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Integrable Systems Via Inverse Integrating Factor

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1. Introduction

We consider two-dimensional autonomous systems of differential equations of the form

$$\dot{x} = -y + X_s(x,y), \quad \dot{y} = x + Y_s(x,y), \tag{1}$$

where

$$X_s(x,y) = \sum_{k=0}^{s} a_k x^k y^{s-k}, \quad Y_s(x,y) = \sum_{k=0}^{s} b_k x^k y^{s-k},$$

are homogeneous polynomials of degree s, with $s \geq 2$; being a_k and b_k, $k = 0, 1, \ldots, s$, arbitrary real coefficients. Recently, these systems have been studied by several authors (see for instance [1], [2], [4], [5], [11] and [12]), especially in order to obtain information about the number of small amplitude limit cycles and to determine the cyclicity of the origin (see for instance [1] and [12]). Our aim is to find solutions $V(x,y) = 0$ of system (1) where $V(x,y)$ is an inverse integrating factor (this notion will be defined below). The method consists in characterizing the systems which have an inverse integrating factor. This paper contains a natural generalization of the results developed in [3]. Theorem 1 gives an explicit method for obtaining such inverse integrating factor, which is used in Theorem 2 to construct some particular class of integrable vector fields. The method shows that if $V(x,y)$ is a product of n linear factors, elevated each factor to α_i, we always arrive to a partial differential

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equation of $n-$th order, where only appear partial differentials with respect to one variable, and we can consider this partial differential equation as a linear differential equation respect to this variable. In this work we only study the case $n = 1$ (see Theorem 2) and some particular cases for $n = 2$. In the other cases, $n = 3$, we can not totally characterize the systems which have this inverse integrating factor, but we show that the known cases verify the equations we obtain.

We can write the system (1) (see [2]) in polar coordinates $x = r\cos(\varphi)$ and $y = r\sin(\varphi)$ as

$$\begin{aligned} \dot{r} &= P_s(\varphi)r^s, \\ \dot{\varphi} &= 1 + Q_s(\varphi)r^{s-1}, \end{aligned} \tag{2}$$

where $P_s(\varphi)$ and $Q_s(\varphi)$ are trigonometrical polynomials of the form

$$\begin{aligned} P_s(\varphi) &= R_{s+1}\cos\left((s+1)\varphi + \varphi_{s+1}\right) + R_{s-1}\cos\left((s-1)\varphi + \varphi_{s-1}\right) + \dots \\ &\quad + \begin{cases} R_1\cos(\varphi+\varphi_1) & \text{if s is even;} \\ R_0 & \text{if s is odd;} \end{cases} \\ Q_s(\varphi) &= -R_{s+1}\sin\left((s+1)\varphi + \varphi_{s+1}\right) + \bar{R}_{s-1}\sin\left((s-1)\varphi + \bar{\varphi}_{s-1}\right) + \dots \\ &\quad + \begin{cases} \bar{R}\sin(\varphi+\bar{\varphi}_1) & \text{if s is even;} \\ \bar{R}_0 & \text{if s is odd;} \end{cases} \end{aligned}$$

where $R_i, \bar{R}_i, \varphi_i$ and $\bar{\varphi}_i$ are real constants.

If we do the change $R = r^{s-1}$, the system (2) becomes

$$\begin{aligned} \dot{R} &= (s-1)P_s(\varphi)R^2, \\ \dot{\varphi} &= 1 + Q_s(\varphi)R. \end{aligned} \tag{3}$$

In the study and determination of the first integrals for homogeneous systems (see [2-5]), we used a technique consisting in the research of polynomial particular solutions of system (3) of the form

$$V(R,\varphi) = 1 + V_1(\varphi)R + V_2(\varphi)R^2 + \dots + V_p(\varphi)R^p = 0, \tag{4}$$

where $V_k(\varphi)$, $k = 1, 2, \dots, p$, are homogenous trigonometrical polynomials of degree $k(s-1)$ in the variables $\cos\varphi$ and $\sin\varphi$.

By using the functions x_i, $i = 1, 2, \dots, p$ defined implicity by

$$V_1 = \sum_{j=1}^{p} x_j,\ V_2 = \sum_{\substack{j,k=1 \\ j<k}}^{p} x_j x_k, \dots,\ V_p = x_1 x_2 \dots x_p, \tag{5}$$

function (4) can be written $V(R,\varphi)=\prod_{i=1}^{p}(1+x_i(\varphi)R)$. The following result appears in [3].

PROPOSITION 1. *Function* (4) *is a particular solution of system* (3) *if the homogeneous trigonometric polynomials* $V_k(\varphi)$, $k=1,2,\ldots,p$, *verify the following differential system*

$$(6)\qquad \begin{aligned} &V'_{k+1}+V'_kQ_s+k(s-1)V_kP_s=V_kV'_1, \quad k=1,2,\ldots,p-1,\\ &V'_pQ_s+p(s-1)V_pP_s=V_pV'_1, \end{aligned}$$

where $'=\frac{d}{d\varphi}$.

2. INVERSE INTEGRATING FACTORS

A function $V(x,y)$ will be called an *inverse integrating factor* for system (1) if $V(x,y)=0$ is a particular solution for this system and the divergence of the vector field

$$C=\left[\frac{-y+X_s(x,y)}{V(x,y)},\frac{x+Y_s(x,y)}{V(x,y)}\right]$$

defined at $\mathbb{R}^2\setminus\{(x,y):V(x,y)=0\}$ is zero.

We notice that if the divergence of a vector field is zero, then system (1) defined for this vector field is integrable. In particular, if system (1) has an inverse integrating factor, then the system is integrable and the origin is a center. In [6-10] it has been shown that the inverse integrating factor plays a fundamental role in the problem of the center and in the determination of the limit cycles.

If system (1) is written in polar coordinates, see (2), then the function $V(R,\varphi)$ is an inverse integrating factor for system (1) if

$$(7)\qquad \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{P_s(\varphi)r^{s+1}}{V(R,\varphi)}\right]+\frac{\partial}{\partial\varphi}\left[\frac{1+Q_s(\varphi)r^{s-1}}{V(R,\varphi)}\right]=0.$$

THEOREM 1. *If* $V(R,\varphi)=\prod_{i=1}^{p}(1+x_i(\varphi)R)^{\alpha_i}$ *is an inverse integrating factor for system (1) with* α_i *are real numbers, then the functions* $x_i(\varphi)$, $i=1,2,\ldots,p$, *must verify the following system of differential equations*

$$(8)\qquad \frac{dx_i}{dz}=\frac{\frac{x_i}{z-x_i}}{\frac{s+1}{s-1}+\sum_{j=1}^{p}\frac{\alpha_jx_j}{z-x_j}},\qquad i=1,2,\ldots p,$$

being $z=Q_s(\varphi)$.

Proof. We develop the expression (7) for $V(R,\varphi) = \prod_{i=1}^{p}(1 + x_i(\varphi)R)^{\alpha_i}$ with respect to the powers of R; we have

$$(x_i - Q_s)x_i' = (s-1)P_s x_i, \quad i = 1,2,\dots,p, \tag{9}$$

with the condition

$$(s+1)P_s + Q_s' - \sum_{i=1}^{p} \alpha_i x_i' = 0. \tag{10}$$

If we take $z = Q_s$ as independent variable instead of φ, then we have

$$\frac{dx_i}{d\varphi} = \frac{dx_i}{dQ_s}\frac{dQ_s}{d\varphi}, \quad i = 1,2,\dots,p,$$

and (10) transform into

$$(s+1)P_s + Q_s' - Q_s' \sum_{i=1}^{p} \alpha_i \frac{dx_i}{dz} = 0,$$

which gives

$$P_s' = \frac{1}{(s+1)}\left[\sum_{i=1}^{p} \alpha_i \frac{dx_i}{dz} - 1\right] Q_s'. \tag{11}$$

By inserting expression (11) in system (9), and considering the change of variable $z = Q_s$ we can write

$$\frac{dx_i}{dz} = \frac{(s-1)}{(s+1)} \frac{\left[\sum_{j=1}^{p} \alpha_j \frac{dx_j}{dz} - 1\right]}{x_i - z} x_i, \quad i = 1,2,\dots,p.$$

Then, isolating $\frac{dx_i}{dz}$, $i = 1,\dots,p$, in the above system we get

$$\frac{dx_i}{dz} = \frac{\prod_{\substack{j=1\\ j\neq i}}^{p}(z - x_j)}{\frac{s+1}{s-1}\prod_{j=1}^{p}(z - x_j) + \sum_{j=1}^{p}\left[\prod_{\substack{k=1\\ k\neq j}}^{p}(z - x_k)\right]\alpha_j x_j} x_i, \quad i = 1,2,\dots,p.$$

If we divide the numerator and denominator of the previous fraction by the product $\prod_{j=1}^{p}(z - x_j)$, we obtain the system (8). ∎

Note that the system (8) is symmetric with respect to the variables x_i, $i = 1,2,\dots,p$, with the corresponding α_i.

We want to find functions of the form

$$U(x_1,x_2,\dots,x_p,z) \equiv H(x_1,x_2,\dots,x_p) + zG(x_1,x_2,\dots,x_p), \tag{12}$$

so that, for system (8), $\frac{dU}{dz} = 0$.

PROPOSITION 2. *In order to find the fuctions of the form* (12) *for the system* (8) *it is sufficient to find solutions of the partial differential system*

$$(13)\qquad \begin{aligned} &\frac{\partial H}{\partial x_i} + x_i \frac{\partial G}{\partial x_i} + \alpha_i G = 0, \quad i = 1, 2, \ldots, p, \\ &\sum_{k=1}^{p} x_i \frac{\partial G}{\partial x_i} + \frac{s+1}{s-1} G = 0. \end{aligned}$$

Proof. If we differentiate (12) with respect to z, we have

$$\frac{dU}{dz} = \sum_{i=1}^{p} \frac{\partial H}{\partial x_i} \frac{dx_i}{dz} + z \sum_{i=1}^{p} \frac{\partial G}{dx_i} \frac{dx_i}{dz} + G = 0.$$

By replacing in the previous expression the value of $\frac{dx_i}{dz}$, $i = 1, 2, \ldots, p$, given in (8) it becomes

$$\sum_{i=1}^{p} \frac{\partial H}{\partial x_i} \frac{x_i}{z - x_i} + z \sum_{i=1}^{p} \frac{\partial G}{\partial x_i} \frac{x_i}{z - x_i} + \left(\frac{s+1}{s-1} + \sum_{i=1}^{p} \frac{\alpha_i x_i}{z - x_i} \right) G = 0,$$

or

$$\begin{aligned} &\sum_{i=1}^{p} \frac{\partial H}{\partial x_i} \frac{x_i}{z - x_i} + \sum_{i=1}^{p} \frac{\partial G}{\partial x_i} (x_i + \frac{x_i^2}{z - x_i}) + (\frac{s+1}{s-1} + \sum_{i=1}^{p} \frac{\alpha_i x_i}{z - x_i}) G \\ &= \sum_{i=1}^{p} \frac{x_i \frac{\partial H}{\partial x_i} + x_i^2 \frac{\partial G}{\partial x_i} + \alpha_i x_i G}{z - x_i} + \sum_{i=1}^{p} x_i \frac{\partial G}{\partial x_i} + \frac{s+1}{s-1} G = 0. \end{aligned}$$

For this last expression to be null it is sufficient that conditions (13) hold. Notice that these conditions are not necessary in order that the previous expression be null. ∎

If we make the change of variable $y_i = \frac{x_{i+1}}{x_1}$, $i = 1, 2, \ldots, p-1$, and we take the functions G and H as follows:

$$\begin{aligned} G(x_1, x_2, \ldots, x_p) &= x_1^{-a} g(y_1, y_2, \ldots, y_{p-1}), \\ H(x_1, x_2, \ldots, x_p) &= x_1^{1-a} h(y_1, y_2, \ldots, y_{p-1}), \end{aligned}$$

then the last equation of system (13) is satisfied identically, and system (13) takes the form

$$(14)\qquad \begin{aligned} &\alpha_i g + y_i \frac{\partial g}{\partial y_i} + \frac{\partial h}{\partial y_i} = 0, \quad i = 1, 2, \ldots, p-1, \\ &(1-a)h - (a - \sum_{i=1}^{p} \alpha_i) g + \sum_{i=1}^{p-1} (1 - y_i) \frac{\partial h}{\partial y_i} = 0, \end{aligned}$$

where $a = \frac{s+1}{s-1}$.

3. Linear inverse integrating factors

We first consider the case $p = 1$; in this case system (8) is

$$\frac{dx}{dz} = \frac{\frac{x}{z-z}}{a + \frac{\alpha x}{z-x}}. \tag{15}$$

If we make the change of variable $x = \lambda z$ the expression (15) takes the form

$$\frac{d\lambda}{dz} = \frac{(1 - a + (a - \alpha)\lambda)\lambda}{(a - (a - \alpha)\lambda)z}.$$

The general solution of this equation is given by

$$\frac{\lambda^a}{1 - a + (a - \alpha)\lambda} = C_1 z^{1-a},$$

where C_1 is an arbitrary constant. By going back through the change of variables it is easy to see that

$$z = Q_s = \frac{1}{C_1(1-a)}(x^a - C_1(a - \alpha)x).$$

Since Q_s is a homogeneous trigonometric polynomial of $s - 1$ degree, then x must be a homogeneous trigonometric polynomial of first degree elevated to the power of $s - 1$ if s is even and a homogeneous trigonometric polynomial of second degree elevated to the power of $\frac{s-1}{2}$ if s is odd. We take $x = C_2^{s-1} \cos^{s-1}(\varphi + \varphi_0)$ for s even and $x = (C_0 + C_2 \cos(2\varphi + 2\varphi_0))^{\frac{s-1}{2}}$ for s odd where C_0 and C_2 are arbitrary constants. Using the last expression of Q_s and equation (10) we have the following result.

THEOREM 2. *For $s \in \mathbb{N}$ with $s \geq 2$, system (2) with*

$$P_s(\varphi) \begin{cases} \sin(\varphi + \varphi_0)(K_1 C_2^{s+1} \cos^s(\varphi + \varphi_0) \\ \qquad + (K_2 - 1)C_2^{s-1} \cos^{s-2}(\varphi + \varphi_0)) & \text{if } s \text{ is even}, \\ C_2 \sin(2\varphi + 2\varphi_0)(K_1(C_0 + C_2 \cos(2\varphi + 2\varphi_0))^{\frac{s-1}{2}} \\ \qquad + (K_2 - 1)(C_0 + C_2 \cos(2\varphi + 2\varphi_0))^{\frac{s-3}{2}}) & \text{if } s \text{ is odd}, \end{cases}$$

$$Q_s(\varphi) \begin{cases} K_1(C_2 \cos(\varphi + \varphi_0))^{s+1} + K_2(C_2 \cos(\varphi + \varphi_0))^{s-1} & \text{if } s \text{ is even}, \\ K_1(C_0 + C_2 \cos(2\varphi + 2\varphi_0))^{\frac{s+1}{2}} \\ \qquad + K_2(C_0 + C_2 \cos(2\varphi + 2\varphi_0))^{\frac{s-1}{2}} & \text{if } s \text{ is odd}, \end{cases}$$

where $K_1 = \frac{1-s}{2C_1}$ *and* $K_2 = \frac{1+\alpha+(1-\alpha)s}{2}$, *it is integrable and the inverse integrating factor is* $V(R,\varphi) = (1 + C_2^{s-1}\cos^{s-1}(\varphi+\varphi_0)R)^\alpha$ *if* s *is even and* $V(R,\varphi) = (1 + (C_0 + C_2\cos(2\varphi+2\varphi_0))^{\frac{s-1}{2}}R)^\alpha$ *if* s *is odd.*

4. Quadratic inverse integrating factors

We now consider the case $p = 2$, in this case system (14) is

$$\begin{aligned} &\alpha_1 g + y\frac{dg}{dy} + \frac{dh}{dy} = 0, \\ &(1-a)h - (a-\alpha_1-\alpha_2)g + (1-y)\frac{dh}{dy} = 0, \end{aligned} \tag{16}$$

or

$$\begin{aligned} &\alpha_1 g + y\frac{dg}{dy} + \frac{dh}{dy} = 0, \\ &(1-a)h = [(a-\alpha_2) - \alpha_1 y]g + y(1-y)\frac{dg}{dy}. \end{aligned} \tag{17}$$

If we differentiate the second equation of (17) respect to y, we have

$$(1-a)\frac{dh}{dy} = -\alpha_1 g + [(a-\alpha_2+1) - (\alpha_1+2)y]\frac{dg}{dh} + y(1-y)\frac{d^2g}{dy^2}.$$

Replacing the value of $\frac{dh}{dy}$ obtained from the first equation of system (17) in the previous expression, we find

$$y(1-y)\frac{d^2g}{dy^2} + [(1+a-\alpha_2) - (1+\alpha_1+a)y]\frac{dg}{dy} - a\alpha_1 g = 0. \tag{18}$$

Relation (18) is a hypergeometrical second order linear differential equation. In [3], the particular case $\alpha_1 = \alpha_2 = \alpha$ and $a - \alpha = 1/2$ was studied. This last relation is verified by certain integrable systems (1) in the quadratic case $s = 2$. In this case if s is odd we obtain reversible systems.

Other important case is $\alpha_1 = \alpha_2 = \alpha$ and $a - \alpha = -1/2$. This last relation is verified by certain integrable systems (1) in the homogeneous cubic case $s = 3$.

Since $\alpha_1 = \alpha_2 = \alpha$ $a - \alpha = -1/2$, equation (18) can be written as

$$y(1-y)\frac{d^2g}{dy^2} + \left[\frac{1}{2} - \left[2a + \frac{3}{2}\right]y\right]\frac{dg}{dy} + a\left[a + \frac{1}{2}\right]g = 0.$$

The general solution of this equation is given by

$$g(y) = \frac{1}{2}\left[C_1(1+\sqrt{y})^{-2a} + C_2(1-\sqrt{y})^{-2a}\right],$$

where C_1 and C_2 are arbitrary constants. We will have for this solution of $g(y)$ that

$$h(y) = C_1\frac{(1+2a\sqrt{y}+y)}{2(a-1)}(1+\sqrt{y})^{-2a} + C_2\frac{(1-2a\sqrt{y}+y)}{2(a-1)}(1-\sqrt{y})^{-2a}.$$

By going back throughout the change of variables it is easy to see that

$$G(x_1,x_2) = C_1\left(\sqrt{x_1}+\sqrt{x_2}\right)^{-2a} + C_2\left(\sqrt{x_1}-\sqrt{x_2}\right)^{-2a},$$

and

$$H(x_1,x_2) = C_1\frac{(x_1+2a\sqrt{x_1x_2}+x_2)}{2(a-1)}\left(\sqrt{x_1}+\sqrt{x_2}\right)^{-2a}$$
$$C_2\frac{(x_1-2a\sqrt{x_1x_2}+x_2)}{2(a-1)}\left(\sqrt{x_1}-\sqrt{x_2}\right)^{-2a}.$$

Therefore

$$U_1(x_1,x_2,z) = \left(\sqrt{x_1}+\sqrt{x_2}\right)^{-2a}\left[\frac{x_1+2a\sqrt{x_1x_2}+x_2}{2(a-1)}+z\right],$$

and

$$U_2(x_1,x_2,z) = \left(\sqrt{x_1}-\sqrt{x_2}\right)^{-2a}\left[\frac{x_1-2a\sqrt{x_1x_2}+x_2}{2(a-1)}+z\right],$$

are two independent functions of the form (12) for system (8), which can be written into the from

$$U_1(x_1,x_2,z) = (x_1+x_2+2\sqrt{x_1x_2})^{-a}\left[\frac{x_1+2a\sqrt{x_1x_2}+x_2}{2(a-1)}+z\right],$$
$$U_2(x_1,x_2,z) = (x_1+x_2-2\sqrt{x_1x_2})^{-a}\left[\frac{x_1-2a\sqrt{x_1x_2}+x_2}{2(a-1)}+z\right].$$

As $V_1 = x_1+x_2$, $V_2 = x_1x_2$, $z = Q_s$ and $a = \frac{s+1}{s-1}$ we can write

$$U_1(V_1,V_2,z) = \left(V_1+2\sqrt{V_2}\right)^{-a}\left[\frac{V_1+2a\sqrt{V_2}}{2(a-1)}+Q_s\right],$$
$$U_2(V_1,V_2,z) = \left(V_1-2\sqrt{V_2}\right)^{-a}\left[\frac{V_1-2a\sqrt{V_2}}{2(a-1)}+Q_s\right].$$

that is,

$$(19)\qquad \begin{aligned} \left[\frac{V_1 + 2a\sqrt{V_2}}{2(a-1)} + Q_s\right] &= K_1\left(V_1 + 2\sqrt{V_2}\right)^a, \\ \left[\frac{V_1 - 2a\sqrt{V_2}}{2(a-1)} + Q_s\right] &= K_2\left(V_1 - 2\sqrt{V_2}\right)^a, \end{aligned}$$

where K_1 and K_2 are arbitrary constants.

By multiplying the two previous equations, we have

$$\frac{V_1^2 - 4a^2V_2}{4(a-1)^2} + \frac{QV_1}{a-1} + Q^2 = K_1K_2(V_1^2 - 4V_2)^a.$$

As V_1 and V_2 are homogeneous trigonometrical polynomials of degrees $(s-1)$ and $2(s-1)$, respectively, then $V_1^2 - 4V_2$ is a trigonometrical polynomial of degree $2(s-1)$. As $a = \frac{s+1}{s-1}$ if s is even, then $V_1^2 - 4V_2$ must be a homogeneous trigonometric polynomial of second degree elevated to the power of $s-1$ and if s is odd, then $V_1^2 - 4V_2$ must be a homogeneous trigonometric polynomial of fourth degree elevated to the power of $\frac{s-1}{2}$. Hence, as $V_1^2 - 4V_2 = (V_1 + 2\sqrt{V_2})(V_1 - 2\sqrt{V_2})$, we can take $V_1 + 2\sqrt{V_2}$ and $V_1 - 2\sqrt{V_2}$ homogenous trigonometric polynomials of first degree elevated to the power of $s-1$ if s is even an homogeneous trigonometric polynomials of second degree elevated to the power of $\frac{s-1}{2}$ if s is odd. Therefore we have $V_1 + 2\sqrt{V_2} = A^{s-1}$ and $V_1 - 2\sqrt{V_2} = B^{s-1}$ where $A = a_1 \sin\varphi + b_1 \cos\varphi$, and $B = a_2 \sin\varphi + b_2 \cos\varphi$, if s is even, and we have $V_1 + 2\sqrt{V_2} = C^{\frac{s-1}{2}}$ and $V_1 - 2\sqrt{V_2} = D^{\frac{s-1}{2}}$ where $C = a_1 \sin 2\varphi + b_1 \cos 2\varphi + c_1$ and $D = a_2 \sin 2\varphi + b_2 \cos 2\varphi + c_2$ if s is odd, then we have

$$V_1 = \tfrac{A^{s-1}+B^{s-1}}{2} \quad \text{and} \quad \sqrt{V_2} = \tfrac{A^{s-1}-B^{s-1}}{4}, \quad \text{if } s \text{ is even},$$

$$V_1 = \tfrac{C^{\frac{s-1}{2}}+D^{\frac{s-1}{2}}}{2} \quad \text{and} \quad \sqrt{V_2} = \tfrac{C^{\frac{s-1}{2}}-D^{\frac{s-1}{2}}}{4}, \quad \text{if } s \text{ is odd}.$$

Subtracting the equations of system (19) and using the previous expressions we can write the following conditions

$$(20)\qquad \begin{aligned} \frac{a}{2(a-1)}\left(A^{s-1} - B^{s-1}\right) &= \left(K_1A^{s+1} - K_2B^{s+1}\right), \quad \text{if } s \text{ is even}, \\ \frac{a}{2(a-1)}\left(C^{\frac{s-1}{2}} - D^{\frac{s-1}{2}}\right) &= \left(K_1C^{\frac{s+1}{2}} - K_2D^{\frac{s+1}{2}}\right), \quad \text{if } s \text{ is odd}. \end{aligned}$$

We can write these conditions into the form

$$
(21)\quad
\begin{aligned}
&A^{s-1}\left[\frac{a}{2(a-1)}-K_1A^2\right]=B^{s-1}\left[\frac{a}{2(a-1)}-K_2B^2\right], && \text{if } s \text{ is even},\\
&C^{\frac{s-1}{2}}\left[\tfrac{a}{2(a-1)}-K_1C\right]=D^{\frac{s-1}{2}}\left[\tfrac{a}{2(a-1)}-K_2D\right], && \text{if } s \text{ is odd}.
\end{aligned}
$$

The first condition of (21) is only satisfied for $s = 2$ if A do not divide B, in the case that A divide B we have that $A = B$ and in this case $V_2 = 0$ and we obtain particular cases of Theorem 2.

The second condition is only satisfied for $s = 3$ and $s = 5$ if C do not divide D, in the case that C divides D we have that $C = D$ and in this case $V_2 = 0$, and we obtain particular cases of Theorem 2. Using the condition for $s = 3$ we obtain the integrable homogeneous cubic system which verifies $a - \alpha = -1/2$. In the case $s = 5$ C and D have a linear common factor.

Remark 1. Let $a - \alpha = -1/2$, for $s = 3$ we have $\alpha = 5/3$ from condition (21) we obtain

$$C = -\frac{b_1^2\sqrt{a_2^2+b_2^2}}{a_2\sqrt{a_2^2+b_1^2}} - \frac{b_1b_2}{a_2}\cos 2\varphi + b_1\sin 2\varphi$$

and

$$D = \frac{a_2\sqrt{a_2^2+b_2^2}}{\sqrt{a_2^2+b_1^2}} + a_2\cos 2\varphi + b_2\sin 2\varphi.$$

In this case $V_1 = \frac{C+D}{2}$, $V_2 = \frac{(C+D)^2}{16}$, $Q = K_1C^2 - \frac{V_1+4V_2}{2}$ and $P = \frac{5}{8}\frac{dV_1}{d\varphi} - \frac{1}{4}\frac{dQ}{d\varphi}$. Taking $a_2 = b_2 = \sin t, b_1 = \cos t, t = T - \frac{\pi}{4}$ and $\varphi = \psi + \frac{\pi}{4}$ we obtain the following system

$$P(\psi) = \cos 4\psi + 2\cos(2\psi + T), \quad Q(\psi) = -\sin 4\psi + \sin(2\psi + T),$$

which corresponds to integrable case (iii) of [2].

Remark 2. Let $a - \alpha = -1/2$, for $s = 5$ we have $\alpha = 2$ from condition (21) we obtain

$$C = \frac{\sqrt{b_1^2 - b_1b_2 + b_2^2}}{\sqrt{3}} + \frac{b_1 - 2b_2}{\sqrt{3}}\cos 2\varphi + b_1\sin 2\varphi$$

and

$$D = -\frac{\sqrt{b_1^2 - b_1 b_2 + b_2^2}}{\sqrt{3}} + \frac{2b_1 - b_2}{\sqrt{3}} \cos 2\varphi + b_2 \sin 2\varphi.$$

In this case $V_1 = \frac{C^2+D^2}{2}$, $V_2 = \frac{(C^2+D^2)^2}{16}$, $Q = K_1 C^3 - (V_1 + 3V_2)$ and $P = \frac{1}{3}\frac{dV_1}{d\varphi} - \frac{1}{6}\frac{dQ}{d\varphi}$. Taking $b_1 = \frac{\cos t}{2} + \frac{\sin t}{2\sqrt{3}}$, $b_2 = \frac{\sin t}{\sqrt{3}}$, $\varphi = \psi + \frac{t}{2} + \frac{\pi}{3}$ we obtain the following system

$$\begin{aligned} P(\psi) &= -\sin 6\psi + \sin 4\psi - \sin 2\psi, \\ Q(\psi) &= -\cos 6\psi - \frac{1}{2}\cos 4\psi + \cos 2\psi + \frac{1}{2}, \end{aligned} \tag{22}$$

which is integrable and the inverse integrating factor is

$$V(r,\psi) = \left[1 + \frac{1}{3}\left(\cos^2\psi(2-\cos 2\psi)\right) r^4 + \frac{1}{3}(\cos^6\psi \sin^2\psi) r^8\right]^2.$$

In cartesian coordinates system (22) takes the form

$$\dot{x} = \frac{y}{3}(3 + x^4), \qquad \dot{y} = \frac{x}{3}(3 + 4x^2y^2 - 3y^4),$$

which is a time-reversible system (see [11]) because it is invariant under the transformation $x \to -x$, $y \to y$, and $t \to -t$.

5. Cubic inverse integrating factors

We now consider the case $p = 3$; in this case system (14) is

$$\begin{aligned} &\alpha_1 g + y_1 \frac{\partial g}{\partial y_1} + \frac{\partial h}{\partial y_1} = 0, \\ &\alpha_2 g + y_2 \frac{\partial g}{\partial y_2} + \frac{\partial h}{\partial y_2} = 0, \\ &(1-a)h - (a - \alpha_1 - \alpha_2 - \alpha_3)g + (1-y_1)\frac{\partial h}{\partial y_1} + (1-y_2)\frac{\partial h}{\partial y_2} = 0. \end{aligned} \tag{23}$$

If we differentiate equations (23) respect to y_1 and y_2 up to third order, we obtain 18 linear relations, which are not independent, in the 20 variables

$$h, g, \frac{\partial h}{\partial y_1}, \frac{\partial h}{\partial y_2}, \frac{\partial g}{\partial y_1}, \frac{\partial g}{\partial y_2}, \frac{\partial^2 h}{\partial y_1^2}, \frac{\partial^2 h}{\partial y_1 y_2}, \frac{\partial^2 h}{\partial y_2^2}, \frac{\partial^2 g}{\partial y_1^2}, \frac{\partial^2 g}{\partial y_1 y_2}, \frac{\partial^2 g}{\partial y_2^2},$$
$$\frac{\partial^3 h}{\partial y_1^3}, \frac{\partial^3 h}{\partial y_1^2 y_2}, \frac{\partial^3 h}{\partial y_1 y_2^2}, \frac{\partial^3 h}{\partial y_2^3}, \frac{\partial^3 g}{\partial y_1^3}, \frac{\partial^3 g}{\partial y_1^2 y_2}, \frac{\partial^3 g}{\partial y_1 y_2^2} \text{ and } \frac{\partial^3 g}{\partial y_2^3}.$$

We can eliminate all the variables except h, $\frac{\partial h}{\partial y_1}$, $\frac{\partial^2 h}{\partial y_1^2}$, and $\frac{\partial^3 h}{\partial y_1^3}$ which are leagued by the following partial differential equation of third order

$$
\begin{aligned}
(24)\quad & \frac{\partial^3 h}{\partial y_1^3} + \left[\frac{1+\alpha_1+\alpha_3}{y_1-1} + \frac{1+a-\alpha_2-\alpha_3}{y_1} + \frac{1+\alpha_1+\alpha_2}{y_1-y_2}\right]\frac{\partial^2 h}{\partial y_1^2} \\
& + (1+\alpha_1)\left[\frac{a-\alpha_2}{y_1(y_1-1)} + \frac{\alpha_3-a}{y_1(y_2-y_1)} - \frac{\alpha_1+\alpha_2+\alpha_3}{(y_1-1)(y_2-y_1)}\right]\frac{\partial h}{\partial y_1} \\
& + \frac{\alpha_1(1+\alpha_1)(a-1)}{y_1(1-y_1)(y_2-y_1)}h = 0.
\end{aligned}
$$

In this partial differential equation does not appear partial differentials respect to the variable y_2 does not appear, therefore we can consider the equation (24) as a linear differential equation respect to the variable y_1. Using the equation (24) we are going to find some integrable cases that have a cubic inverse integrating factor.

First we will study an integrable system in the quadratic case $s = 2$, which verifies $a = 3$ and $\alpha_1 = \alpha_2 = \alpha_3 = \frac{5}{3}$ (see [2]). In this case equation (24) takes the form

$$
\begin{aligned}
(25)\quad & 9y_1(y_1-1)(y_2-y_1)\frac{d^3 h}{dy_1^3} + 3(15y_1 - 28y_1^2 - 2y_2 + 15y_1y_2)\frac{d^2 h}{dy_1^2} \\
& + 8(4 - 23y_1 + 4y_2)\frac{dh}{dy_1} - 80h = 0.
\end{aligned}
$$

Integrating twice the previous expression and making the change $\frac{dp}{dy_1} = h$, we have

$$
\begin{aligned}
(26)\quad & 9y_1(y_1-1)(y_2-y_1)\frac{d^2 p}{dy_1^2} + 3(3y_1 - 10y_1^2 + 4y_2 + 3y_1y_2)\frac{dp}{dy_1} \\
& - (4 + 10y_1 - 4y_2)p = 0.
\end{aligned}
$$

If we make the change of variable $z = \frac{y_1(1-y_2)}{y_1-y_2}$ and the change of function $p(z) = \frac{(1-y_2-z)^{2/3}}{(1-z)^{4/3}} f(z)$, equation (26) results

$$
(27)\qquad z(1-z)\frac{d^2 f}{dz^2} + \left(-\frac{4}{3} + \frac{5}{3}z\right)\frac{df}{dz} - \frac{4}{3}f = 0.
$$

The relation (27) is a hypergeometrical second order linear differential equation which has a polynomial particular solution which allow us to make the change $f(z) = (1 - z - \frac{z^2}{2})w(z)$. By using this change expression (27) takes the form

$$(28) \qquad (8 - 6z + 6z^2 - 7z^3)\frac{dw}{dz} + 3z(z-1)(2 - 2z - z^2)\frac{d^2w}{dz^2} = 0.$$

From (28) we obtain

$$w(z) = \frac{(z-2)(z-1)^{1/3}z^{1/3}}{6(-2+2z+z^2)} + \frac{1}{18}\int (z(z-1))^{-2/3}dz$$

and

$$p(z) = \frac{(1 - y_2 - z)^{2/3}(1 - z - \frac{z^2}{2})}{(1-z)^{4/3}} \cdot \left[\frac{(z-2)(z-1)^{1/3}z^{1/3}}{6(-2+2z+z^2)} + \frac{1}{18}\int (z(z-1))^{-2/3}dz\right].$$

Using this particular solution and making a change of variables in equation (25) we finally obtain the three independent solutions of equation (25). By going back throughtout all the changes of variables, these three independent solutions are

$$h_1(y_1, y_2) = \frac{A_1(y_1, y_2)}{((y_1 - 1)(y_2 - y_1)(y_2 - 1))^{\frac{7}{3}}},$$

$$h_2(y_1, y_2) = \frac{(y_1y_2)^{\frac{1}{3}}A_2(y_1, y_2)}{18(1-y_1)^2(y_2-y_1)^2(y_2-1)^2} - \frac{y_2^{\frac{1}{3}}A_1(y_1, y_2)}{54(y_2-1)^2(y_1-1)^{\frac{7}{3}}(y_2-y_1)^{\frac{7}{3}}}\int (y_1(y_1-1)(y_2-y_1))^{-\frac{2}{3}}dy_1,$$

$$h_3(y_1, y_2) = \frac{A_3(y_1, y_2)}{10(1-y_1)^2(y_2-y_1)^2(y_2-1)^2} - \frac{A_1(y_1, y_2)}{60(y_2-1)^2(y_1-1)^{\frac{7}{3}}(y_2-y_1)^{\frac{7}{3}}}\int ((y_1-1)(y_2-y_1))^{-\frac{2}{3}}dy_1,$$

where

$$A_1(y_1, y_2) = -y_1^2 - y_1^3 + 4y_1y_2 - 2y_1^2y_2 + 4y_1^3y_2 - y_2^2 - 2y_1y_2^2 - 2y_1^2y_2^2 - y_1^3y_2^2 - y_2^3 + 4y_1y_2^3 - y_1^3y_2^3,$$

$$A_2(y_1, y_2) = y_1 + y_1^2 + y_2 - 6y_1y_2 + y_1^2y_2 + y_2^2 + y_1y_2^2,$$

$$A_3(y_1, y_2) = y_1^2 - y_1y_2 - y_1^2y_2 + y_2^2 - y_1y_2^2 + y_1^2y_2^2,$$

and from equations (23) we have

$$
\begin{aligned}
g_1(y_1,y_2) &= \frac{B_1(y_1,y_2)}{((y_1-1)(y_2-y_1)(y_2-1))^{\frac{7}{3}}},\\
g_2(y_1,y_2) &= \frac{(y_1y_2)^{\frac{1}{3}}B_2(y_1,y_2)}{9(y_1-1)^2(y_2-y_1)^2(y_2-1)^2}\\
&\quad - \frac{y_2^{\frac{1}{3}}B_1(y_1,y_2)}{54(y_2-1)^2(y_1-1)^{\frac{7}{3}}(y_2-y_1)^{\frac{7}{3}}}\int (y_1(y_1-1)(y_2-y_1))^{-\frac{2}{3}}dy_1,\\
g_3(y_1,y_2) &= \frac{B_3(y_1,y_2)}{20(y_1-1)^2(y_2-y_1)^2(y_2-1)^2}\\
&\quad - \frac{B_1(y_1,y_2)}{60(y_2-1)^2(y_1-1)^{\frac{7}{3}}(y_2-y_1)^{\frac{7}{3}}}\int ((y_1-1)(y_2-y_1))^{-\frac{2}{3}}dy_1,
\end{aligned}
$$

where

$$
\begin{aligned}
B_1(y_1,y_2) &= -y_1+4y_1^2-y_1^3-y_2-2y_1y_2-2y_1^2y_2-y_1^3y_2+4y_2^2-2y_1y_2^2\\
&\quad +4y_1^2y_2^2-y_2^3-y_1y_2^3,\\
B_2(y_1,y_2) &= -1+y_1-y_1^2+y_2+y_1y_2-y_2^2,\\
B_3(y_1,y_2) &= -y_1-y_1^2-y_2+6y_1y_2-y_1^2y_2-y_2^2-y_1y_2^2.
\end{aligned}
$$

If we recall that $y_1=\frac{x_2}{x_1}$ and $y_2=\frac{x_3}{x_1}$ and we take $V_1=x_1+x_2+x_3$, $V_2=x_1x_2+x_1x_3+x_2x_3$ and $V_3=x_1x_2x_3$, the functions H and G are

$$
\begin{aligned}
H_1(V_1,V_2,V_3) &= \frac{6V_1^2V_3-V_1V_2^2-9V_2V_3}{D^{\frac{7}{6}}},\\
G_1(V_1,V_2,V_3) &= \frac{6V_2^2-V_1^2V_2-9V_1V_3}{D^{\frac{7}{6}}},\\
H_2(V_1,V_2,V_3) &= \frac{3V_3^{\frac{1}{3}}(V_1V_2-9V_3)}{D}-H_1(V_1,V_2,V_3)L_1,\\
G_2(V_1,V_2,V_3) &= \frac{6V_3^{\frac{1}{3}}(3V_2-V_1^2)}{D}-G_1(V_1,V_2,V_3)L_1,
\end{aligned}
$$

where $L_1=(y_2(y_2-1))^{\frac{1}{3}}\int (y_1(y_1-1)(y_2-y_1))^{-\frac{2}{3}}\,dy_1$ and

$$
\begin{aligned}
H_3(V_1,V_2,V_3) &= \frac{6(V_2^2-3V_1V_3)}{D}-H_1(V_1,V_2,V_3)L_2,\\
G_3(V_1,V_2,V_3) &= \frac{3(9V_3-V_1V_2)}{D}-G_1(V_1,V_2,V_3)L_2,
\end{aligned}
$$

where $L_2 = (y_2-1)^{\frac{1}{3}} \int ((y_1-1)(y_2-y_1))^{-\frac{2}{3}} dy_1$, where $D = V_1^2V_2^2 - 4V_2^3 - 4V_1^3V_3 + 18V_1V_2V_3 - 27V_3^2$.

We have checked that the values of V_1, V_2 and V_3 of the quadratic integrable case (case (iii) in appendix of [2]) verify that functions (12) are constants, that is, $H_1 + Q_2G_1 = K_1$, $H_2 + Q_2G_2 = K_2$ and $H_3 + Q_2G_3 = K_3$. In this case

$$\begin{aligned} V_1 &= 6R_3 \sin(\varphi+\varphi_1),\\ V_2 &= 12R_3^2 \cos\left(\frac{\varphi_3-3\varphi_1}{2}\right)\sin\left(\varphi+\frac{\varphi_3-\varphi_1}{2}\right)\sin(\varphi+\varphi_1),\\ V_3 &= 8R_3^3 \sin\left(\frac{\varphi_3-3\varphi_1}{2}\right)\cos^3\left(\varphi+\frac{\varphi_3-\varphi_1}{2}\right), \end{aligned}$$

and it is easy to see that

$$\begin{aligned} K_1 &= 0,\\ K_2 &= \frac{1}{4R_3^2}\cos^{\frac{2}{3}}\left(\frac{\varphi_3-3\varphi_1}{2}\right)\sin^{-2}\left(\frac{\varphi_3-3\varphi_1}{2}\right) \quad \text{and}\\ K_3 &= \frac{1}{16R_3^2}. \end{aligned}$$

Second we will study a integrable system in the quadratic case $s = 3$, which verifies $a = 2$ and $\alpha_1 = \alpha_2 = \alpha_3 = \frac{5}{3}$ (see [2]). In this case equation (24) takes the form

$$\begin{aligned} (29)\quad 9y_1(y_1-1)(y_2-y_1)\frac{d^3h}{dy_1^3} &+ 3(12y_1 - 25y_1^2 + y_2 + 12y_1y_2)\frac{d^2h}{dy_1^2}\\ &+ 8(1-17y_1+y_2)\frac{dh}{dy_1} - 40h = 0. \end{aligned}$$

Integrating the previous expression we have

$$\begin{aligned} (30)\quad 9y_1(y_1-1)(y_2-y_1)\frac{d^2h}{dy_1^2} &+ 6(3y_1 - 8y_1^2 + 2y_2 + 3y_1y_2)\frac{dh}{dy_1}\\ &- 10(1+4y_1+y_2)h = 0. \end{aligned}$$

If we make the change of variable $z = \frac{y_1(1-y_2)}{y_1-y_2}$ and the change of function $h(z) = \frac{(1-y_2-z)^{\frac{5}{3}}}{(z-1)^{\frac{7}{3}}} f(z)$, equation (30) results

$$(31)\qquad z(1-z)\frac{d^2f}{dz^2} + \left(-\frac{4}{3}+\frac{8}{3}z\right)\frac{df}{dz} - 2f = 0.$$

Relation (31) is a hypergeometrical second order linear differential equation which has a polynomial particular solution given by $f(z) = (2-3z-3z^2+2z^3)$. From (29) we find three independent particular solution which, by going back throughout all the changes of variables, have the form

$$
\begin{aligned}
h_1(y_1,y_2) &= \frac{C_1(y_1,y_2)}{((y_1-1)(y_2-y_1)(y_2-1))^{\frac{7}{3}}},\\
h_2(y_1,y_2) &= \frac{6(y_1y_2)^{\frac{1}{3}}C_2(y_1,y_2)}{(y_1-1)^2(y_2-y_1)^2(y_2-1)^2}\\
&\quad + \frac{y_2^{\frac{1}{3}}C_1(y_1,y_2)}{(y_2-1)^2(y_1-1)^{\frac{7}{3}}(y_2-y_1)^{\frac{7}{3}}}\int (y_1(y_1-1)(y_2-y_1))^{-\frac{2}{3}}\,dy_1,\\
h_3(y_1,y_2) &= \frac{3C_3(y_1,y_2)}{(y_1-1)^2(y_2-y_1)^2(y_2-1)^2}\\
&\quad + \frac{C_1(y_1,y_2)}{(y_2-1)^2(y_1-1)^{\frac{7}{3}}(y_2-y_1)^{\frac{7}{3}}}\int ((y_1-1)(y_2-y_1))^{-\frac{2}{3}}\,dy_1,
\end{aligned}
$$

where

$$
\begin{aligned}
C_1(y_1,y_2) &= (y_1-2y_2+y_1y_2)(-2y_1+y_2+y_1y_2)(-y_1-y_2+2y_1y_2),\\
C_2(y_1,y_2) &= y_1^2-y_1y_2-y_1^2y_2+y_2^2-y_1y_2^2+y_1^2y_2^2,\\
C_3(y_1,y_2) &= y_1^2+y_1^3-2y_1y_2-2y_1^3y_2+y_2^2+y_1^3y_2^2+y_2^3-2y_1y_2^3+y_1^2y_2^3,
\end{aligned}
$$

and from equations (23) we have

$$
\begin{aligned}
g_1(y_1,y_2) &= \frac{D_1(y_1,y_2)}{((y_1-1)(y_2-y_1)(y_2-1))^{\frac{7}{3}}}\\
g_2(y_1,y_2) &= \frac{3(y_1y_2)^{\frac{1}{3}}D_2(y_1,y_2)}{9(y_1-1)^2(y_2-y_1)^2(y_2-1)^2}\\
&\quad + \frac{y_2^{\frac{1}{3}}D_1(y_1,y_2)}{(y_2-1)^2(y_1-1)^{\frac{7}{3}}(y_2-y_1)^{\frac{7}{3}}}\int (y_1(y_1-1)(y_2-y_1))^{-\frac{2}{3}}dy_1,\\
g_3(y_1,y_2) &= \frac{6D_3(y_1,y_2)}{(y_1-1)^2(y_2-y_1)^2(y_2-1)^2}\\
&\quad + \frac{D_1(y_1,y_2)}{(y_2-1)^2(y_1-1)^{\frac{7}{3}}(y_2-y_1)^{\frac{7}{3}}}\int ((y_1-1)(y_2-y_1))^{-\frac{2}{3}}dy_1,
\end{aligned}
$$

where

$$
\begin{aligned}
D_1(y_1,y_2) &= -y_1^2 - y_1^3 + 4y_1y_2 - 2y_1^2y_2 + 4y_1^3y_2 - y_2^2 - 2y_1y_2^2 - 2y_1^2y_2^2 - y_1^3y_2^2 \\
&\quad - y_2^3 + 4y_1y_2^3 - y_1^2y_2^3, \\
D_2(y_1,y_2) &= -y_1 + y_1^2 - y_2 + 6y_1y_2 - y_1^2y_2 - y_2^2 - y_1y_2^2, \\
D_3(y_1,y_2) &= -y_1^2 + y_1y_2 + y_1^2y_2 - y_2^2 + y_1y_2^2 - y_1^2y_2^2.
\end{aligned}
$$

If we recall that $y_1 = \frac{x_2}{x_1}$ and $y_2 = \frac{x_3}{x_1}$ and we take $V_1 = x_1 + x_2 + x_3$, $V_2 = x_1x_2 + x_1x_3 + x_2x_3$ and $V_3 = x_1x_2x_3$, the functions H and G are

$$
\begin{aligned}
H_1(V_1,V_2,V_3) &= \frac{2V_2^3 - 9V_1V_2V_3 + 27V_3^2}{D^{\frac{7}{6}}}, \\
G_1(V_1,V_2,V_3) &= \frac{6V_3V_1^2 - 9V_3V_2 - V_1V_2^2}{D^{\frac{7}{6}}}, \\
H_2(V_1,V_2,V_3) &= \frac{6V_3^{\frac{1}{3}}(V_2^2 - 3V_1V_3)}{D} + H_1(V_1,V_2,V_3)L_1, \\
G_2(V_1,V_2,V_3) &= \frac{3V_3^{\frac{1}{3}}(9V_3 - V_1V_2)}{D} + G_1(V_1,V_2,V_3)L_1,
\end{aligned}
$$

where $L_1 = (y_2(y_2-1))^{\frac{1}{3}} \int (y_1(y_1-1)(y_2-y_1))^{-\frac{2}{3}} dy_1$ and

$$
\begin{aligned}
H_3(V_1,V_2,V_3) &= \frac{3V_1V_2^2 - 12V_1^2V_3 + 9V_2V_3}{D} - H_1(V_1,V_2,V_3)L_2, \\
G_3(V_1,V_2,V_3) &= \frac{18V_1V_3 - 6V_2^2}{D} - G_1(V_1,V_2,V_3)L_2,
\end{aligned}
$$

where $L_2 = (y_2-1)^{\frac{1}{3}} \int ((y_1-1)(y_2-y_1))^{-\frac{2}{3}} dy_1$, where $D = V_1^2V_2^2 - 4V_2^3 - 4V_1^3V_3 + 18V_1V_2V_3 - 27V_3^2$.

We have checked that the values of V_1, V_2 and V_3 of the cubic homogeneous integrable case (case (iii) in [2]) verify that functions (12) are constants, that is, $H_1 + Q_3G_1 = K_1$, $H_2 + Q_3G_2 = K_2$ and $H_3 + Q_3G_3 = K_3$. In this case

$$
\begin{aligned}
V_1 &= 3R_4(\sin^{-1}(\varphi_4 - 2\varphi_2) + \sin(2\varphi + \varphi_2)), \\
V_2 &= 12R_4^2 \sin^3(\varphi + \frac{\varphi_4 - \varphi_2}{2}) \cos(\varphi + \frac{3\varphi_2 - \varphi_4}{2}) \sin^{-1}(\varphi_4 - 2\varphi_2), \\
V_3 &= 8R_4^3 \sin^6(\varphi + \frac{\varphi_4 - \varphi_2}{2}) \sin^{-1}(\varphi_4 - 2\varphi_2),
\end{aligned}
$$

and it is easy to see that

$$K_1 = 0,$$
$$K_2 = -\frac{\tan(2\varphi_2 - \varphi_4)}{R_4} \text{ and}$$
$$K_3 = -\frac{\sin^{\frac{5}{3}}(2\varphi_2 - \varphi_4)}{R_4 \cos^2(2\varphi_2 - \varphi_4)}.$$

Finally we will study a integrable system in the quadratic case $s = 5$, which verifies $a = \frac{3}{2}$ and $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = \frac{1}{4}$ (see [5]). In this case equation (24) takes the form

$$4y_1(y_1-1)(y_2-y_1)\frac{d^3h}{dy_1^3} + (17y_1 - 26y_1^2 - 5y_2 + 14y_1y_2)\frac{d^2h}{dy_1^2} + 2(5 - 16y_1 + 2y_2)\frac{dh}{dy_1} - 4h = 0. \tag{32}$$

Integrating twice the previous expression we have

$$4y_1(y_1-1)(y_2-y_1)\frac{dh}{dy_1} + (y_1 - 2y_1^2 + 3y_2 - 2y_1y_2)h = 0. \tag{33}$$

Relation (33) is a first order linear differential equation which has a particular solution given by

$$h(y_1, y_2) = y_1^{\frac{4}{3}}(y_1 - 1)^{-\frac{1}{4}}(y_2 - y_1)^{-1}.$$

From (32) we find only two independent particular solutions which have the form

$$h_1(y_1, y_2) = \frac{y_2^{\frac{3}{4}}}{(y_2-1)^{\frac{1}{4}}(y_2-y_1)},$$
$$h_2(y_1, y_2) = \frac{y_1^{\frac{3}{4}}}{(y_1-1)^{\frac{1}{4}}(y_2-y_1)},$$

and from equations (23) we have

$$g_1(y_1, y_2) = \frac{1}{(y_2-1)^{\frac{1}{4}}y_2^{\frac{1}{4}}(y_2-y_1)},$$
$$g_2(y_1, y_2) = \frac{1}{(y_1-1)^{\frac{1}{4}}y_1^{\frac{1}{4}}(y_2-y_1)}.$$

Since $y_1 = \frac{x_2}{x_1}$ and $y_2 = \frac{x_3}{x_1}$, the functions H and G are

$$H_1(x_1,x_2,x_3) = \frac{x_3^{\frac{3}{4}}}{(x_2-x_3)(x_3-x_1)^{\frac{1}{4}}},$$
$$G_1(x_1,x_2,x_3) = \frac{1}{(x_2-x_3)(x_3-x_1)^{\frac{1}{4}}x_3^{\frac{1}{4}}},$$
$$H_2(x_1,x_2,x_3) = \frac{x_3^{\frac{3}{4}}}{(x_3-x_2)(x_2-x_1)^{\frac{1}{4}}},$$
$$G_2(x_1,x_2,x_3) = \frac{1}{(x_3-x_2)(x_2-x_1)^{\frac{1}{4}}x_2^{\frac{1}{4}}}.$$

In this case the inverse integrating factor is $V(R,\varphi) = (1+W_1R+W_2R^2)(1+U_1R)^{\frac{1}{4}}$ where $W_1 = x_2+x_3$, $W_2 = x_2x_3$ and $U_1 = x_1$. It is difficult and not interesting to find the fuctions H and G in function of W_1, W_2 and U_1. We know that these functions verify $H_1+Q_5G_1 = K_1$ and $H_2+Q_5G_2 = K_2$ where K_1 and K_2 are constants. It is easier to use these two algebraic combinations $K_1K_2 = (H_1+Q_5G_1)(H_2+Q_5G_2)$ and $K_1^4+K_2^4 = (H_1+Q_5G_1)^4+(H_2+Q_5G_2)^4$ in order to find their expression in function of W_1, W_2 and U_1, and we obtain

$$(34) \qquad K_1K_2 = \frac{Q_5^2 - Q_5W_1 + W_2}{(W_1^2-4W_2)(W_2(W_2-U_1W_1+U_1^2))^{\frac{1}{4}}},$$

$$\begin{aligned} K_1^4+K_2^4 &= \frac{Q_5^4(W_1^2-2W_2-U_1W_1)+Q_5^3(4W_2(2U_1-W_1)}{W_2(W_2-U_1W_1+U_1^2)(W_1^2-4W_2)^2} \\ &+ \frac{Q_5^2(6W_2(2W_2-U_1W_1))+Q_5(4W_2(U_1(W_1^2-2W_2)-W_2W_1))}{W_2(W_2-U_1W_1+U_1^2)(W_1^2-4W_2)^2} \\ &+ \frac{W_2(W_2(W_1^2-2W_2)-U_1W_1(W_1^2-3W_2))}{W_2(W_2-U_1W_1+U_1^2)(W_1^2-4W_2)^2}. \end{aligned}$$

We have checked that the values of U_1, W_1 and W_2 of the integrable case (case (iii) in [5]) verify the functions (34) are constants, in this case

$$\begin{aligned} U_1 &= 4R_6\sin^2(2\varphi+\frac{\varphi_4}{2}+\frac{\pi}{4}), \\ W_1 &= 2R_6(1+2\sin(2\varphi+\varphi_2)+\sin(4\varphi+\varphi_4)), \\ W_2 &= 4R_6^2\sin^2(2\varphi+\frac{\varphi_4}{2}+\frac{\pi}{4})\left[\cos(\varphi+\frac{\varphi_2}{2})+\sin(\varphi+\frac{\varphi_2}{2})\right]^4, \end{aligned}$$

and it is easy to see that

$$K_1K_2 = \frac{1}{2\sqrt{2}R_6\left[\sin(\varphi_2 - \frac{\varphi_4}{2}) - \cos(\varphi_2 - \frac{\varphi_4}{2})\right]} \quad \text{and}$$

$$K_1^4 + K_2^4 = \frac{1}{4R_6^2(1 - \sin(2\varphi_2 - \varphi_4))}.$$

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