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On the Spectra of Elements in Certain Algebras of Vector Valued Functions and Sequences

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1. TERMINOLOGY AND NOTATIONS.

By an algebra A we will always mean an associative algebra over the field $\mathbb{C}$ of complex numbers.

$\sigma(A) := \{\chi\colon A \longrightarrow \mathbb{C}$ linear, multiplicative, $\neq 0\}$ denotes the set of all characters on A and for $x \in A$ let $\sigma_A(x)$ denote the spectrum of x with respect to A. An element $x \in A$ is called quasiinvertible element in A, if there is a (so-called quasiinverse) element $y \in A$ such that $xy = yx = x + y$ and the quasiinverse element is uniquely determined. Let $Q(A)$ denote the set of quasiinvertible elements in A, and $q\colon Q(A) \longrightarrow Q(A)$ the map which assigns to each $x \in Q(A)$ its quasiinverse element; we will call it the quasiinversion in A. If A has a unit element e, then $e - Q(A) = G(A)$ (group of invertible elements in A) and for each $x \in$ Q(A) one has $(e-x)^{-1} = e - q(x)$. Let $A_e (= A \times \mathbb{C})$ denote the algebra which arises from A by the formal adjunction of a unit element. Then for $(x, \lambda) \in A_e$ we have $(x,\lambda) \in Q(A_e) \Longleftrightarrow \lambda \neq 1$ and $\frac{1}{1-\lambda}x \in Q(A)$, and if $(x,\lambda) \in Q(A_e)$, then its quasiinverse element in A_e is equal to $\left(\frac{1}{1-\lambda}q(\frac{1}{1-\lambda}x), \frac{-\lambda}{1-\lambda}\right)$. An algebra A provided with a locally convex topology is called a locally convex algebra, if multiplication is jointly continuous. A locally convex algebra A is called locally m-convex, if its 0-nbhd-filter has a basis of sets U satisfying $U^2 \subset U$.

LEMMA. *Let A be an algebra and $I \subset A$ a proper ideal. Then*

$$\sigma_I(x) \cup \{0\} = \sigma_A(x), \quad \forall x \in I.$$

If I does not contain a unit element, then

$$\sigma_I(x) = \sigma_A(x), \quad \forall x \in I.$$

Proof. Clearly, as I is a proper ideal in A, $\sigma_A(x)$ contains 0 for every $x \in I$. Now let $\lambda \in \mathbb{C}\backslash\{0\}$ and $x \in I$. If $\frac{1}{\lambda}x$ is quasiinvertible in I, then it is also quasiinvertible in A. Conversely, let $y \in A$ such that $\frac{1}{\lambda}xy = y(\frac{1}{\lambda}x) = \frac{1}{\lambda}x + y$. But then $y = \frac{1}{\lambda}xy - \frac{1}{\lambda}x$ is already contained in I. The last assertion is obvious. ∎

As a first easy application of the lemma we obtain a description of the spectrum of elements in a product of algebras:

Let $(A_\iota)_{\iota\in I}$ be a family of algebras and $x = (x_\iota)_{\iota\in I} \in A := \prod_{\iota\in I} A_\iota$. Then

$$\sigma_A(x) = \bigcup_{\iota\in I} \sigma_{A_\iota}(x_\iota).$$

In fact, if each A_ι has a unit element, then G(A) $=\prod_{\iota\in I} G(A_\iota)$, which yields the assertion. Otherwise, put $J = \{\iota \in I\colon A_\iota$ does not have a unit $\}$; then A is a proper ideal without unit element in $\prod_{\iota\in J}(A_\iota)_e \times \prod_{\iota\in I\backslash J} A_\iota$ and the lemma yields the assertion.

Next we are going to study the spectrum of elements in algebras $C(T, A)$ of all continuous functions $f: T \longrightarrow A$ where T is a completely regular Hausdorff space and A a locally convex algebra (provided with pointwise operations). In [2] the set $\sigma(C(T, A))$ of characters on $C(T, A)$ was described, in the case that A is metrizable and realcompact (as a topological space): The characters on $C(T, A)$ are exactly of the form $\chi \circ \delta_x$, where $\chi \in \sigma(A)$ and δ_x is the evaluation (of the continuous extension) in a point x in the realcompactification νX of X. We will now describe the spectrum of an element in such an algebra.

PROPOSITION 1. *Let T be a completely regular Hausdorff space, A a locally convex algebra with continuous quasiinversion q and let $f \in C(T, A)$. Then*

$$\sigma_{C(T,A)}(f) = \bigcup_{t\in T} \sigma_A(f(t)).$$

Proof. We may assume that A contains a unit element e. In fact, if A does not, $C(T, A)$ is a proper ideal wihout unit in $C(T, A_e)$, which by the lemma gives $\sigma_{C(T,A)}(f) = \sigma_{C(T,A_e)}(f)$ and clearly, $\sigma_{A_e}(f(t)) = \sigma_A(f(t))$ for all $t \in T$. Moreover, quasiinversion and hence inversion are continuous in A_e.

We will show that for any $g \in G(C(T, A))$,

$$g \in G(C(T, A)) \iff g(t) \in G(A),\ \forall t \in T,$$

(which yields the assertion). In fact, if $g(t) \in G(A)$ for all $t \in T$, then $h\colon T \longrightarrow A, t \longmapsto g(t)^{-1}$ is continuous, hence inverse to g in $C(T, A)$. The converse implication is trivial. ∎

Remarks. 1) The hypothesis about continuity of quasiinversion in A is essential. In fact, there exist even complete metrizable locally convex algebras with unit and discontinuous inversion hence quasiinversion, e.g. the Arens-algebra $\bigcap_{p\geq 1} L^p([0,1])$ (see [1]). $T := G(A)$ provided with the relative topology is metrizable hence completely regular and Hausdorff, and the inclusion $j: T \hookrightarrow A$ is not invertible in $C(T, A)$, but of course $j(t) \in G(A)$ for all $t \in T$. Thus $0 \in \sigma_{C(T,A)}(j) \setminus \bigcup_{t\in T} \sigma_A(j(t))$. On the other hand, every locally m-convex algebra has continuous quasiinversion.

2) Let T be a completely regular Hausdorff space and A a locally convex algebra with continuous quasiinversion such that

$$\{\chi(x): \chi \in \sigma(A)\} \subset \sigma_A(x) \subset \{\chi(x): \chi \in \sigma(A)\} \cup \{0\}$$

for all $x \in A$. (All commutative Banach algebras have this last property). Then $C(T, A)$ has the same properties. In fact, we must only show that

$$\sigma_{C(T,A)}(f) \subset \{\chi(f): \chi \in \sigma(C(T, A))\} \cup \{0\}.$$

Let $\lambda \in \sigma_{C(T,A)}(f) \setminus \{0\}$. Then, by proposition 1, there is $t \in T$ such that $\lambda \in \sigma_A(f(t))$. By hypothesis, there is $\psi \in \sigma(A)$ such that $\lambda = \psi(f(t))$.

Certainly $\chi: C(T, A) \longrightarrow \mathbb{C}, g \longmapsto \psi(g(t))$ is a character on $C(T, A)$.

Next we will study the spectrum of elements in algebras of vector-valued sequences.

Let $(\lambda, \|.\|)$ be a normal Banach sequence space, i.e. $(\lambda, \|.\|)$ is a Banach space such that $\bigoplus_{\mathbb{N}} \mathbb{C} \subset \lambda \subset \mathbb{C}^{\mathbb{N}}$ such that for all $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \lambda$ and all $\beta = (\beta_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}$,

$$(|\beta_n| \leq |\alpha_n|, \ \forall n \in \mathbb{N}) \quad \Rightarrow \quad (\beta \in \lambda \text{ and } \|\beta\| \leq \|\alpha\|).$$

For every $n \in \mathbb{N}$ the number $\rho_n := \|(\delta_{kn})_{k\in\mathbb{N}}\|$ is positive. Provided with the multiplication $\lambda \times \lambda \longrightarrow \lambda, ((\alpha_n)_{n\in\mathbb{N}}, (\beta_n)_{n\in\mathbb{N}}) \longmapsto (\rho_n\alpha_n\beta_n)_{n\in\mathbb{N}}$, the Banach space $(\lambda, \|.\|)$ is a Banach algebra. If A is a locally convex algebra and $cs(A)$ the set of continuous seminorms on A, then

$$\lambda(A) := \{(a_n)_{n\in\mathbb{N}} \in A^{\mathbb{N}}: \left(p(a_n)\right)_{n\in\mathbb{N}} \in \lambda, \ \forall p \in cs(A)\}$$

provided with the locally convex topology generated by the seminorms

$$\hat{p}: \lambda(A) \longrightarrow [0, \infty), (a_n)_{n\in\mathbb{N}} \longmapsto \|\left(p(a_n)\right)_{n\in\mathbb{N}}\|$$

is a locally convex algebra with respect to the multiplication $(a_n)_{n\in\mathbb{N}} \cdot (b_n)_{n\in\mathbb{N}} := \left(\rho_n a_n b_n\right)_{n\in\mathbb{N}}$. For the case that $(\lambda, \|.\|)$ has sectional convergence (i.e., $\|((0)_{k<n},$

$(\alpha_k)_{k\geq n})\| \stackrel{n\to\infty}{\longrightarrow} 0$ for all $(\alpha_n)_{n\in\mathbb{N}} \in \lambda$), the set of characters on $\lambda(A)$ was characterized in [2], namely

$$\sigma(\lambda(A)) = \{\chi \circ pr_n \colon n \in \mathbb{N}, \chi \in \sigma(A)\}.$$

We will now investigate the spectrum of elements in such algebras.

PROPOSITION 2. *Let $(\lambda, \|.\|)$ be a normal Banach sequence space with sectional convergence, let A be a locallly convex algebra with sequentially continuous quasiinversion and let $x = (x_n)_{n\in\mathbb{N}} \in \lambda(A)$. Then*

$$\sigma_{\lambda(A)}(x) = \bigcup_{n\in\mathbb{N}} \sigma_A(\rho_n x_n) \cup \{0\}$$

(where $\rho_n := \|(\delta_{kn})_{k\in\mathbb{N}}\|, \quad (n \in \mathbb{N})$).

Proof. The map $\lambda(A) \longrightarrow c_0(A), (a_n)_{n\in\mathbb{N}} \longmapsto (\rho_n a_n)_{n\in\mathbb{N}}$ is injective, linear, multiplicative and its range is an ideal in $c_0(A)$ without unit element. Then by the lemma, we may assume that $(\lambda, \|.\|) = (c_0, \|.\|_\infty)$, and we must prove that for each $y = (y_n)_{n\in\mathbb{N}} \in c_0(A)$ one has $\sigma_{c_0(A)}(y) = \bigcup_{n\in\mathbb{N}} \sigma_A(y_n) \cup \{0\}$.

As, clearly, $0 \in \sigma_{c_0(A)}(y)$, we have to show that $z \in Q(c_0(A))$ if and only if $z_n \in Q(A)(n \in \mathbb{N})$, whenever $z = (z_n)_{n\in\mathbb{N}} \in c_0(A)$. The only if part is obvious. So let $z = (z_n)_{n\in\mathbb{N}} \in c_0(A)$ be given such that $z_n \in Q(A)$ for all $n \in \mathbb{N}$. We will be done, if we show that $y := \left(q(z_n)\right)_{n\in\mathbb{N}}$ belongs to $c_0(A)$ (q denoting quasiinversion in A, as before). But this is clear, as $(z_n)_{n\in\mathbb{N}}$ tends to 0 in $Q(A) \subset A$ and q is sequentially continuous on $Q(A)$. ∎

Remarks. 1) Again the assumption about sequential continuity of the quasiinversion in A cannot be ommited. In fact, let A again denote the Arens-algebra (see the Remark above). Then by the metrizability of A there is a sequence $(a_n)_{n\in\mathbb{N}} \in Q(A)$ tending to an element a in $Q(A)$ such that $\left(q(a_n)\right)_{n\in\mathbb{N}}$ does not converge to $q(a)$. Then $x = (x_n)_{n\in\mathbb{N}} := \left(a_n + q(a) - a_n q(a)\right)_{n\in\mathbb{N}}$ belongs to $c_0(A), x_n \in Q(A)$ and $q(x_n) = a + q(a_n) - aq(a_n)$. But $(q(x_n))_{n\in\mathbb{N}}$ does not belong to $c_0(A)$, because otherwise $\left(q(a_n)\right)_{n\in\mathbb{N}}$ would converge to $-(e-a)^{-1}a = -(e-q(a))a = q(a)$. Thus $1 \in \sigma_{c_0(A)}(x) \setminus \bigcup_{n\in\mathbb{N}} \sigma_A(x_n)$.

2) Let $(\lambda, \|.\|)$ be a normal Banach sequence space with sectional convergence and A a locally convex algebra with sequentially continuous quasiinversion such that $\{\chi(x) \colon \chi \in \sigma(A)\} \subset \sigma_A(x) \subset \{\chi(x) \colon \chi \in \sigma(A)\} \cup \{0\}$ for all $x \in A$. Then $\lambda(A)$ has the same properties, as is immediately clear from the description of $\sigma_{\lambda(A)}((x_n)_{n\in\mathbb{N}})$.

3) Let $(\lambda, \|.\|)$ be a normal Banach sequence space, $\rho_n := \|(\delta_{kn})_{k\in\mathbb{N}}\| (n \in \mathbb{N})$; then for every locally convex algebra A the map

$$\lambda(A) \longrightarrow l^\infty(A), (a_n)_{n\in\mathbb{N}} \longmapsto (\rho_n a_n)_{n\in\mathbb{N}}$$

is injective, linear, multiplicative and its range is an ideal in $l^\infty(A)$. Thus if one could describe the characters on $l^\infty(A)$ or the spectrum $\sigma_{l^\infty(A)}((a_n)_{n\in\mathbb{N}})$, this would allow such a description for all $\lambda(A)$. Unfortunately, the case $l^\infty(A)$ is not easy to handle (unless the bounded sets in A are relatively compact, which leads to $C(\beta\mathbb{N}, A)$) even if A is a Banach algebra.

We owe the following two observations to L. Frerick and J. Wengenroth (oral communication):

α) Let A be a commutative C^*- algebra with unit element e. Then

$$\sigma_{l^\infty(A)}((x_n)_{n\in\mathbb{N}}) = \overline{\bigcup_{n\in\mathbb{N}} \sigma_A(x_n)}$$

for all $(x_n)_{n\in\mathbb{N}} \in l^\infty(A)$. In fact, it suffices to prove "$\subset$" . Let

$$\lambda \in \mathbb{C}\backslash \overline{\bigcup_{n\in\mathbb{N}} \sigma_A(x_n)}.$$

Then $\lambda e - x_n \in G(A)$ for all $n \in \mathbb{N}$. Assume that the sequence $(\|(\lambda e - x_n)^{-1}\|)_{n\in\mathbb{N}}$ is unbounded. Let $\varepsilon > 0$ be given. Then there is $n \in \mathbb{N}$ such that $\|(\lambda e - x_n)^{-1}\| > \frac{1}{\varepsilon}$ and therefore (A being a C^*-algebra) there is $\mu \in \sigma_A((\lambda e - x_n)^{-1})$ such that $|\mu| > \frac{1}{\varepsilon}$. Consequently

$$\begin{aligned}\mu e - (\lambda e - x_n)^{-1} &= (\mu(\lambda e - x_n) - e)(\lambda e - x_n)^{-1} \\ &= \mu((\lambda - \tfrac{1}{\mu})e - x)(\lambda e - x_n)^{-1} \notin G(A)\end{aligned}$$

and thus $\lambda - \frac{1}{\mu} \in \sigma_A(x_n) \subset \bigcup_{m\in\mathbb{N}} \sigma_A(x_m)$. As $|\frac{1}{\mu}| < \varepsilon$, we obtain that $\lambda \in \overline{\bigcup_{m\in\mathbb{N}} \sigma_A(x_m)}$, a contradiction. Thus $((\lambda e - x_n)^{-1})_{n\in\mathbb{N}} \in l^\infty(A)$, hence $\lambda(e)_{n\in\mathbb{N}} - (x_n)_{n\in\mathbb{N}} \in G(l^\infty(A))$ and $\lambda \notin \sigma_{l^\infty(A)}((x_n)_{n\in\mathbb{N}})$.

β) The description in α) for the spectrum of elements in $l^\infty(A)$ is no longer true for non-commutative C^*-algebra A. In fact, let A be the C^*-algebra of linear bounded operators in l^2. For each $n \in \mathbb{N}$ let $x_n \in A$ be defined by

$$x_n((\delta_{kl})_{l\in\mathbb{N}}) := \begin{cases} (\delta_{k+1,l})_{l\in\mathbb{N}} & \text{if} \quad k \leq n+1 \\ (0)_{l\in\mathbb{N}} & \text{if} \quad k > n+1 \end{cases}$$

Then $\|x_n\|_{op} = 1$ for all $n \in \mathbb{N}$, hence $(x_n)_{n\in\mathbb{N}} \in l^\infty(A)$. Since each x_n is nilpotent, $\sigma_A(x_n) = \{0\}$ for all $n \in \mathbb{N}$. On the other hand, $\sqrt[k]{\sup_{n\in\mathbb{N}} \|x_n{}^k\|} = 1$ for each $k \in \mathbb{N}$, as $\|x_k{}^k\| = 1$. Therefore there is $\lambda \in \sigma_{l^\infty(A)}((x_n)_{n\in\mathbb{N}})$ such that $|\lambda| = 1$.

References

[1] ARENS, R., The space L^ω and convex topological rings, *Bull. Amer. Math. Soc.*, **52** (1946), 931–935.

[2] DIEROLF, S., SCHRÖDER, K.H., WENGENROTH, J., Characters on certain function algebras, to appear in *Functiones et Approximatio*, 1998.