EXTRACTA MATHEMATICAE Vol. **12**, Núm. 3, 309 – 313 (1997)

R–Schauder Decompositions. Some Applications *

P. Galindo, M. Maestre and P. Rueda

Dpto. de Análisis Matemático, Univ. de Valencia, 46100-*Burjassot, Valencia, Spain*

(Research announcement presented by J.M.F. Castillo)

AMS *Subject Class.* (1991): 46B20 *Received May* 30, 1997

1. Introduction and notation

In recent years several authors have been interested in describing the bidual of some subspaces of $\mathcal{P}(X)$ (the space of continuous polinomials on a Banach space X) as subspaces of $\mathcal{P}(X^{**})$. See, for instance, [2], [5], [6] and [10]. The aim of this work is to extend these results to holomorphic functions. Related to this, Prieto obtains some interesting results in [9]. According to her one has the following situation: $(\mathcal{P}(^mX))_m$ and $(\mathcal{P}_{wu}(^mX)^{**})_m$ are Schauder decompositions of $\mathcal{H}_b(X)$ and $\mathcal{H}_{wu}(X)^{**}$ respectively; hence, topological isomorphisms between $\mathcal{P}(^mX)$ and $\mathcal{P}_{wu}(^mX)^{**}$ for all $n \in \mathbb{N}$ apparently yield to a topological isomorphism between $\mathcal{H}_b(X)$ and $\mathcal{H}_{wu}(X)^{**}$ (Theorem 12 of [9]). However, $\mathcal{H}(\mathbb{C})$ and $\mathcal{H}(\Delta)$, where Δ is the open unit ball of $\mathbb{C}$, have the same Schauder decomposition, $(\mathcal{P}(^m\mathbb{C}))_m$, but they are not topologically isomorphic (see the remark after Corollary 10.6.12 of [7] or Theorem 2.3). This example shows that to obtain a topological isomorphism between Fréchet spaces it is not enough that they have the same Schauder decomposition. In order to clarify this situation we introduce a new class of Schauder decompositions: the R-Schauder decompositions. Some applications to the study of the bidual space of some closed subspaces of $\mathcal{H}_b(U)$ are given in Section 3.

In the sequel we use the notation E^* for the strong dual of an arbitrary Fréchet space E, X for an arbitrary complex Banach space and B, B^* and B^{**} for the open unit ball of X, X^* and X^{**} respectively. For a balanced open subset U of X let $\mathcal{H}_b(U)$ be the space of all holomorphic functions of bounded type on U, that is, the space of all holomorphic functions on U which are bounded on all U-bounded sets. We recall that the U-bounded sets are, in

*The first author was partially supported by DGICYT (Spain) pr. 91-0326; the second one was partially supported by DGICYT (Spain) pr. 91-0326 and pr. 91-0538

the case $U = X$, the bounded subsets of X, whereas, in the case of an arbitrary open set U, they are the bounded subsets of U whose distance to the boundary of U is greater than zero. If A is a U-bounded set, we set $\|f\|_A = \sup_{x\in A} |f(x)|$, for $f \in \mathcal{H}_b(U)$. $\mathcal{H}_b(U)$ will be endowed with the topology τ_b defined by the seminorms $\|.\|_A$. It is well known that $(\mathcal{H}_b(U), \tau_b)$ is a Fréchet space. Let $\mathcal{H}_{wu}(U)$ denote the closed subspace of $\mathcal{H}_b(U)$ of all holomorphic functions on U which are weakly uniformly continuous on all U-bounded sets. If G is a balanced open subset of X^*, $\mathcal{H}_{w^*}(G)$ is the closed subspace of $\mathcal{H}_b(G)$ of all holomorphic functions on G which are *weak**-uniformly continuous on all G-bounded sets. Let $\mathcal{P}(^mX)$ be the space of all continuous m-homogeneous polynomials on X. Let $\mathcal{P}_{wu}(^mX) = \mathcal{P}(^mX) \cap \mathcal{H}_{wu}(X)$ and $\mathcal{P}_{w^*}(^mX^*) = \mathcal{P}(^mX^*) \cap \mathcal{H}_{w^*}(X^*)$.

2. Main results

Definition 2.1. Let E be a Fréchet space and let $(E_n, \|\cdot\|_n)_n$ be a sequence of Banach spaces that is a Schauder decomposition of E. The sequence $(E_n)_n$ is said to be an R-Schauder decomposition of E, $0 < R \leq \infty$, if whenever $x_n \in E_n$, the series $\sum_{n=0}^{\infty} x_n$converges if and only if $\limsup_n \|x_n\|_n^{1/n} \leq \frac{1}{R}$.

Example 2.2. By using Cauchy inequalities we obtain that the family $(\mathcal{P}(^mX), \|\cdot\|_B)_m$ is, at the same time, an ∞-Schauder decomposition of $\mathcal{H}_b(X)$ and an R-Schauder decomposition of $\mathcal{H}_b(RB)$. Moreover, given a bounded balanced open set $U \subset X$, the sequence $(\mathcal{P}(^mX), \|\cdot\|_U)_m$ is a 1-Schauder decomposition of $\mathcal{H}_b(U)$. Analogously for $\mathcal{P}_{wu}(^mX)$, $\mathcal{P}_{w^*}(^mX^*)$ and their corresponding spaces of holomorphic functions.

Every R-Schauder decomposition is $\mathcal{S}$-absolute (Definition 3.7 of [3]). However, the converse is not true. There exist essentially two types of R-Schauder decompositions: the ∞-Schauder decompositions and the 1-Schauder decompositions. Indeed, if $(E_n, \|\cdot\|_n)_n$ is an R-Schauder decomposition of E, $0 < R < \infty$, then $(E_n, R^n\|\cdot\|_n)_n$ is a 1-Schauder decomposition of E. Then a natural question arises: is it possible to establish a topological isomorphism between two Fréchet spaces, one having a 1-Schauder decomposition and the other one an ∞-Schauder decomposition? Or, better: is it possible to find a Banach space X such that $\mathcal{H}_b(X)$ is topologically isomorphic to $\mathcal{H}_b(B)$? The answer to both questions is negative and has been told us by José Bonet in a personal communication, which we gratefully acknowledge, where he pointed out the following power series approach to R-Schauder decompositions: Every Fréchet space E with an R-Schauder decomposition can be identified with the power series space $\lambda^1(A_R; (E_n)_n)$ (where $A_R = \{(r^n)_n : 0 < r < R\}$) defined by $\lambda^1(A_R; (E_n)_n) := \{x = (x_n) \in \prod_n E_n \colon p_r(x) := \sum_{n=0}^{\infty} \|x_n\|_n r^n < \infty, \forall r : 0 < r < R\}$, endowed with the locally convex topology given by the family of seminorms $\{p_r : 0 < r < R\}$. He obtains the following theorem.

THEOREM 2.3. *If E and F are Fréchet spaces having a R-Schauder, $0 < R < \infty$, and an ∞-Schauder decomposition respectively, then there exists no topological isomorphism between E and F.*

Therefore, given a Banach space X, the space $\mathcal{H}_b(X)$ (resp. $\mathcal{H}_{wu}(X)$, $\mathcal{H}_{w^*}(X^*)$) is not topologically isomorphic to $\mathcal{H}_b(B)$ (resp. $\mathcal{H}_{wu}(B), \mathcal{H}_{w^*}(B^*)$).

Our main theorem characterizes when a topological isomorphism occurs between spaces E and F having R-Schauder decompositions of the same type:

THEOREM 2.4. *Let $(E_n, \|.\|_n)_n$ and $(F_n, \|.\|_n)_n$ be R-Schauder decompositions of the Fréchet spaces E and F respectively ($0 < R \leq \infty$). Assume that there exist algebraic isomorphisms $T_m : E_m \longrightarrow F_m$ for all $m \in \mathbb{N}$ so that:*

(i) (Condition I) *In case $0 < R < \infty$, for each $t > 1$ there exist $a_t, b_t > 0$ such that, for every $m \in \mathbb{N}$ and every $x_m \in E_m$,*

$$\|T_m(x_m)\|_m \leq a_t t^m \|x_m\|_m \quad \text{and} \quad \|x_m\|_m \leq b_t t^m \|T_m(x_m)\|_m.$$

(ii) (Condition II) *In case $R = \infty$, there exist $t, t' > 0$ and $a_t, b_{t'} > 0$ such that, for every $m \in \mathbb{N}$ and every $x_m \in E_m$,*

$$\|T_m(x_m)\|_m \leq a_t t^m \|x_m\|_m \text{and} \|x_m\|_m \leq b_{t'} (t')^m \|T_m(x_m)\|_m.$$

Then the map $T : x = \sum_{m=0}^{\infty} x_m \in E \longrightarrow T(x) := \sum_{m=0}^{\infty} T_m(x_m) \in F$ is a topological isomorphism.

Conversely, if there exists a topological isomorphism $T : E \longrightarrow F$ so that $T(E_m) \subset F_m$, for all $m \in \mathbb{N}$, then $T(E_m) = F_m$and $T_m := T|_{E_m}$ are topological isomorphisms satisfying Condition I in case $0 < R < \infty$ and Condition II in case $R = \infty$.

COROLLARY 2.5. *Let $(E_n, \|.\|_n)_n$ and $(F_n, \|.\|_n)_n$ be R-Schauder decompositions of E and F respectively ($0 < R \leq \infty$). If E_n is isometrically isomorphic to F_n for every $n \in \mathbb{N}$, then E and F are topologically isomorphic.*

3. Applications

We now state some applications of these results to the study of biduals of spaces of holomorphic functions.

Corollary 3.1. *Let $G \subset X^*$ be either B^* or X^*. If $\mathcal{P}_{w^*}(^mX^*)$ contains no copy of ℓ^1, for every $m \in \mathbb{N}$, then $\mathcal{H}_{w^*}(G)^{**}$ is topologically isomorphic to $\overline{\mathcal{H}_{w^*}(G)}^{\tau_0}$, the closure of $\mathcal{H}_{w^*}(G)$ in $(\mathcal{H}_b(G), \tau_0)$. In particular, the isomorphism holds whenever X is an Asplund space.*

Corollary 3.2. *Let X be a Banach space such that X^* has the approximation property. Let $G \subset X^*$ be either B^* or X^*. If $\mathcal{P}_{w^*}(^mX^*)$ contains no copy of ℓ^1 for all $m \in \mathbb{N}$, then $\mathcal{H}_{w^*}(G)^{**}$ is topologically isomorphic to $\mathcal{H}_b(G)$. In particular, the isomorphism holds whenever X is an Asplund space such that X^* has the approximation property.*

Corollaries 3.1 and 3.2 have been obtained by Valdivia in [11] for entire functions under the assumption that ℓ^1 is not contained in the space of entire functions. J.C. Diaz pointed out to us that this assumption is equivalent to the non-containement of ℓ^1 in $\mathcal{P}_{w^*}(^mX^*)$ for all $m \in \mathbb{N}$ (Corollary 1.25 of [8]).

Corollary 3.3. *Let $U \subset X$ be either a bounded convex balanced open subset of X or $U = X$. Assume that $\mathcal{P}_{wu}(^mX)$ contains no copy of ℓ^1 for all $m \in \mathbb{N}$ (for example when X^* is an Asplund space). Then*

(a) *$\mathcal{H}_{wu}(U)^{**}$ is topologically isomorphic to $\overline{\mathcal{H}_{w^*}(U^{**})}^{\tau_0}$, where U^{**} is the interior on X^{**} for the norm topology of the closure of U for the weak*-topology on X^{**}. In particular, $\mathcal{H}_{wu}(B)^{**}$ is topologically isomorphic to $\overline{\mathcal{H}_{w^*}(B^{**})}^{\tau_0}$ and $\mathcal{H}_{wu}(X)^{**}$ is topologically isomorphic to $\overline{\mathcal{H}_{w^*}(X^{**})}^{\tau_0}$.*

(b) *Moreover, if X^{**} has the approximation property then $\mathcal{H}_{wu}(U)^{**}$ is topologically isomorphic to $\mathcal{H}_b(U^{**})$.*

Corollary 3.4. *Let X be a Banach space and let $U \subset X$ be either B or X. If for every $m \in \mathbb{N}$ $\mathcal{P}_{wu}(^mX)^{**}$ is isometrically isomorphic to $\mathcal{P}(^mX)$, then $\mathcal{H}_{wu}(U)^{**}$ is isomorphic to $\mathcal{H}_b(U)$.*

Corollary 3.4 and Corollary 3.5 below clarify Theorem 12 of [9].

Let us now consider the map $\widetilde{\delta}_m : z \in X^{**} \longrightarrow \widetilde{\delta}_{m,z} \in \mathcal{P}(^mX)^*$ given by $\widetilde{\delta}_{m,z}(P) = \widetilde{P}(z)$, where $\widetilde{P}$ denotes the Aron-Berner extension [1] of P to X^{**}. González in [5] has defined, extending an earlier definition of Aron and Dineen [2], a Banach space X to be Q-reflexive if the adjoint map $\widetilde{\delta}_m^* : \mathcal{P}(^mX)^{**} \longrightarrow$

$\mathcal{P}(^mX^{**})$ of $\widetilde{\delta}_m$ is bijective and hence, a topological isomorphism for every $m \in \mathbb{N}$. Since $\|\widetilde{\delta}_m^*\| \leq 1$, in order to satisfy the remaining inequalities in the hypothesis of Theorem 2.4 one has to assume that the maps $\widetilde{\delta}_m^*$ have some additional properties, for example to be isometries (in this case let us call X to be isometrically Q-reflexive).

COROLLARY 3.5. *Let X be an isometrically Q-reflexive Banach space and let $U \subset X$ be either B or X. Then the space $\mathcal{H}_b(U)^{**}$ is topologically isomorphic to $\mathcal{H}_b(U^{**})$.*

Compare with Proposition 16 of [2].

THEOREM 3.6. *Let X be a Banach space and let $U \subset X$ be either the open unit ball of X or $U = X$. Then the space $\mathcal{H}_b(U)^{**}$ is topologically isomorphic to $\mathcal{H}_b(U^{**})$ if, and only if, X is Q-reflexive and the sequence $(\widetilde{\delta_m})_m$ satisfies either Condition I if $U \neq X$ or Condition II if $U = X$.*

The proofs of these results are detailed in [4].

REFERENCES

[1] ARON, R., BERNER, P., A Hahn-Banach extension theorem for analytic mappings, *Bulletin de la Société Mathématique de France,* **106** (1978), 3-24.

[2] ARON, R., DINEEN, S., *Q*-reflexive Banach spaces, *Rocky Mountain J. Math.,* to appear.

[3] DINEEN, S., "Complex analysis in locally convex spaces", North-Holland Math. Stud., Vol.57, North-Holland, Amsterdam, 1981.

[4] GALINDO, P., MAESTRE, M., RUEDA, P., Biduality in spaces of holomorphic functions, preprint.

[5] GONZÁLEZ, M., Remarks on *Q*-reflexive Banach spaces, *Proc. Roy. Irish Acad.,* **96** (1996), 195-201.

[6] JARAMILLO, J., PRIETO, A., ZALDUENDO, I., The bidual of the space of polynomials on a Banach space, *Math. Proc. of the Cambridge Phil. Soc.,* to appear.

[7] JARCHOW, H., "Locally convex spaces", B. G. Teubner Stuttgart, 1981.

[8] MIÑARRO, M.A., "Descomposiciones de espacios de Fréchet. Aplicación al producto tensorial proyectivo", Tesis Doctoral, Universidad de Sevilla, Sevilla, Spain, 1991.

[9] PRIETO, A., The bidual of spaces of holomorphic functions in infinitely many variables, *Proc. R. Ir. Acad.,* **92A** (1) (1992), 1-8.

[10] VALDIVIA, M., Banach spaces of polynomials without copies of l^1, *Proc. Amer. Math. Soc.,* **123** (10) (1995), 3143-3150.

[11] VALDIVIA, M., Fréchet spaces of holomorphic functions without copies of l^1, *Math. Nachr.,* **181** (1996), 277-287.