EXTRACTA MATHEMATICAE Vol. **13**, Núm. 1, 1 – 19 (1998)

Similarity and Rescaling

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(Survey paper presented by J. Llibre)

AMS *Subject Class.* (1991): 34Axx, 35Axx *Received September* 15, 1997

1. Introduction

Similarity is one of the most fundamental concepts, both in physics and mathematics. We meet it, formally, for the first time when, as a school boy (or schoolgirl) we are exposed to the theory of similar triangles. But, informally, children can pretty well say if the toys, cars and trains with which they play are really similar!

This first aspect, geometrical similitude, is the best known and the best understood. Another one, more abstract, deals with the physical similitude. Since all systems must obey the same physical laws, in addition to the geometrical scaling factors, relations between different physical quantities must be fulfilled in order to make two systems really similar. An amusing example is given by Swift's Gulliver travels. For example, on the earth, his giants, say 10 times bigger that Gulliver (a human being), would collapse under their own weight. Being made of the same flesh and bones as you and me, their muscles will be characterised by the same coefficient (expressed in Newton/m^2). The section of their muscles varying as the square of the dimension and their weight as the cube, they won't be able to survive on earth. Of course, everything will be allright if we put them on a planet, where the gravity is $g/10$. Then, the natural frequency of the motion of their leg (number of steps in unit of time), will be ten times smaller. We recognize how central will be these ideas in the theory of modelling. Such reduced models play a central role in shipbuilding, aeronautical engineering, oceanography, etc... In engineering, quite often, many different phenomena, belonging to different branches of science take place simultaneously and conflicts are possible. Moreover, changing gravity on earth is impossible (we should go to a satellite).

But other aspects of similarity can be found in the logic of a machine or in an algorithm. A good example is provided by the scheme of a massively parallel computer known as the hypercube (Figure 1). Each horizontal line represents a processor. A vertical bar represents a connection between these processors. In order to build a computer with twice as many processors (8 for example), we take two 4 processors computers and connect the processors bearing the same number. Such a structure will be called self–similar (i.e., similar to itself). It has nice properties of optimisation. In the hypercube, the number of connections go as $(1/2)N\log_2 N$, while the maximum number of transfers to bring information from any processor to another one is $\log_2 N$.

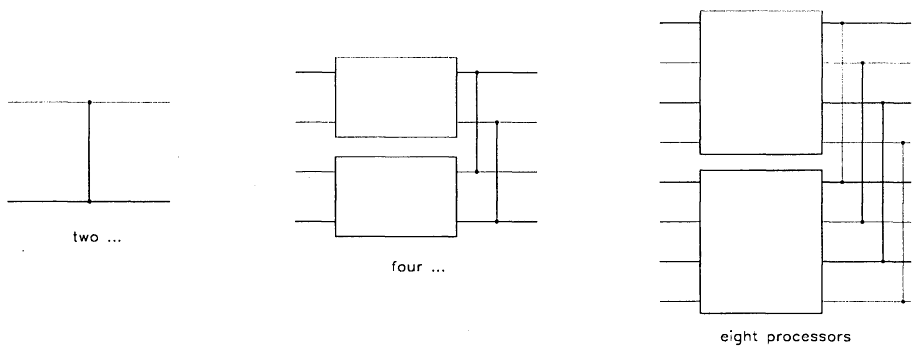

Figure 1

Under its geometrical and logical aspects, similarity and self–similarity appear as rather regular, easy to distinguish patterns. Nature has more fantasy and in some cases it likes to add some randomness. Self–similarity, in that case, is more difficult to distinguish but is still there. Mandelbrot has widely written on this topic and from the galactic structure to the one found in the Brownian motion (via the coast of Brittany) has shown many beautiful pictures. Before we go to more mathematical details a brief historical approach is also helpful.

Besides geometrical similarity, the first one to recognize a coherent structure in a physical phenomena was Fourier with his study of the heat propagation. Then, the idea of physical similarity between different experiments and the possibility of comparing their results after the introduction of properly chosen dimensionless quantities is recognized mostly by the fluid dynamists of the late 19 - beginning 20 century. From these works, it emerges the concept of Dimensional Analysis [1] with the *Pi* theorem of Vaschy-Buckingham. At the same time, reduced models are used in engineering.

On the other hand, Lie [2] in an effort to solve ordinary differential and partial differential equations introduces transformations, in which similarity transformations are specially simple. But the success of this generalization is very limited. On the other hand, dimensional analysis is badly presented with the accent put on the notion of "physical dimension" and not on the invariance properties of the modelling equations, which, precisely, define the dimensions.

The Soviet School with Sedov [3], Zeldovich [4] and Barenblatt [5] extends considerably the mathematical analysis by similarity transformation of different problems while Ovsjannikov [6], Eisenhart [2], Bluman and Cole [7] pick up and push further the mathematical theory of group transformation.

Later on, Wilson introduces the concept of renormalization in field theory which is quickly used by condensed matter physicists to study critical phenomena. An introduction of the renormalization group in physics is given in [8] while a more mathematically oriented treatment is given by Feigenbaum [9, 10]. As mentioned previously in the meantime, Mandelbrot and co–workers develop the concept of fractals on a very simple basis and, at least at the beginning, as a tool describing nature specially in the so called "soft science" as geology, botany, geography. The logical aspect of similarity (and the self–similar character of optimal devices) is the work of people in information theory, connecting networks and computer structures. A good introduction on this topic will be found in [11].

The purpose of this long introduction was to show that similarity is a very fruitful concept, with many aspects and many possible approaches. In the rest of the paper, we will concentrate on what can be considered as the core of the problem namely the invariant properties under similarity transformations of the model's equations. Rescaling, which is mentionned in the tittle is a simple tool (but very important to understand and appreciate the nature of self–similarity, embedding it in a more general concept). It will be introduced in due time. The paper is organized as follows:

2. The Similarity Transformation
3. Ordinary Differential Equations Invariant under Similarity Transformations
4. Partial Differential Equations Invariant under Similarity Transformation
5. Rescaling and the Asymptotic Nature of the Self–similar Solutions
6. Conclusions

2. The Similarity Transformation

Let us show, on an example, what is the similarity transformation and, first, how it is used to find out the most reduced form of the modelling equations and what are the essential parameters. This is important in the theory of reduced models and in computer calculations. We consider a prey–predator problem governed by the system of two equations

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= x(a_1 + b_{11}x + b_{12}y), \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= y(a_2 + b_{21}x + b_{22}y). \end{aligned} \tag{1}$$

Usually x designates the prey with $a_1 > 0$, b_{11} and b_{12} negatives and y designates the predator with a_2 and b_{22} negatives and b_{21} positive. We introduce the similarity transformations both on the variables and the parameters as follows

$$\begin{aligned} t = a^{\alpha}\bar{t}, \quad & x = a^{\beta}\bar{x}, \quad y = a^{\gamma}\bar{y}, \\ a_1 = a^{\lambda_1}\bar{a}_1, \quad b_{11} &= a^{\mu_{11}}\bar{b}_{11}, \quad b_{12} = a^{\mu_{12}}\bar{b}_{12}, \\ a_2 = a^{\lambda_2}\bar{a}_2, \quad b_{21} &= a^{\mu_{21}}\bar{b}_{21}, \quad b_{22} = a^{\mu_{22}}b_{22}. \end{aligned} \tag{2}$$

The invariance of (1) under (2) implies

$$\begin{aligned} \beta - \alpha &= \beta + \lambda_1 = 2\beta + \mu_{11} = \beta + \gamma + \mu_{12}, \\ \gamma - \alpha &= \gamma + \lambda_2 = \gamma + \beta + \mu_{21} = 2\gamma + \mu_{22}. \end{aligned} \tag{3}$$

From (3) we deduce

$$\alpha + \lambda_1 = \alpha + \lambda_2 = 0, \tag{4}$$

$$\beta = \lambda_1 - \mu_{11} = \lambda_2 - \mu_{21}, \tag{5}$$

$$\gamma = \lambda_2 - \mu_{22} = \lambda_1 - \mu_{12}. \tag{6}$$

From (5) we deduce the dimensionless $X = b_{11}x/a_1$ and the existence of the essential parameter $R_2 = a_1 b_{21}/a_2 b_{11}$. From (6) we deduce the dimensionless $Y = b_{22}y/a_2$ and the essential parameter $R_1 = a_2 b_{12}/a_1 b_{22}$. Equation (1) can now be written

$$\begin{aligned} \frac{1}{a_1}\frac{\mathrm{d}X}{\mathrm{d}t} &= X(1 + X + R_1 Y), \\ \frac{1}{a_2}\frac{\mathrm{d}Y}{\mathrm{d}t} &= Y(1 + R_2 X + Y). \end{aligned} \tag{7}$$

Finally (4) points out a third essential parameter a_1/a_2 (in addition to R_1 and R_2). Consequently the parameter's space has been reduced from **6** to **3**. Two problems with the same R_1, R_2 and a_1/a_2 are strictly similar. The determination of these essential parameters is always the first step to be performed when we meet a new problem.

3. Ordinary Differential Equations Invariant under Similarity Transformations

To avoid writing lengthy equations we will consider only first, second and third order equations. The generalization is usually straightforward. The dependent variable will be denoted by x and the independent by t (since quite often in Physics this variable is the time). The most general first, second and third order ODE can be written as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,\, t), \quad \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = f\left(\frac{\mathrm{d}x}{\mathrm{d}t},\, x,\, t\right), \quad \frac{\mathrm{d}^3x}{\mathrm{d}t^3} = f\left(\frac{\mathrm{d}^2x}{\mathrm{d}t^2},\, \frac{\mathrm{d}x}{\mathrm{d}t},\, x,\, t\right), \tag{8}$$

respectively. In the preceding section we were trying to compare the solutions of two models M and $\bar{M}$ with different parameters and, consequently, we have considered all the possible modifications both in the variables (independent and dependent) and in the parameters. Now we want to solve a fixed system and consequently the parameters do not change any more, and we just consider the two transformations

$$t = a^{\alpha}\bar{t}, \quad x = a^{\beta}\bar{x}. \tag{9}$$

It is easily demonstrated that a necessary and sufficient condition for (8) to be invariant under (9) is that the three equations can be written as

$$\begin{gathered}\frac{\mathrm{d}x}{\mathrm{d}t} = t^{k-1}F\left(\frac{x}{t^k}\right), \quad \frac{\mathrm{d}^2x}{\mathrm{d}t^2} = t^{k-2}F\left(\frac{x}{t^k},\, \frac{1}{t^{k-1}}\frac{\mathrm{d}x}{\mathrm{d}t}\right), \\ \frac{\mathrm{d}^3x}{\mathrm{d}t^3} = t^{k-3}F\left(\frac{x}{t^k},\, \frac{1}{t^{k-1}}\frac{\mathrm{d}x}{\mathrm{d}t},\, \frac{1}{t^{k-2}}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right).\end{gathered} \tag{10}$$

In (10) $k = \beta/\alpha$, which is usually given by the equation. What are the consequences of the forms (10) taken by the equations? The first one is the existence of a so-called self-similar solution given by $x = \xi t^k$ where ξ is an invariant of the transformation (a constant). Indeed introducing $x = \xi t^k$ in

(10) we see that time cancels, obtaining the three algebraic equations

$$\begin{gathered} k\xi = F(\xi), \quad k(k-1)\xi = F(\xi,\, k\xi), \\ k(k-1)(k-2)\xi = F(\xi,\, k\xi,\, k(k-1)\xi). \end{gathered} \tag{11}$$

We note that the solution $x = \xi t^k$ is a very special one, with no degree of freedom. An extra degree of freedom is obtained if, in addition to the invariance under self–similar transformations as given by (9) the equations are invariant under time translation with a solution $x = \xi(t - t_0)^k$ which, nevertheless, remains a very special one. Happily, two properties are going to underline the usefulness of this self–similar solution (SSS).

First we use the invariance property to reduce the order of these equations. Second, we will see that the SSS is sometimes the asymptotic solution (or a singular solution). The first property is demonstrated by the introduction of the new variables

$$\xi = \frac{x}{t^k}, \quad \eta = \frac{1}{t^{k-1}}\frac{\mathrm{d}x}{\mathrm{d}t} \quad \text{and} \quad \mu = \frac{1}{t^{k-2}}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}. \tag{12}$$

From $x = \xi t^k$ we deduce by derivation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = kt^{k-1}\xi + t^k\frac{\mathrm{d}\xi}{\mathrm{d}t}. \tag{13}$$

Using the first of (10) and introducing a new time $\mathrm{d}\theta = \mathrm{d}t/t$, i.e., $\theta = \log{(t/t_0)}$, we obtain

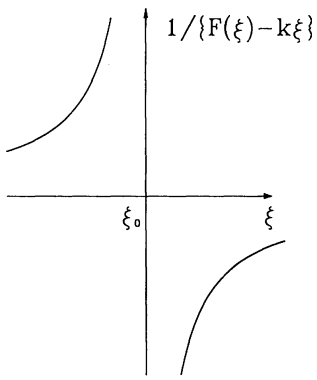

Figure 2

$$\frac{\mathrm{d}\xi}{\mathrm{d}\theta} = F(\xi) - k\xi,$$

and the first order ODE invariant under self–similarity is reduced to a quadrature. Note that if $F(\xi) = k\xi$ has a solution ξ_0 we have $\theta \to \pm\infty$ while $\xi \to \xi_0$ indicating that $\xi = \xi_0$ is either a repulsive or attractive barrier on the ξ axis. Fig. 2 shows the case of an attractive ξ_0. Remember now that ξ_0 is the solution of $F(\xi) = k\xi$, i.e., precisely the SSS.

For the second order ODE the reduction proceeds in a similar way. We compute $\mathrm{d}\xi/\mathrm{d}t$ and $\mathrm{d}\eta/\mathrm{d}t$. Then, using the second of (10) and introducing the same logarithmic compression of time with

$$\theta = \log\left(\frac{t}{t_0}\right), \tag{14}$$

we can write

$$\begin{aligned} \frac{\mathrm{d}\xi}{\mathrm{d}\theta} &= \eta - k\xi, \\ \frac{\mathrm{d}\eta}{\mathrm{d}\theta} &= F(\xi,\, \eta) - (k-1)\eta. \end{aligned} \tag{15}$$

System (15) is quite useful. First of all dividing $\mathrm{d}\eta/\mathrm{d}\theta$ by $\mathrm{d}\xi/\mathrm{d}\theta$ we eliminate θ and reduce by one the order of the equation. Moreover we can consider (15) as giving the time evolution of a two-dimensional dynamical system. The equilibrium point ($\xi = \xi_0$, $\eta = \eta_0$) is given by $\eta_0 = k\xi_0$ and ξ_0 by the equation

$$F(\xi_0,\, k\xi_0) = k(k-1)\xi_0, \tag{16}$$

but (16) is just the second of (11), which defines the SSS of the SODE (i.e., Second ODE). In the same way for a TODE (i.e., Third ODE) we obtain the 3D dynamical system

$$\begin{aligned} \frac{\mathrm{d}\xi}{\mathrm{d}\theta} &= \eta - k\xi, \\ \frac{\mathrm{d}\eta}{\mathrm{d}\theta} &= \mu - (k-1)\eta, \\ \frac{\mathrm{d}\mu}{\mathrm{d}\theta} &= F(\xi,\, \eta,\, \mu) - (k-2)\mu. \end{aligned} \tag{17}$$

Again a reduction of the order of the system can be obtained by dividing $\mathrm{d}\eta/\mathrm{d}\theta$ and $\mathrm{d}\mu/\mathrm{d}\theta$ by $\mathrm{d}\xi/\mathrm{d}\theta$, eliminating consequently the time θ. Also the equilibrium point of this system is given by $\xi = \xi_0$, $\eta = k\xi_0$ and $\mu = k(k-1)\xi_0$, where ξ_0 is given by

$$F(\xi_0,\, k\xi_0,\, k(k-1)\xi_0) = k(k-1)(k-2)\xi_0, \tag{18}$$

in agreement with the third of (11) defining the SSS of the TODE.

This property of the SSS to be the equilibrium point of a dynamical system will be crucial in the study of the properties of the SSS in the section dealing with rescaling. As a first step in this direction let us consider the ODE

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + K\frac{x^{2p+1}}{t^m} = 0, \quad m, p \in \mathbb{R}^+, \quad K > 0, \tag{19}$$

under the transformation $t = a^{\alpha}\bar{t}$, $x = a^{\beta}\bar{x}$, the equation is invariant if $k \equiv \beta/\alpha = (m-2)/2p$, indicating an SSS in $At^{(m-2)/2p}$ with

$$KA^{2p} = \left(1 - \frac{m-2}{2p}\right)\frac{m-2}{2p}. \tag{20}$$

In order to have always A real we will assume $2p + 2 > m > 2$.

To show the asymptotic nature of the SSS let us suppose that $\xi = x/t^k$ is a rescaled coordinate while θ (with $\mathrm{d}\theta = \mathrm{d}t/t$) is the new time. We must call $\mathrm{d}\xi/\mathrm{d}\theta$ the new velocity ω which according to the first equation (15) is given by $\omega = \eta - k\xi$. Introducing in the second of (15) we obtain

$$\frac{\mathrm{d}\omega}{\mathrm{d}\theta} = F(\xi,\, \omega + k\xi) - (2k-1)\omega - k(k-1)\xi. \tag{21}$$

Now we interpret (21) as describing the motion in the new space–time ξ, θ with 3 forces:

1) $F(\xi,\, \omega + k\xi)$ a rescaled physical force.
2) A transformation force deriving from the potential $\phi = k(k-1)\xi^2/2$.
3) A friction $-(2k-1)\omega$ which has the usual sign if $k > 1/2$ (i.e., it dissipates energy and drives the motion to a rest).

Then from (19) and (10) we see that $F(\xi,\, \omega + k\xi) = -Kx^{2p+1}t^{2-k-m}$, which with the definitions of k and ξ it becomes simply $-K\xi^{2p+1}$, i.e., a time independent force deriving from the potential $K\xi^{2p+2}/(2p+2)$. Now we suppose that $1 > k > 1/2$, implying

$$2p + 2 > m > p + 2 \tag{22}$$

and in the new space–time the motion takes place in a total potential Φ_T resulting of

→ a confining potential $K\xi^{2p+2}/(2p+2)$,

→ a repulsive potential in ξ^2.

For small ξ the repulsive potential is dominant while for large ξ it is the rescaled physical potential with an overall potential as shown on Fig. 3.

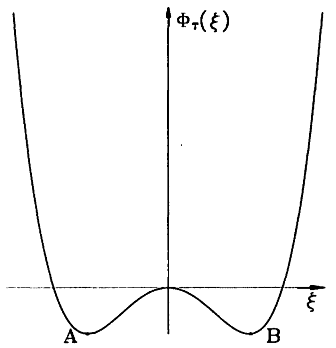

Figure 3

Now our knowledge of the role of the friction allows us to state that for large time the particle will be at rest at one of the minimum points A or B where the two forces cancel, i.e., the SSS. Consequently, in some problems, the evolution comes to rest after a complicated motion in the rescaled space–time. We will find the same property in the evolution of systems described by partial differential equations.

The last section of this paper on rescaling will be a systematic study of these new spaces where more general transformations are considered.

To end this rather long section on ODE we briefly mention further simplifications when, in addition to invariance under self–similarity, the system is also invariant under time translation. Again we consider only SODE and TODE. Their most general forms are for the SODE

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x^{2q+1}f(\xi) = 0 \tag{23}$$

and for the TODE

$$\frac{\mathrm{d}^3x}{\mathrm{d}t^3} + x^{3q+1}f(\xi,\eta) = 0, \tag{24}$$

where ξ and η are now

$$\xi = \frac{1}{x^{q+1}}\frac{\mathrm{d}x}{\mathrm{d}t}, \quad \eta = \frac{1}{x^{2q+1}}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}. \tag{25}$$

The self–similar solution is $x = A(t-t_0)^{-1/q}$, ξ and η being the invariants of the transformation. The SODE (23) having two symmetries can be solved. It exhibits the first integral of motion obtained from

$$\frac{\mathrm{d}x}{x} + \frac{\xi\mathrm{d}\xi}{(q+1)\xi^2 + f(\xi)} = 0. \tag{26}$$

To obtain the SSS we introduce $x = A(t-t_0)^{-1/q}$ in (23) and (25). It gives $\xi_0 = -1/(qA^q)$ and $A(1+q)/q^2 + A^{2q+1}f(\xi_0) = 0$ which can be written

$$(q+1)\xi_0^2 + f(\xi_0) = 0. \tag{27}$$

Now (26) indicates that the motion of the point ξ (on the ξ axis) converges or diverges from the zero of the denominator of the term $\xi \mathrm{d}\xi/[(q+1)\xi^2 + f(\xi)]$. But we have just shown that this zero corresponds to the SSS. Moreover if the integration in (26) can be made, we end up with a relation giving ξ as a function of x, i.e., $\mathrm{d}x/\mathrm{d}t$ as a function of x. A last integration precises the relation between t and x.

The case of TODE is slightly more complex. The two symmetries are not enough to obtain an integral of motion. But they allow the obtention of a first order differential equation, which writes

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \frac{f(\xi,\,\eta) + (2q+1)\xi\eta}{(q+1)\xi^2 - \eta}. \tag{28}$$

Note that the SSS obtained by plugging $x = A(t-t_0)^{-1/q}$ in (24) is given by the solution of

$$(q+1)(2q+1)\xi_0^3 + f(\xi_0,\,(q+1)\xi_0^2) = 0. \tag{29}$$

We immediately recognize that the cancellation of both the numerator and the denominator of (28) gives the SSS. Introducing a time θ with

$$\begin{aligned} \frac{\mathrm{d}\xi}{\mathrm{d}\theta} &= (q+1)\xi^2 - \eta, \\ \frac{\mathrm{d}\eta}{\mathrm{d}\theta} &= f(\xi,\,\eta) + (2q+1)\xi\eta, \end{aligned} \tag{30}$$

we can identify the SSS with the equilibrium point of the 2D dynamical system (30) and the stability properties of this point, i.e., the behavior of the system (30) linearised around the point ($\xi = \xi_0$, $\eta = (q+1)\xi_0^2$) will give information about the attractive or repulsive character of the SSS (at least in its neighbourhood).

4. Partial Differential Equations Invariant under Similarity Transformations

Again we introduce an example, namely the equation describing the nonlinear heat diffusion

$$\frac{\partial \Psi}{\partial t} = \kappa \frac{\partial}{\partial x}\left(\Psi^s \frac{\partial \Psi}{\partial x}\right), \tag{31}$$

where s is a real positive number. We introduce the similarity transformations

$$t = a^{\alpha}\bar{t}, \quad x = a^{\beta}\bar{x}, \quad \Psi = a^{\gamma}\bar{\Psi}. \tag{32}$$

Imposing the invariance of equation (31) under this transformation we see that α and β can be arbitrary chosen with γ given by

$$\gamma = \frac{2\beta - \alpha}{s}. \tag{33}$$

Let us introduce as new independent and dependent variables the invariants of the transformations, i.e.,

$$\xi = \left(\frac{T}{t}\right)^{\beta/\alpha} x, \quad \phi = \left(\frac{T}{t}\right)^{\gamma/\alpha} \Psi. \tag{34}$$

In (34) we have introduced an explicit time T which will be taken as the initial time. Let us now rewrite (31) with the following notations

$$\lambda = \frac{\beta}{\alpha}, \quad \frac{\gamma}{\alpha} = \frac{2\lambda - 1}{s}.$$

We obtain

$$\kappa \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\phi^s \frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right) = \frac{1}{T}\cdot\left[\frac{2\lambda - 1}{s}\phi - \lambda\xi\frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right]. \tag{35}$$

We have decreased by one the number of independent variables (and in that case obtained an ordinary differential equation). Moreover λ is at our disposal and we can ask: what is the price we have to pay?

As in the case of the previous section, we have no choice for the initial conditions. At time $t = T$, $\Psi = \phi$ and $x = \xi$, (35) is just the differential equation giving those initial conditions leading to a self–similar solution. Two questions must be raised. Are these initial conditions physical? What happens for other initial conditions?

To answer the first question, we must precise the type of initial condition we are interested in. For example we consider a Ψ with compact support and a finite amount of initial heat. We can write

$$\begin{aligned} \int \Psi(x, t = T)\mathrm{d}x = \int \phi(\xi)\mathrm{d}\xi = \int \Psi(x, t)\mathrm{d}x \\ = \left(\frac{t}{T}\right)^{\frac{\lambda(s+2)-1}{s}} \int \phi(\xi)\mathrm{d}\xi. \end{aligned} \tag{36}$$

(36) expresses simply that the heat is conserved and uses the form of the self–similar solution to express this conservation at $t = T$ and at any time t,

this imposes $\lambda = 1/(s+2)$. Moreover, this choice turns the second member of (35) into total derivative and leads to the result

$$\phi(\xi) = \left(K - \frac{s}{2(s+2)\kappa T}\xi^2\right)^{1/s}, \tag{37}$$

K being given by integration of $\int \mathrm{d}\phi(\xi)\mathrm{d}\xi$ equal to the initial total amount of heat. If we take such precised initial conditions, the subsequent evolution will be entirely taken care of by the double rescaling both in x and Ψ. Incidently, for the linear case ($s = 0$) we find $\lambda = 1/2$ and $\gamma/\alpha = -1/2$. Let us rewrite (37) as

$$\phi(\xi) = K^{1/s}\left(1 - \frac{s\xi^2}{2(s+2)\kappa T K}\right)^{1/s}. \tag{38}$$

In the limit $s \to 0$ and from its definition, K must go to one and $\phi(\xi)$ becomes

$$\phi(\xi) = \exp\left(-\frac{\xi^2}{4\kappa T}\right) = \exp\left(-\frac{x^2}{4\kappa t}\right), \tag{39}$$

which is the solution given by Fourier who use arguments very similar to those given above, noticing that $x/\sqrt{t}$ leads to a reduction to one variable.

What happens for initial conditions not given by (38)? We must, at this point, enlarge the frame of self–similarity and introduce the concept of rescaling.

5. Rescaling

In fact the concept of self–similarity and the SSS obtained, uses already the concept of rescaling. This is evident on the nonlinear diffusion equation (31). For example for the linear $s = 0$ case, dividing x by $\sqrt{t}$ and multiplying Ψ by $\sqrt{t}$ allows a simplification of the problem and more precisely a formal elimination of the time t. But we were working in a mathematical philosophy. Was it possible to absorb the time in a similarity transformation? What happens for initial conditions different from those forced upon by the transformation? A hint to the solution of these questions was given when we have interpreted equation (21) as describing the motion of a particle in a new space–time and a new phase space. Consequently here we consider rescaling from a physical point of view as a tool to describe a phenomena in a new space–time with appearance of new forces and new phenomena. Then we will select the scales left at our disposal to simplify, if possible, the problems, at least in an asymptotic limit. We forecast two results:

(1) The cases where our rescaling will coincide with the rescaling given by the similarity transformations will certainly describe specially interesting situations.

(2) Systems which do not possess the self–similarity property but which may acquire it asymptotically, if we can prove that in the limit the blocking term goes to zero.

We see, consequently, that rescaling is, a priori, a concept completely different from self–similarity but that the technical problems (especially computational) will sometimes be very analogous if not identical. On the other hand we will see that our scaling factor will be sometimes very special.

5.1. RESCALING OF THE NEWTON LAW. We consider a one dimensional motion $x(t)$ with an acceleration Γ to which we add a friction and, consequently, the Newton law writes

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \beta \frac{\mathrm{d}x}{\mathrm{d}t} = \Gamma(x,\, t). \tag{40}$$

We rescale both space and time with two scales $C(t)$ and $A(t)$ with

$$x = \xi C(t), \quad \mathrm{d}t = A(t)^2 \mathrm{d}\theta. \tag{41}$$

ξ and θ are the new space and time, $C(t)$ is always positive and both $C(t)$ and $A(t)$ never vanish. We must define the new velocity ω as

$$\omega = \frac{\mathrm{d}\xi}{\mathrm{d}\theta}, \tag{42}$$

which taking (41) into account gives the following relation between the old ($v = \mathrm{d}x/\mathrm{d}t$) and the new velocities (omitting from here on the explicit time dependence)

$$v = \omega \frac{C}{A^2} + \xi \frac{\mathrm{d}C}{\mathrm{d}t}. \tag{43}$$

Equations (41) and (43) give the relation between the two elements of the phase space

$$\mathrm{d}x\mathrm{d}v = \begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \omega} \\ \dfrac{\partial v}{\partial \xi} & \dfrac{\partial v}{\partial \omega} \end{vmatrix} \mathrm{d}\xi \mathrm{d}\omega = \begin{vmatrix} C & 0 \\ \dfrac{\mathrm{d}C}{\mathrm{d}t} & \dfrac{C}{A^2} \end{vmatrix} \mathrm{d}\xi \mathrm{d}\omega = \frac{C^2}{A^2} \mathrm{d}\xi\, \mathrm{d}\omega. \tag{44}$$

Equation (44) indicates that $C = A$ provides conservation of the phase space volume element and consequently we can guess that this choice will play a crucial role in problems like the solution of Vlasov equation.

Now we consider the "new equation (40)". A little algebra gives

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}\theta^2} + \left[\beta A^2 + 2\frac{A}{C}\left(A\frac{\mathrm{d}C}{\mathrm{d}t} - C\frac{\mathrm{d}A}{\mathrm{d}t}\right)\right]\frac{\mathrm{d}\xi}{\mathrm{d}\theta} + \frac{A^4}{C}\left(\frac{\mathrm{d}^2C}{\mathrm{d}t^2} + \beta\frac{\mathrm{d}C}{\mathrm{d}t}\right)\xi = \frac{A^4}{C}\Gamma. \tag{45}$$

We have generalized what we found precedently. The forces are

i) A rescaled "Physical force" $(A^4/C)\Gamma$.

ii) A force corresponding to a linear oscillator (see [12] or [13])

$$\frac{A^4}{C}\left(\frac{\mathrm{d}^2C}{\mathrm{d}t^2} + \beta\frac{\mathrm{d}C}{\mathrm{d}t}\right)\xi.$$

iii) A friction which is added to the original one.

We rediscover on (45) a result already suggested by (44), i.e., if $A = C$ then the additional friction term cancels and, correlatively, we have conservation of the phase space volume element.

The choice of C and A depends now of the problem. For example if $\beta \neq 0$, A and C can be selected in order to cancel the total friction in the new space–time with the possibility of obtaining a constant of motion [14]. In [15] the case of the nonlinear time dependent harmonic oscillator described by (19) is extended to all possible values of the parameters m and p. We see that, indeed, the scales $A^2 = t$ and $C = t^{(m-2)/2p}$ play a central role in this problem (these are the scales associated to the self–similarity) but that, also, other scales are introduced.

5.2. Rescaling the Schrödinger Equation. This is a quite important problem. The details of the computation can be found in [16] and [17]. We sketch the main steps.

First of all x and t are rescaled with $C(t)$ and $A(t)$, the new coordinate and time being ξ and θ defined in (41). Then the Schrödinger equation goes over to

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x,t)\Psi. \tag{46}$$

The rescaling of $\Psi(x,t)$ is selected as

$$\Psi(x,t) = Z(x,t)\phi(\xi,\theta). \tag{47}$$

Plugging (47) in (46) we impose for the new Schrödinger equation to begin as (46), i.e.,

$$i\hbar\frac{\partial\phi}{\partial\theta} = -\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial\xi^2} + \cdots, \tag{48}$$

but this implies $Z(\mathrm{d}\theta/\mathrm{d}t) = Z/C^2$ and consequently $A = C$. This is not surprising since we know that quantum mechanics ignore the friction concept. The next step is to get rid in the second member of (48) of the term $\partial\phi/\partial\xi$. We also impose the normalisation of $\phi(\xi,\theta)$, i.e., $\int \phi\phi^*\mathrm{d}\xi = \int \Psi\Psi^*\mathrm{d}x = 1$. We get

$$Z(x,t) = C^{-1/2}\exp\left(\frac{im}{2\hbar}\right)C^{-1}\frac{\mathrm{d}C}{\mathrm{d}t}x^2, \tag{49}$$

and one scale is free $C(t)$. The new Schrödinger equation writes

$$i\hbar\frac{\partial\phi}{\partial\theta} = -\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial\xi^2} + \mathcal{V}(\xi,\theta)\theta, \tag{50}$$

with

$$\mathcal{V}(\xi,\theta) = C^2V(x,t) + \frac{1}{2}C^3\frac{\mathrm{d}^2C}{\mathrm{d}t^2}\xi^2. \tag{51}$$

It is worthwile to note that if in (45), which is the equation giving the new forces, we have $C = A$ and assume that the physical force is the gradient of a potential, we recover (51). Although it is simply a consequence of the correspondence principles it was interesting to see the precised details, given above, on the transformation of the Schrödinger equation.

Finally let us show how rescaling can be used in the case of a quantum harmonic oscillator with time dependent frequency. We consider a potential given by $(1/2)\Omega^2(t)x^2$ and let us introduce it in (51). The new potential is

$$\frac{1}{2}C^3\xi^2\left(\frac{\mathrm{d}^2C}{\mathrm{d}t^2} + \Omega^2C\right), \tag{52}$$

and we take a scale C such that

$$\frac{\mathrm{d}^2C}{\mathrm{d}t^2} + \Omega^2(t)C = \frac{\omega^2}{C^3}. \tag{53}$$

The importance of this equation was pointed out for the first time by Lewis in [18]. The new potential is $(1/2)\omega^2\xi^2$ and is time independent with the well known solution in term of Hermite polynomials. The rescaling with a solution of (53) has transformed the quantum harmonic oscillator with time varying frequency into one with no time variation. Another example can be found in problems of time evolution of populations [19].

5.3. RESCALING PARTIAL DIFFERENTIAL EQUATIONS. Let us take again the nonlinear heat equation (31). We rescale both the dependent and independent variables with time varying scales, i.e., we introduce a new coordinate $\xi = x/C(t)$ with $C(t)$ an arbitrary positive function (i.e., not goint through zero).

A new time θ with $d\theta = dt/A^2(t)$ where $A(t)$ is also arbitrary and a new dependent variable $\phi(\xi,\theta) = \Psi(x,t)/B(t)$ are introduced. The three scales $A(t)$, $B(t)$, $C(t)$ are at our disposal. Equation (31) is now written

$$\frac{\partial\phi}{\partial\theta} - A^2\frac{\dot{C}}{C}\xi\frac{\partial\phi}{\partial\xi} + A^2\frac{\dot{B}}{B}\phi = \kappa\frac{A^2B^s}{C^2}\frac{\partial}{\partial\xi}\left(\phi^s\frac{\partial\phi}{\partial\xi}\right). \tag{54}$$

In (54), the dots indicate the derivatives with respect to t. Now, the essential idea is that, after a transient period where ϕ will be a function of both ξ and θ, it will become independent of θ and the subsequent evolution will be described by the rescaling. The contraction of the two variables x and t into one which was forced upon in the self–similar solutions must now come naturally.

Obviously, the different coefficients in (54) must be time independent. Let us try the following forms for A, B, C

$$A = (1+\Omega t)^\alpha, \quad B = (1+\Omega t)^\beta, \quad C = (1+\Omega t)^\gamma. \tag{55}$$

In (55), Ω is a real positive number. A little algebra indicates that we must take $\alpha = 1/2$, $\beta = (2\gamma-1)/s$, γ being, for the moment, arbitrary. Equation (54) is now written

$$\frac{\partial\phi}{\partial\theta} = \kappa\frac{\partial}{\partial\xi}\left(\phi^s\frac{\partial\phi}{\partial\xi}\right) + \gamma\Omega\frac{\partial}{\partial\xi}(\xi\phi) - \Omega\frac{\gamma(s+2)-1}{s}\phi. \tag{56}$$

What is the physical problem in the new space–time? We have heat diffusion with a heat current given by

$$J = -\kappa\phi^s\frac{\partial\phi}{\partial\xi} - \gamma\Omega\xi\phi, \tag{57}$$

and a source. In fact the last term in the right hand side of (56) indicates the presence of a source given by

$$S = -\Omega \frac{\gamma(s+2)-1}{s}\phi, \tag{58}$$

(56) being now written

$$\frac{\partial \phi}{\partial \theta} + \frac{\partial J}{\partial \xi} = S. \tag{59}$$

The presence of a source proportional to ϕ (as in a nuclear reactor) is not compatible with a steady state. We cancel it taking $\gamma = 1/(s+2)$. A steady state is now possible with an equilibrium between the "physical current" -$\kappa\phi^s \partial\phi/\partial\xi$, which leads to a diffusion and the "transformation current"- $\gamma\Omega\xi\phi$, which is a return current. Now, the final equation

$$\kappa \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\phi^s \frac{\mathrm{d}\phi}{\mathrm{d}\xi}\right) + \frac{\Omega}{s+2}\frac{\mathrm{d}}{\mathrm{d}\xi}(\xi\phi) = 0 \tag{60}$$

is strictly identical to (35). When we take $\lambda = 1/(s+2)$ -cf with $\gamma = 1/(s+2)$! indeed, we have build the self–similar solution through the different requirements on the coefficients and the source term. On diffusion type equation rescaling introduces (see [20] for more details)

- a rescaled "physical current",
- a transformation current,
- source terms (positive or negative).

These terms are used to conjecture on the asymptotic state of the system (usually a steady state). For partial differential equation describing motion of particles we will come back to transformations of the type described in Section 5.2 with rescaled physical force, linear in ξ, transformation forces and friction. An interesting case is given in [21] for the one dimensional expansion of an electron beam.

6. Conclusion

Invariance under similarity transformation and/or under time translation are quite common properties in the equations modelling the physical world. They allow partial or total integration and lead to much simpler equations which can eventually be numerically solved with a much reduced numerical effort (decrease in the dimension of the phase space or the parameters space, or decrease in the number of independent variables).

Nevertheless similarity transformations and time translations put constraints on the initial conditions which can be treated although they often point out these initial conditions or the critical parameters for which the nature of the solution changes.

Embedding these concepts in the physical frame of rescaling can permit to precise the nature of these SSS and give information on their possible asymptotic nature. In that case the knowledge of the physicists complements nicely the more rigorous mathematical treatment.

One application not mentionent here is the use of rescaling in problems where space and time scales vary considerably (like expansion or collapse in astrophysics or plasma physics). In that case it is more convenient to take care of these scale variations analytically by using rescaled space–time rather than reajusting many times the size of the grid's mesh and the time steps. An example is given in [22].

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