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Unconditionally Convergent Polynomials in Banach Spaces and Related Properties

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Introduction

Our aim is to introduce a new notion of unconditionallity, in the context of polynomials in Banach spaces, that looks directly to the polynomial topology defined on the involved spaces. This notion allows us to generalise some well-known relations of duality that appear in the linear context.

For E, F Banach spaces and $k \in \mathbb{N}$, we denote by $\mathcal{P}(^kE, F)$ (resp. $\mathcal{P}(^kE)$) the Banach space of homogeneous polynomials of degree k defined on E with values in F (resp.in $\mathbb{R}$ or $\mathbb{C}$); by θ_k we denote the canonical k-degree polynomial $\theta_k : E \to \hat{\otimes}^k_{s,\pi} E$, defined by $\theta_k(x) = x \otimes \overset{k}{\ldots} \otimes x$; and by τ_k we denote the topology in E induced by $\mathcal{P}(^kE)$.

Unconditionally convergent polynomials

We recall that a formal series $\sum_{i=1}^{\infty} x_i$ in a Banach space is a weakly unconditionally Cauchy series (w.u.C) if $\sum_{i=1}^{\infty} |x^*(x_i)| < \infty$, for every $x^* \in E^*$. Equivalently, if for every subseries $\sum_{i=1}^{\infty} x_{k_i}$, the sequence of partial sums $\{\sum_{i=1}^{n} x_{k_i}\}_n$ is a weakly Cauchy sequence in E.

Definition. A series $\sum_{i=1}^{\infty} x_i$ in a Banach space E is said to be τ_k-Cauchy if for each $P \in \mathcal{P}(^kE)$, $\lim_{n,m} |P(\sum_{i=1}^{n} x_i) - P(\sum_{i=1}^{m} x_i)| = 0$ (i.e, if the sequence of partial sums $\{\sum_{i=1}^{n} x_i\}_n$ is a τ_k-Cauchy sequence). A series is said to be τ_k-unconditionally Cauchy (τ_k-u.C) if every of its subseries is τ_k-Cauchy.

LEMMA. *A series $\sum_{i=1}^{\infty} x_i$ is τ_k-unconditionally Cauchy if and only if it is weakly unconditionally Cauchy.*

This lemma justifies the following definition:

DEFINITION. A polynomial $P \in \mathcal{P}({}^kE, F)$ is said to be unconditionally convergent if for every w.u.C series $\sum_{i=1}^{\infty} x_i$ in E, the sequence $\{P(\sum_{i=1}^{n} x_i)\}_n$ converges in norm. The subspace of unconditionally convergent polynomials will be denoted by $\mathcal{P}_{uc}({}^kE, F)$.

THEOREM. (1) *A Banach space E does not contain a copy of c_0 if and only for every $k \in \mathbb{N}$ and for every Banach space F, $\mathcal{P}_{uc}({}^kE, F) = \mathcal{P}({}^kE, F)$.*

(2) *For all Banach spaces E, F and all $k \in \mathbb{N}$, every weakly compact polynomial $P \in \mathcal{P}({}^kE, F)$ is an unconditionally convergent polynomial.*

The "only if" part of (1) is clear, and the "if" part is established using the canonical polynomial θ_k, for each k.

The proof of (2) follows from the following result which, in the linear case, is a consequence of the *Orlicz-Pettis Theorem*:

LEMMA. *A polynomial $P \in \mathcal{P}({}^kE, F)$ is unconditionally convergent if and only if for every w.u.C series $\sum_{i=1}^{\infty} x_i$ in E, the sequence $\{P(\sum_{i=1}^{n} x_i)\}_n$ is weakly convergent in F.*

In order to prove the lemma, we introduce a class of subsets in the dual space $\mathcal{P}({}^kE) \simeq (\hat{\otimes}^k_{s,\pi} E)^*$ that will be usefull also in further results:

DEFINITION. Given a bounded subset A of $(\hat{\otimes}^k_{s,\pi} E)^*$, we say that $A \in \mathcal{V}_k((\hat{\otimes}^k_{s,\pi} E)^*)$ if for every w.u.C series $\sum_{i=1}^{\infty} x_i$ in E,

$$\lim_{n,m} \sup\{|Q(\sum_{i=1}^{n} x_i) - Q(\sum_{i=1}^{m} x_i)|; Q \in A\} = 0.$$

Two important properties of this class of subsets are that it contains every relatively weakly compact set of the dual space $(\hat{\otimes}^k_{s,\pi} E)^*$ and that it is preserved by closed absolutelly convex hulls.

POLYNOMIAL PROPERTY $\mathcal{V}$

We denote by $\mathcal{P}_{wco}(^kE,F)$ the space of weakly compact polynomials. We have said that $\mathcal{P}_{wco}(^kE,F) \subset \mathcal{P}_{uc}(^kE,F)$ always holds, but in general, the inverse inclusion is not true: as an example we can take the polynomial $\theta_k : \ell_k \to \hat{\otimes}^k_{s,\pi}\ell_k$ which is unconditionally convergent for every k, because ℓ_k has no copy of c_0, but it is not weakly compact. The achievement of this non-trivial relation gives rise to the definition of a new property, that generalises a linear one:

DEFINITION. A Banach space E is said to have the *property* k-$\mathcal{V}$ for $k \in \mathbb{N}$, if for every Banach space F, every $P \in \mathcal{P}_{uc}(^kE,F)$ is a weakly compact polynomial.

The next result had been established for $k = 1$ in [3]:

THEOREM. *Given a Banach space E, the following assertions are equivalent:*

(1) *E has the property k-$\mathcal{V}$.*

(2) *Every set $A \in \mathcal{V}_k((\hat{\otimes}^k_{s,\pi}E)^*)$ is a relatively weakly compact set.*

Every Banach space E such that $\hat{\otimes}^k_{s,\pi}E$ is reflexive has the property k-$\mathcal{V}$, and c_0 has it for every $k \in \mathbb{N}$.

There are links between this property and others that have been studied recently, like the property k-$\mathcal{V}^*$ defined in [1]. In particular, there is another definition of unconditionallity and polynomial property $\mathcal{V}$ in [2] that is quite different from ours: they define an unconditionally convergent polynomial P as a polynomial for which $\sum_{i=1}^{\infty} P(x_i)$ is unconditionally convergent, for every w.u.C series $\sum_{i=1}^{\infty} x_i$. It can be proved that this class of polynomials contains every $P \in \mathcal{P}_{uc}(^kE,F)$, and there are examples, like c_0, where the two classes are not the same. So, in general the definition of the k-$\mathcal{V}$ property above is less restrictive than the polynomial property $\mathcal{V}$ in [2].

REFERENCES

[1] BOMBAL, F., On polynomial properties in Banach spaces, *Atti. Sem. Mat. Fis. Univ. Modena,* **XLIV** (1996), 135–146.

[2] GONZÁLEZ, M. AND GUTIÉRREZ, J.M., Unconditionally converging polynomials on Banach spaces, *Math. Proc. Camb. Phil. Soc.,* **117** (1995), 321–331.

[3] PELCZYNSKI, A., On Banach spaces on which every unconditionally operator is weakly compact, *Bull. Acad. Pol. Sci.,* **10** (1962), 641–648.