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On Operators T such that Weyl's Theorem holds for $f(T)$

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1. Introduction and notations

Throughout this paper let X be an infinite dimensional complex Banach space and let $\mathcal{L}(X)$ denote the Banach algebra of all bounded linear operators on X. For $T \in \mathcal{L}(X)$ let $\sigma(T)$ denote the spectrum of T. We denote by $\pi_{00}(T)$ the set of isolated points of $\sigma(T)$ which are eigenvalues of finite multiplicity. Let $N(T)$ and $T(X)$ denote the kernel and the range of T, respectively. An operator $T \in \mathcal{L}(X)$ is called Fredholm operator if dim $N(T)$ and codim $T(X)$ are finite. The index of a Fredholm operator T is defined by

$$\text{ind}\,(T) = \dim N(T) - \text{codim}\, T(X).$$

A Fredholm operator T with ind $(T) = 0$ is called a Weyl operator.

The Weyl spectrum of $T \in \mathcal{L}(X)$ is defined to be

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is not a Weyl operator}\}$$

It is well known that $\sigma_W(T)$ is non empty and compact (see [1], [3], [9]).

Following L.A. Coburn [3], we say that Weyl's theorem holds for $T \in \mathcal{L}(X)$ if

$$\sigma_W(T) = \sigma(T) \setminus \pi_{00}(T).$$

There are several classes of operators, including normal and hyponormal operators on a Hilbert space, for which Weyl's theorem holds (see e.g. [1], [3], [9]).

For an operator T in $\mathcal{L}(X)$ we will use the following notations:

$$\Phi(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is a Fredholm operator}\}$$

and

$$\mathcal{H}(T) = \{f : \triangle(f) \to \mathbb{C} : \triangle(f) \text{ is open}, \triangle(f) \subseteq \sigma(T),\ f \text{ is holomorphic}\}.$$

For $f \in \mathcal{H}(T)$ the operator $f(T)$ is defined by the well-known analytic calculus (see [5]).

In [11] we have introduced (in the more general context of Fredholm elements in Banach algebras) the following class:

$$\begin{aligned} \mathcal{W}(X) = \{T \in \mathcal{L}(X) :\ & \operatorname{ind}(T - \lambda I) \leq 0 \quad \text{for all } \lambda \in \Phi(T) \\ \textit{or}\ & \operatorname{ind}(T - \lambda I) \geq 0 \quad \text{for all } \lambda \in \Phi(T)\}. \end{aligned}$$

An operator $T \in \mathcal{L}(X)$ is called isoloid if isolated points of $\sigma(T)$ are eigenvalues of T.

The main result of this paper reads now as follows:

THEOREM 1. *Let $T \in \mathcal{L}(X)$ be an isoloid operator and let Weyl's theorem hold for T. The following assertions are equivalent:*

(a) $T \in \mathcal{W}(X)$.

(b) *For each $f \in \mathcal{H}(T)$, Weyl's theorem holds for $f(T)$.*

(c) *For each polynomial p, Weyl's theorem holds for $p(T)$.*

The proof of Theorem 1 will be given in Section 3.

If X is a Hilbert space then $T \in \mathcal{L}(X)$ is called hyponormal if $T^*T \geq TT^*$. Let T be hyponormal. Then it is easy to see that $T - \lambda I$ is hyponormal for each $\lambda \in \mathbb{C}$ and that $\operatorname{ind}(T - \lambda I) \leq 0$ for each $\lambda \in \Phi(T)$. Thus each hyponormal operator belongs to $\mathcal{W}(X)$. We have already mentioned that Weyl's theorem holds for hyponormal operators. Furthermore, it is known that hyponormal operators are isoloid (see [12]).

The following corollary is therefore an immediate consequence of Theorem 1 (cf. [7, Theorem 2]).

COROLLARY 1. *If T is hyponormal and $f \in \mathcal{H}(T)$ then Weyl's theorem holds for $f(T)$.*

Remark. Corollary 1 answers an old question of K.K. Oberai [10] (see also [8, Theorem 3.3]):

If T is hyponormal then does Weyl's theorem hold for T^2 ?

EXAMPLE. If U is the unilateral shift on l_2 define $T : l_2 \oplus l_2 \to l_2 \oplus l_2$ by

$$T = \begin{pmatrix} U+I & 0 \\ 0 & U^*-I \end{pmatrix}$$

Then we have (see [8, Example 3.4])

$$\sigma(T) = \sigma_W(T) \ , \quad \pi_{00}(T) = \emptyset \ ,$$

T is isoloid, Weyl's theorem holds for T, $1 \notin \sigma_W(T^2)$, $1 \in \sigma(T^2)$ and $1 \notin \sigma_W(T^2) \cup \pi_{00}(T^2)$. Thus Weyl's theorem does not hold for T^2.

2. THE SPECTRAL MAPPING THEOREM FOR $\sigma_W(T)$

In this section we show that for $T \in \mathcal{L}(X)$ we have

$$f(\sigma_W(T)) = \sigma_W(f(T)) \text{ for all } f \in \mathcal{H}(T) \Longleftrightarrow T \in \mathcal{W}(X).$$

This characterisation will be used in Section 3 for the proof of Theorem 1.

The Weyl spectrum satisfies the one-way spectral mapping theorem ([4, Theorem 2]):

$$\text{(1)} \qquad f \in \mathcal{H}(T) \Longrightarrow \sigma_W(f(T)) \subseteq f(\sigma_W(T)).$$

The example 3.3 in [1] shows that this inclusion may be proper.

In [11] we have shown the following theorem in the more general context of Fredholm elements in Banach algebras. For the convenience of the reader we give a proof.

THEOREM 2. *For $T \in \mathcal{L}(X)$ the following assertions are equivalent:*

(a) $T \in \mathcal{W}(X)$.

(b) $\sigma_W(f(T)) = f(\sigma_W(T))$ *for each* $f \in \mathcal{H}(T)$.

(c) $\sigma_W(p(T)) = p(\sigma_W(T))$ *for each polynomial* p.

For the proof of Theorem 2 we need the following proposition.

PROPOSITION 1. *Let $T, S \in \mathcal{L}(X)$*

(a) *If T and S are Fredholm operators then TS is a Fredholm operator and*

$$\text{ind}\,(TS) = \text{ind}\,(T) + \text{ind}\,(S).$$

(b) *If* $TS = ST$ *then*

$$TS \text{ is Fredholm} \iff T \text{ and } S \text{ are Fredholm.}$$

Proof. (a) [5, Satz 71.3]. (b) [5, Problems 3 and 4 in §82]. ∎

Proof of Theorem 2. (a) $\Longrightarrow$ (b): Suppose that $f \in \mathcal{H}(T)$ and $\lambda \in \mathbb{C}$. Let $g(z) = f(z) - \lambda$.

Assume first that g is not identically 0 in any component of its domain containing $\sigma(T)$. Let $c_1, \dots, c_n$ denote the zeros of g in $\sigma(T)$, with multiplicities $k_1, \dots, k_n$. Define p by $p(z) = \Pi_{j=1}^n (z - c_j)^{k_j}$ and write $g(z) = p(z)h(z)$, where $h \in \mathcal{H}(T)$ has no zeros in $\sigma(T)$. Then we have

$$g(T) = f(T) - \lambda I = p(T)h(T) \quad \text{with } h(T) \text{ invertible.}$$

Now suppose that $\lambda \notin \sigma_W(f(T))$. Thus $g(T)$ is a Weyl operator. Proposition 1 then gives

$$c_1, c_2, \dots, c_n \in \Phi(T)$$

and

$$\begin{aligned} 0 &= \operatorname{ind}(g(T)) = \operatorname{ind}(p(T)) + \underbrace{\operatorname{ind}(h(T))}_{=0} \\ &= \sum_{j=1}^{n} k_j \operatorname{ind}(T - c_j I) \ . \end{aligned}$$

Since $T \in \mathcal{W}(X)$, we derive $\operatorname{ind}(T - c_j I) = 0$ for $j = 1, \dots, n$, thus $c_j \notin \sigma_W(T)$ $(j = 1, \dots, n)$ and therefore $\lambda \notin f(\sigma_W(T))$. Hence we have shown that $f(\sigma_W(T)) \subseteq \sigma_W(f(T))$. By (1) we get

$$f(\sigma_W(T)) = \sigma_W(f(T)) \ .$$

In the general case, g is defined on an open set $V = V_1 \cup V_2$ with V_1, V_2 open, $V_1 \cap V_2 = \emptyset$, $g \equiv 0$ on V_1 and g is not identically 0 in any component of V_2. Thus $\sigma(T) = \sigma_1 \cup \sigma_2$ with σ_i compact and $\sigma_i \subseteq V_i$ $(i = 1, 2)$. Let P be the spectral projection associated with σ_2. Take $X_1 = N(P)$, $X_2 = P(X)$ and $T_i = T|_{X_i}$ for $i = 1, 2$. By [5, Theorem 100.1], we get $X = X_1 \oplus X_2$, $T_i(X_i) \subseteq X_i$ and $\sigma(T_i) = \sigma_i$ $(i = 1, 2)$. Since $g \equiv 0$ on σ_1, $g(T_1) = 0$, thus

$$g(T) = 0 \oplus g(T_2) \tag{2}$$

and

$$g(T) = g(T)P = Pg(T) \ . \tag{3}$$

Further we have

$$\begin{aligned} & P \text{ is a Weyl operator} \\ \iff & \dim X_1 < \infty \\ \iff & \sigma_1 \text{ is finite and consists of eigenvalues of } T \\ & \text{of finite multiplicity} \\ \iff & \sigma_W(T) \cap V_1 = \emptyset \ . \end{aligned} \tag{4}$$

Since $\operatorname{codim} P(X) = \dim N(P)$, we get from (2), (3), (4) and Proposition 1 that

$$g(T) \text{ is a Weyl operator } \iff P \text{ and } g(T_2) \text{ are Weyl operators.}$$

Thus the previous arguments imply that

$$\lambda \in f(\sigma_W(T)) \iff \lambda \in \sigma_W(f(T)) \ .$$

(b) $\Rightarrow$ (c) : Clear.
(c) $\Rightarrow$ (a) : Assume to the contrary that $T \notin \mathcal{W}(X)$. Then there are $\lambda_1, \lambda_2 \in \Phi(T)$ with

$$\operatorname{ind}(T - \lambda_1 I) > 0 \textit{ and } \operatorname{ind}(T - \lambda_2 I) < 0.$$

Put $k := \operatorname{ind}(T - \lambda_1 I)$ and $m := -\operatorname{ind}(T - \lambda_2 I)$.
Put $p(\lambda) := (\lambda - \lambda_1)^m (\lambda - \lambda_2)^k$. Proposition 1 gives that $p(T)$ is a Fredholm operator with

$$\operatorname{ind}(p(T)) = mk + k(-m) = 0,$$

thus $0 \notin \sigma_W(p(T))$. Since $\lambda_1 \in \sigma_W(T)$ we get $0 = p(\lambda_1) \in p(\sigma_W(T)) = \sigma_W(p(T))$, a contradiction. ∎

Remark. If T is a hyponormal operator on a Hilbert space X, then $T \in \mathcal{W}(X)$ (see Section 1). Thus Theorem 2 is a generalisation of [7, Theorem 1].

THEOREM 3. *Let $T \in \mathcal{L}(X)$. If $f \in \mathcal{H}(T)$ is injective on $\sigma_W(T)$ then*

$$\sigma_W(f(T)) = f(\sigma_W(T)).$$

Proof. By (2), we only have to show that $f(\sigma_W(T)) \subseteq \sigma_W(f(T))$. Let $\mu_0 \in f(\sigma_W(T))$. Put $\lambda_0 \in \sigma_W(T)$ with $f(\lambda_0) = \mu_0$. Define $g \in \mathcal{H}(T)$ by

$$g(\lambda) = \begin{cases} \dfrac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0}, & \text{for } \lambda \neq \lambda_0 \\ f'(\lambda_0), & \text{for } \lambda = \lambda_0 \end{cases} \qquad (\lambda \in \Delta(f)).$$

Since f is injective on $\sigma_W(T)$, g does not vanish on $\sigma_W(T)$ hence $0 \notin g(\sigma_W(T))$. Thus by (2), $0 \notin \sigma_W(g(T))$. This shows that $g(T)$ is a Weyl operator. Since $g(T)(T-\lambda_0 I) = f(T)-\mu_0 I$ and $T-\lambda_0 I$ is not a Weyl operator, we derive from Proposition 1 that $f(T) - \mu_0 I$ is not a Weyl operator. Thus $\mu_0 \in \sigma_W(f(T))$. ∎

3. The Proof of Theorem 1.

Before proving Theorem 1 we deal with some preliminary results.

PROPOSITION 2. *Let $T \in \mathcal{L}(X)$ be isoloid. If $f \in \mathcal{H}(T)$ then*

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T)). \tag{5}$$

Proof. The assertion is a modification of [10, Lemma 1 and Proposition 1], see also [7, (2.1)]. ∎

Let $T \in \mathcal{L}(X)$ be isoloid and let Weyl's theorem hold for T . It follows from (5) that

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma_W(T)) \tag{6}$$

for each $f \in \mathcal{H}(T)$.

The next theorem is an immediate consequence of (6).

THEOREM 4. *Let T be isoloid and suppose that Weyl's theorem holds for T. If $f \in \mathcal{H}(T)$ then*

Weyl's theorem holds for $f(T) \Longleftrightarrow \sigma_W(f(T)) = f(\sigma_W(T))$.

Theorem 3 and Theorem 4 have the following corollary.

COROLLARY 2. *If $T \in \mathcal{L}(X)$ is isoloid and if Weyl's theorem holds for T, then Weyl's theorem holds for $f(T)$ whenever $f \in \mathcal{H}(T)$ is injective on $\sigma_W(T)$.*

The proof of Theorem 1 is now very short: use Theorem 2 and Theorem 4.

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