\nabla u|^p\, dx + S_N b^p.$$

*Verification of* (EP2): Let $(u^0_\varepsilon)_\varepsilon$ be a weakly convergent sequence in $W^{1,p}_0(\Omega)$ to some $u$.

If $\int_\Omega |u|^{p^*}\, dx > 1$, we have $\int_\Omega |u^0_\varepsilon|^{p^*-\varepsilon}\, dx \neq 1$ for all $\varepsilon$ near 0 and then

$$\liminf_{\varepsilon\to 0} I^0_\varepsilon(u^0_\varepsilon) = +\infty \geq I^0_0(u).$$

Assume now that $\int_\Omega |u|^{p^*}\, dx \leq 1$ and that $\int_\Omega |u^0_\varepsilon|^{p^*}\, dx = 1$ for at least a subsequence of $(u^0_\varepsilon)_\varepsilon$. By the concentration compactness principle there exists two positive measures $d\mu$ and $d\nu$ such that

(i) $(|\nabla u^0_\varepsilon|^p\, dx)_\varepsilon$ converges weakly in the sense of measures to $d\mu$,

(ii) $(|u^0_\varepsilon|^{p^*}\, dx)_\varepsilon$ converges weakly in the sense of measures to $d\nu$,

(iii) $d\mu \geq |\nabla u|^p dx + \sum_{j\in\Lambda} \mu_j \delta_{x_j}$,

(iv) $d\nu = |u|^{p^*} dx + \sum_{j\in\Lambda} \nu_j \delta_{x_j}$, where $\Lambda$ is at most countable, $x_j \in \bar{\Omega}$, $\mu_j$ and $\nu_j$ are nonnegative constants such that $\mu_j \geq S_N \nu_j^{\frac{p}{p^*}}$ for all $j \in \Lambda$.

If we set $\Lambda_0 = \{j \in \Lambda : x_j \in \Omega\}$ we obtain from (i) and (ii) that

$$\lim_{\varepsilon \to 0} \int_\Omega |\nabla u_\varepsilon^0|^p \, dx = \int_\Omega d\mu \geq \int_\Omega |\nabla u|^p \, dx + \sum_{j \in \Lambda_0} \mu_j,$$

and,

$$1 = \lim_{\varepsilon \to 0} \int_\Omega |u_\varepsilon^0|^{p^*} \, dx = \int_\Omega d\nu = \int_\Omega |u|^{p^*} \, dx + \sum_{j \in \Lambda_0} \nu_j.$$

Consequently,

$$\sum_{j \in \Lambda_0} \nu_j = 1 - \int_\Omega |u|^{p*} \, dx = b^{p^*},$$

and

$$\begin{aligned}\liminf_{\varepsilon \to 0} \int_\Omega |\nabla u_\varepsilon^0|^p \, dx = \int_\Omega d\mu &\geq \int_\Omega |\nabla u|^p \, dx + \sum_{j \in \Lambda_0} \mu_j \\ &\geq \int_\Omega |\nabla u|^p \, dx + S_N \sum_{j \in \Lambda_0} \nu_j^{\frac{p}{p^*}} \\ &\geq \int_\Omega |\nabla u|^p \, dx + S_N \Big( \sum_{j \in \Lambda_0} \nu_j \Big)^{\frac{p}{p^*}} \\ &\geq \int_\Omega |\nabla u|^p \, dx + S_N b^p = I_0^0(u).\end{aligned}$$

∎

PROPOSITION 3.10. *When $\lambda$ goes to $\lambda_1$, the sequence $(I_\lambda^0)_\lambda$ epi-converges to the functional $I_{\lambda_1}^0$, in the strong topology of $W_0^{1,p}(\Omega)$.*

*Remark* 3.11. If $\lambda = \lambda(\varepsilon)$ with $\lim_{\varepsilon \to 0} \lambda(\varepsilon) = \lambda_1$, then the functional $J_\varepsilon = I_{\lambda(\varepsilon)\varepsilon}$ epi-converges to $I_{\lambda_1}^0$ in the weak topology of $W_0^{1,p}(\Omega)$, but not in the strong topology of this space.

Indeed, let $u \in W_0^{1,p}(\Omega)$ such that $\int_\Omega |u|^{p^*} \, dx < 1$. If there exists a sequence $(u_\varepsilon)_\varepsilon$ which converges strongly to $u$ in $W_0^{1,p}(\Omega)$, we have $\int_\Omega |u_\varepsilon|^{p^*} \, dx < 1$ for every $\varepsilon$ near 0. Then

$$\limsup_{\varepsilon \to 0} J_\varepsilon(u_\varepsilon) = +\infty > I_{\lambda_1}^0(u).$$

It follows that (EP1) is not satisfied.

From the above results we can deduce the following commutative diagram

$$\begin{array}{ccc} I_{\lambda\varepsilon} & \stackrel{\varepsilon\to 0}{\rightharpoonup} & I^0_\lambda \\ \downarrow \lambda\to\lambda_1 & & \downarrow \lambda\to\lambda_1 \\ I_{\lambda_1\varepsilon} & \stackrel{\varepsilon\to 0}{\rightharpoonup} & I^0_{\lambda_1} \end{array}$$

where the arrow $\rightharpoonup$(resp.$\to$) means an epi-convergence for the weak (resp. strong) topology of $W_0^{1,p}(\Omega)$.

## 4. Asymptotic behavior of $(\bar{u}_{\lambda\varepsilon})_\lambda$ and $(u_{\lambda\varepsilon})_\lambda$

Let $\varphi_1$ be the first eigenfunction of the problem $(P)$ normalized by the condition $\|\varphi_1\|_\infty = 1$. We may assume that $\varphi_1$ is positive on $\Omega$ and we set $\bar{u}_{\lambda_1\varepsilon} = \varphi_1/\|\varphi_1\|_{p^*-\varepsilon}$, for all nonnegative $\varepsilon$.

Proposition 4.1. *For every nonnegative $\varepsilon$, when $\lambda$ goes to $\lambda_1$, the sequence $(\bar{u}_{\lambda\varepsilon})_\lambda$ converges to $\bar{u}_{\lambda_1\varepsilon}$, in the strong topology of $W_0^{1,p}(\Omega)$.*

*Proof.* The sequence $(\bar{u}_{\lambda\varepsilon})_\lambda$ is bounded in $W_0^{1,p}(\Omega)$; then it contains a subsequence which converges weakly to a minimizer of the functional $I_{\lambda\varepsilon}$, namely $\bar{u}_{\lambda_1\varepsilon}$. We have

$$\int_\Omega |\nabla u_{\lambda\varepsilon}|^p\,dx = \lambda\int_\Omega |\bar{u}_{\lambda\varepsilon}|^p\,dx + I_{\lambda\varepsilon}(\bar{u}_{\lambda\varepsilon}),$$

which converges to

$$\lambda_1\int_\Omega |\bar{u}_{\lambda_1\varepsilon}|^p\,dx + I_{\lambda_1\varepsilon}(\bar{u}_{\lambda_1\varepsilon}) = \lambda_1\int_\omega |\bar{u}_{\lambda_1\varepsilon}|^p\,dx = \int_\Omega |\nabla\bar{u}_{\lambda_1\varepsilon}|^p\,dx,$$

when $\lambda$ goes to $\lambda_1$. On the other hand the p-laplacian being strongly monotone, there exists a constant $c$ such that

$$\int_\Omega (|\nabla\bar{u}_{\lambda\varepsilon}|^{p-2}\nabla\bar{u}_{\lambda\varepsilon} - |\nabla\bar{u}_{\lambda_1\varepsilon}|^{p-2}\nabla\bar{u}_{\lambda_1\varepsilon})(\nabla\bar{u}_{\lambda\varepsilon} - \nabla\bar{u}_{\lambda_1\varepsilon})\,dx \geq c\|\bar{u}_{\lambda\varepsilon} - \bar{u}_{\lambda_1\varepsilon}\|^p_{W_0^{1,p}(\Omega)}.$$

We have

$$\int_\Omega |\nabla\bar{u}_{\lambda\varepsilon}|^{p-2}\nabla\bar{u}_{\lambda\varepsilon}\nabla\bar{u}_{\lambda_1\varepsilon}\,dx = \lambda\int_\Omega |\bar{u}_{\lambda\varepsilon}|^{p-1}\bar{u}_{\lambda_1\varepsilon}\,dx + I_{\lambda\varepsilon}(\bar{u}_{\lambda\varepsilon})\int_\Omega |\bar{u}_{\lambda\varepsilon}|^{p^*-1-\varepsilon}\bar{u}_{\lambda_1\varepsilon}\,dx,$$

which converges to $\lambda_1 \int_\Omega |\bar{u}_{\lambda_1\varepsilon}|^p\,dx$, when $\lambda$ goes to $\lambda_1$; and,

$$\int_\Omega |\nabla \bar{u}_{\lambda_1\varepsilon}|^{p-2}\nabla \bar{u}_{\lambda_1\varepsilon}\nabla \bar{u}_{\lambda\varepsilon}\,dx = \lambda_1 \int_\Omega |\bar{u}_{\lambda_1\varepsilon}|^{p-1}\bar{u}_{\lambda\varepsilon}\,dx,$$

which converges to $\lambda_1 \int_\Omega |\bar{u}_{\lambda_1\varepsilon}|^p\,dx$, when $\lambda$ goes to $\lambda_1$. This results yields $\lim_{\lambda\to\lambda_1} \|\bar{u}_{\lambda\varepsilon} - \bar{u}_{\lambda_1\varepsilon}\|_{W_0^{1,p}(\Omega)} = 0$. ∎

COROLLARY 4.2. *For all nonnegative $\varepsilon$, when $\lambda$ goes to $\lambda_1$, the sequence $(u_{\lambda\varepsilon})_\lambda$ converges to 0, strongly in the topology of $W_0^{1,p}(\Omega)$.*

From the above diagram and Theorem 3.2, we obtain the following results.

THEOREM 4.3. *We have the following commutative diagram*

$$\begin{array}{ccc} \bar{u}_{\lambda\varepsilon} & \overset{\varepsilon\to 0}{\longrightarrow} & \bar{u}_{\lambda 0} \\ \downarrow {\scriptstyle \lambda\to\lambda_1} & & \downarrow {\scriptstyle \lambda\to\lambda_1} \\ \bar{u}_{\lambda\varepsilon} & \overset{\varepsilon\to 0}{\longrightarrow} & \bar{u}_{\lambda_1 0} \end{array}$$

*and then, the following convergences*

$$\begin{array}{ccc} u_{\lambda\varepsilon} & \overset{\varepsilon\to 0}{\longrightarrow} & u_{\lambda 0} \\ \downarrow {\scriptstyle \lambda\to\lambda_1} & & \downarrow {\scriptstyle \lambda\to\lambda_1} \\ 0 & \overset{\varepsilon\to 0}{\longrightarrow} & 0 \end{array}$$

Let us recall the following result (see for instance [8, Prop. 13]).

PROPOSITION 4.4. *Let $u \in W_0^{1,p}(\Omega)$ such that $-div(|\nabla u|^{p-2}\nabla u) = f$, where $f \in L^s(\Omega)$. If $s > N/p$, then $u \in L^\infty(\Omega)$ and we have $\|u\|_\infty \le c\|f\|_s^{\frac{1}{p-1}}$, where $c = c(N, p, meas(\Omega))$.*

PROPOSITION 4.5. *For all nonnegative $\varepsilon$, the sequence $(u_{\lambda\varepsilon})_\lambda$ is bounded in $L^\infty(\Omega)$ and, when $\lambda$ goes to $\lambda_1$, converges uniformly to 0.*

*Proof.* Let $f$, $g$ be the functions defined by

$$f(x) = \lambda x^{p-1} + x^{p^*-1-\varepsilon}, \qquad g(x) = \lambda_1 x^{p-1} + x^{p^*-1-\varepsilon},$$

for every nonnegative $\varepsilon$. Choose $r$ such that $r/p^* > N/p$. If we multiply $(P_{\lambda\varepsilon})$ by $u_{\lambda\varepsilon}^r$ and integrate on $\Omega$, we obtain

$$r\beta^{-p}\int_\Omega |\nabla u_{\lambda\varepsilon}^\beta|^p\,dx = \int_\Omega f(u_{\lambda\varepsilon})u_{\lambda\varepsilon}^r\,dx, \tag{4.1}$$

where $\beta = (r-1+p)/p$. Since

$$\lim_{x \to +\infty} \frac{g(x)}{x^{p^*-1}} \leq 1,$$

there exists a positive constant $\gamma$ such that

$$g(x)x^r \leq 2x^{p^*-1+r} + \gamma, \tag{4.2}$$

for all nonnegative real $x$. From (4.2) we deduce

$$\int_\Omega f(u_{\lambda\varepsilon})u_{\lambda\varepsilon}^r \, dx \leq \int_\Omega g(u_{\lambda\varepsilon})u_{\lambda\varepsilon}^r \, dx \leq 2\int_\Omega u_{\lambda\varepsilon}^{p^*-1+r} \, dx + \gamma meas(\Omega). \tag{4.3}$$

By Hölder inequality we obtain

$$\int_\Omega u_{\lambda\varepsilon}^{p^*-1+r} \, dx \leq (\int_\Omega u_{\lambda\varepsilon}^{\beta p^*} \, dx)^{\frac{p}{p^*}} (\int_\Omega u_{\lambda\varepsilon}^{p^*} \, dx)^{1-\frac{p}{p^*}}. \tag{4.4}$$

And by the Sobolev inequality we have

$$S_N(\int_\Omega u_{\lambda\varepsilon}^{\beta p^*} \, dx)^{\frac{p}{p^*}} \leq \int_\Omega |\nabla u_{\lambda\varepsilon}^\beta|^p \, dx. \tag{4.5}$$

Combining (4.1)-(4.5), we obtain

$$\left(S_N - 2\left(\int_\Omega u_{\lambda\varepsilon}^{p^*} \, dx\right)^{1-\frac{p}{p^*}}\right) \int_\Omega u_{\lambda\varepsilon}^{\beta p^*} \, dx \leq \gamma meas(\Omega).$$

Since $(u_{\lambda\varepsilon})_\lambda$ converges to 0 in $W_0^{1,p}(\Omega)$, we have

$$S_N - 2(\int_\Omega u_{\lambda\varepsilon}^{p^*} \, dx)^{1-\frac{p}{p^*}} > 0,$$

for all $\lambda$ near $\lambda_1$. And, then we conclude that $(u_{\lambda\varepsilon})_\lambda$ is bounded in $L^s(\Omega)$, where $s = \beta p^*$.

Since $s > N/p$, $(u_{\lambda\varepsilon})_\lambda$ is bounded in $L^\infty(\Omega)$ and then, in $C^{1,\alpha}(\Omega)$, for some $\alpha \in ]0,1[$; which gives the required result. ∎

PROPOSITION 4.6. *We obtain the following results*

i) *For every* $x \in \Omega$, $\lim_{\lambda \to \lambda_1} \|u_{\lambda\varepsilon}\|_\infty^{-1} u_{\lambda\varepsilon}(x) = \varphi_1(x)$.

ii) $\lim_{\lambda \to \lambda_1} \int_\Omega |\nabla(\|u_{\lambda\varepsilon}\|^{-1} u_{\lambda\varepsilon} - \varphi_1)|^p \, dx = 0$

*Proof.* Let $v_{\lambda\varepsilon} = \|u_{\lambda\varepsilon}\|_\infty^{-1} u_{\lambda\varepsilon}$, then we have

$$-div(|\nabla v_{\lambda\varepsilon}|^{p-2}\nabla v_{\lambda\varepsilon}) - \lambda v_{\lambda\varepsilon}^{p-1} = \|u_{\lambda\varepsilon}\|_\infty^{p^*-p-\varepsilon} v_{\lambda\epsilon}^{p^*-1-\epsilon} \quad \text{in} \quad \Omega.$$

Since $(v_{\lambda\varepsilon})_\lambda$ is bounded in $L^\infty(\Omega)$, when $\lambda$ goes to $\lambda_1$, it converges uniformly to the solution of the equation of $-div(|\nabla u|^{p-2}\nabla u) = \lambda_1 u^{p-1}$, namely $\varphi_1$; which establishes (i).

For (ii) it suffices to remark that $(v_{\lambda\varepsilon})_\lambda$ converges weakly to $\varphi_1$ in $W_0^{1,p}(\Omega)$ and that $\int_\Omega |\nabla v_{\lambda\varepsilon}|^p\,dx$ converges to $\lambda_1 \int_\Omega \varphi_1^p\,dx = \int_\Omega |\nabla\varphi_1|^p\,dx$. ∎

PROPOSITION 4.7. *Assume that $p = 2$. Then, we have*

i) $\lim_{\lambda\to\lambda_1} (\lambda_1 - \lambda)\|u_{\lambda\varepsilon}\|_\infty^{\varepsilon+2-2^*} = \dfrac{\|\varphi_1\|_{2^*-\varepsilon}^{2^*-\varepsilon}}{\|\varphi_1\|_2^2}.$

ii) *For every $x$ in $\Omega$,*

$$\lim_{\lambda\to\lambda_1} (\lambda_1 - \lambda)(u_{\lambda\varepsilon})^{\varepsilon+2-2^*}(x) = \frac{\|\varphi_1\|_{2^*-\varepsilon}^{2^*-\varepsilon}}{\|\varphi_1\|_2^2}(\varphi_1(x))^{\varepsilon+2-2^*}.$$

iii) $\lim_{\lambda\to\lambda_1} (\lambda_1 - \lambda)^{-\frac{2}{2^*-2-\varepsilon}} \int_\Omega |\nabla u_{\lambda\varepsilon}|^2\,dx = \lambda_1 \left\|\dfrac{\varphi_1}{\|\varphi_1\|_{2^*-\varepsilon}}\right\|_2^4.$

*Proof.* (i) Let $v_{\lambda\varepsilon}$ defined as in the proof of the preceding proposition. We have

$$(\lambda_1 - \lambda)\|u_{\lambda\varepsilon}\|_\infty^{\varepsilon+2-2^*} \int_\Omega v_{\lambda\varepsilon}\varphi_1\,dx = \int_\Omega (v_{\lambda\varepsilon})^{2^*-1-\varepsilon}\varphi_1\,dx.$$

If $\lambda$ goes to $\lambda_1$ we obtain the required result.
(ii) and (iii) follow from (i) and the Proposition 4.6. ∎

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