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Restricted Three-Body Problems and the Non-Regularization of the Elliptic Collision Restricted Isosceles Three-Body Problem

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1. Introduction

The main problem in Celestial Mechanics is the so called *n-body problem*, i.e. to describe the motion of n point particles of positive mass moving under Newton's law of gravitation when we know their positions and velocities at a given time.

The two-body problem is completely solved in the sense that we can describe explicitly all its solutions. For $n \geq 3$ we have many partial result about the n-body problem, but we are far to understand all its solutions.

A *restricted three-body problem* consists in describing the motion of a massless particle m_3 under the gravitational action of two positive masses m_1 and m_2, called *primaries.* Since m_3 does not perturb the motion of the primaries, then these are moving following a solution γ of the two body problem. Then we call this particular restricted three-body problem the γ restricted three-body problem.

In Section 2 we prove that there are exactly 30 different restricted three-body problems in dimension 1, 2 and 3. At least 10 of these problems have been studied by several authors.

The rest of the paper is dedicated to study the *elliptic collision restricted isosceles three-body problem*, i.e. the restricted three-body problem when the

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two primaries with equal positive masses are moving in a collision elliptic orbit solution of the two-body problem, and the infinitesimal body m_3 is moving on the perpendicular straight line to the line containing the motion of the primaries which pass through the center of mass of the system.

In Section 3 we present the equations of motion of the elliptic collision restricted isosceles three-body problem, and the proof that its solutions either are defined for all time $t \in \mathbb{R}$, or begin or end in triple collision.

The elliptic collision restricted isosceles three-body problem has three different kinds of singularities, namely triple collision, escape to infinity, and escape to infinity with infinite velocity. Since binary collisions are regularizable we do not consider them as singularities. In Sections 4 and 5 we prove that the unique non-regularizable singularity of the collision restricted isosceles three-body problem is the triple collision.

Using the blow-up techniques introduced by McGehee, ElBialy in [16] also proved the non-regularization of the triple collision of the restricted isosceles three-body problem. As far as we know all studies about the regularization or non-regularization of triple collision are based in the McGehee variables, with the exception of the topological surgery regularization introduced by Easton [15]. Our study of the triple collision in the elliptic collision restricted isosceles three-body problem neither use McGehee variables, nor surgery techniques.

2. The restricted three-body problems in dimension 1, 2, 3

We describe all restricted three-body problems in $\mathbb{R}^n$ with $n = 1, 2, 3$.

PROPOSITION 2.1. *There are 13, 10 and 7 restricted three-body problems in dimension 1, 2 and 3 respectively.*

Proof. Let γ be a fixed solution of the two-body problem. We assume that two mass points m_1 and m_2 are describing the solution γ. If γ is not a collision orbit, let Π be the plane containing it. If γ is a collision orbit, let l be the straight line containing it.

First suppose that $m_1 = m_2$. If $\gamma \subset \Pi$, let R be the unique perpendicular straight line to plane Π passing through the center of mass of m_1 and m_2 fixed at the origin of the coordinate system. If $\gamma \subset l$, let R be a straight line perpendicular to l passing through the center of mass of m_1 and m_2 fixed at the origin of the coordinate system. We assume that the mass point $m_3 \approx 0$ is on the straight line R.

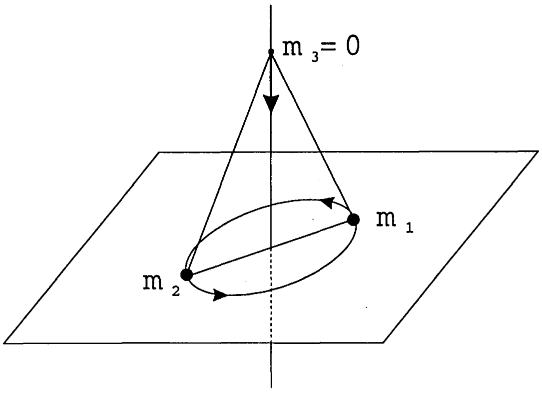

Figure 1: The circular Sitnikov problem.

Due to the symmetric position of the masses $m_1 = m_2$ with respect to the line R, the gravitational force on m_3 is given by a vector containted on the line R. Therfore, if m_3 has its initial velocity on the line R, the In consemotion of m_3 will remain on the line R forever. Thus the motion of m_3 is given by a γ *restricted three-body problem* in dimension 1. We note that the three masses are forming an isosceles triangle which eventually can degenerate to a segment. So these γ restricted three-body problems are known as one-dimensional γ *restricted isosceles three-body problems.*

According with the kind of the orbit γ and being $m_1 = m_2$ we obtain in dimension 1 seven differents restricted isosceles three-body problems: *circular Sitnikov* if γ is a circle [30], [28], [4] (see Figure 1); *elliptic Sitnikov* if γ is an ellipse [30], [1], [28]; *parabolic* if γ is a parabola [14]; *hyperbolic* if γ is a branch of a hyperbola [11]; *elliptic collision* if γ is a collision elliptic orbit [25]; *parabolic collision* if γ is a collision parabolic orbit [12]; and *hyperbolic collision* if γ is a collision hyperbolic orbit [26], [13].

The restricted circular Sitnikov problem is integrable and its solution are explicitly known, see for example [24] and [4].

In the study of the qualitative dynamic of a three-body problem an important fact is to know where end or start its solutions. Let α be a solution of the three-body problem (restricted or not). If α is defined for all $t \geq 0$ (respectively $t \leq 0$) then the behavior of the particles when time tends to $+\infty$ (respectively $-\infty$) is called the *final evolution* of α. A final evolution is *oscillatory* if the upper limit of the distances between the particles is infinite, while the lower limit is finite.

In 1922-32 Chazy [7], [8], [9] (see also [2]) gave a classification of the final evolutions of the three-body problem with masses $m_i > 0$, getting seven possible kinds of final evolution, and proved that six of these are realizable. However the existence or non-existence of oscillatory final evolutions remained an open question until the 1960's. In 1960 Sitnikov [30] gave heuristic arguments for showing the existence of such orbits for the restricted three-body problem that nowdays is known as the elliptic Sitnikov problem. Alekseev [1] in 1968-69 gave the first proof for the existence of these final evolutions in the elliptic Sitnikov problem ($m_3 \approx 0$) and after he extended it to $m_3 > 0$; that is, to the isosceles three-body problem. In 1973 Moser [28] simplified the Alekseev's proof on the existence of oscillatory final evolutions for the elliptic Sitnikov problem giving a more geometrical proof. Later on Llibre and Simó [21] shown the existence of oscillatory final evolutions in the planar circular restricted three-body problem and after extended this result to the planar three-body problem.

Final evolutions for the restricted three-body problems can be different with respect to final evolutions for the general three-body problem with positive masses. Thus in the hyperbolic collision restricted isosceles three-body problem there exist final evolutions that do not appear in the three-body problem with positive masses. More specifically, Meyer and Wang [26] shown for the hyperbolic collision restricted isosceles three-body problem the existence of orbits with final evolution such that m_1 and m_2 escapes hyperbolically to infinity, while $m_3 \approx 0$ tends to an arbitrary finite distance to the origin.

For the parabolic and hyperbolic restricted three-body problems of dimension 1, it is possible to describe the qualitative global flow (see for example [11], [12], [13], [14]). However for the elliptic (respectively elliptic collision) restricted three-body problem in dimension 1, the motion of the primaries is periodic (respectively periodic after a regularization) and consequently the study of the global flow becomes more difficult. For instance, in these last two problems there are periodic orbits that they do not exist in the parabolic and hyperbolic restricted three-body problems (see [6]).

Let γ be a collision orbit of the two-body problem, and let l be the line which contains it. We choose appropiate units in order that the masses of the primaries m_1 and m_2 become $1-\mu$ and μ respectively, with $\mu \in (0,1)$. We can take $m_3 \approx 0$ on the line l between the primaries, or not. This double option and the fact that γ can be a collision elliptic, parabolic or hyperbolic orbit give place to six *restricted collinear three-body problems*.

If m_1 and m_2 are moving in a collision elliptic orbit and m_3 is not between the primaries, we call it the *restricted elliptic collinear problem.* This problem was studied in [22] and [17].

In short, there are thirteen restricted three-body problems in dimension 1.

Now we consider the restricted three-body problems in dimension 2. Consider that m_3 is moving on the plane Π defined by the γ orbit of the primaries, this motion defines a γ *restricted planar three-body problem.* As γ can be a circular, elliptic, parabolic or a branch from a hyperbola orbit, we have four restricted planar three-body problems.

When m_1 and m_2 are moving around their center of mass in circular (respectively elliptic) orbits, and m_3 moves in the plane defined by m_1 and m_2 we have the *circular (respectively elliptic) restricted planar three-body problem* [32].

The circular restricted planar three-body problem is the most classical and studied restricted three-body problem. The elliptic restricted planar three-body problem is a model used, for instance, for studing the motion of an asteroid $m_3 \approx 0$ inside the solar system just formed by the Sun and Jupiter which are moving in elliptic orbits, see for example [10].

Suppose now that the motion of the primaries is contained in the straight line l. Let Π' be a plane such that contains the line l. Consider that $m_3 \approx 0$ is moving in the plane Π' under the influence of the gravitational attraction of the primaries m_1 and m_2 that are describing the collision orbit $\gamma \subset l$. So the motion of m_3 is given by a γ restricted planar three-body problem. As γ can be an elliptic, parabolic or hyperbolic collision orbit, we get three additonal restricted planar three-body problems.

Let γ be a collision orbit, and let l be the straight line that contains this orbit. We take $m_1 = m_2$ and $m_3 \approx 0$ moving on the perpendicular plane to line l passing through center of mass of the primaries m_1 and m_2. The fact that γ can be an elliptic collision, parabolic or hyperbolic collinear orbit allows us to say that there exist three *restricted planar isosceles three-body problems* (see for example [5]).

In short, there are ten restricted three-body problems in dimension 2.

In dimension 3 we consider that the primaries m_1 and m_2 are moving in a γ orbit solution of the two-body problem and the infinitesimal particle m_3 is moving freely in $\mathbb{R}^3$. As Kepler's problem has seven differents solutions, we have that there are seven *restricted three-body problems* in dimension 3. The circular restricted spacial three-body problems and the elliptic one have been studied for several authors, see for example [32] and [20]. ∎

In summary, we can say that there are thirty restricted three-body problems for dimension ≤ 3.

In this work we will study the global flow of the collision elliptic restricted three-body problem in dimension 1.

3. Equations of motion

We consider two equal mass points $m_1 = m_2$ (called *primaries*) and a third infinitesimal mass point $m_3 \approx 0$ at the vertices of an isosceles triangle, in such way that the infinitesimal mass is on the symmetry axis of the triangle. We suppose that the two primaries are moving under Newton's law of gravitation in an elliptic collision orbit on the x-axis with their center of mass fixed at the origin of coordinates. The mass m_3 is moving under the gravitational attraction of the primaries on the z-axis (the symmetry line of the triangle). Taking the initial velocity vector of $m_3 \approx 0$ on the z-axis, the particles will remain always at the vertices of some isosceles triangle. Since $m_3 \approx 0$ the motion of the primaries is not affected by the third, and from the symmetry of the motion it is clear that the third mass point will rest on the z-axis. The elliptic collision restricted isosceles three-body problem in dimension 1 consists in describing the motion of m_3. In what follows will call it simply *the restricted isosceles three-body problem.*

Let x be the distance between the center of mass and m_1, and let z be the distance of m_3 at the center of mass. So $(x, 0)$ denotes the position of m_1, consequently $(-x, 0)$ denotes that of m_2, and $(0, z)$ the position of m_3. In these coordinates (x, z) the equations of motion are

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\frac{1}{x^2}, \\ \frac{d^2z}{dt^2} &= -\frac{z}{(x^2+z^2)^{3/2}}. \end{aligned} \tag{1}$$

We have taken the units of length, mass and time in such way that $m_1 = m_2 = 1/2$, the time between two consecutive binary collision of the primaries equals to 2π and the gravitational constant equals to 1. Then the energy integral for the primaries is

$$\frac{1}{2}\dot{x}^2 - \frac{1}{x} = h.$$

In order that the primaries describe an elliptic collision solution of the two-body problem we must take $h < 0$. The first equation of (1) defines a vector

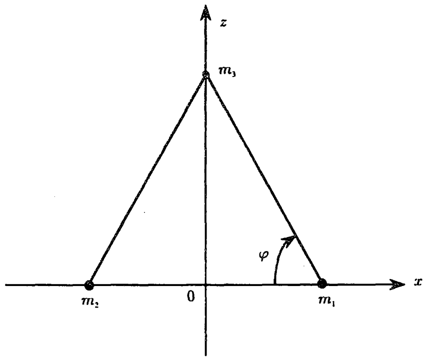

Figure 2: The restricted isosceles three-body problem.

field with a singularity when $x = 0$, due to the binary collision between the primaries. However, this singularity can be removed through a new independent time variable s defined by $dt = xds$. The solution of the differential equation

$$xx'' - x'^2 + x = 0,$$

where $' = d/ds$, gives the position of the primaries explicitly in function of time t through

$$x(E) = \frac{1}{2}(1 - \cos E), \quad t = E - \sin E,$$

where we have a choosen $x(0) = t(0) = 0$, being E the eccentric anomaly.

Note that binary collisions between m_1 and m_2 are only possible when $E = 0 \pmod{2\pi}$ and $x(E) = x(E + 2\pi)$ for all $E \in \mathbb{R}$.

The equation of motion for m_3 is

$$\frac{d^2 z}{dt^2} = -\frac{z}{(x^2(t) + z^2)^{3/2}}, \tag{2}$$

where $x(t) = x(E(t))$. So (2) is the *equation of motion* for the restricted isosceles three-body problem.

We have a binary collision precisely when $x = 0$, i.e. when $t = 0 \pmod{2\pi}$, and we have a total collapse or *triple collision* whenever $x = z = 0$, i.e. when $z = 0$ and $t = 0 \pmod{2\pi}$.

Notice that the phase space of differential equation (2) is $\{(z, \dot{z}, t) \in \mathbb{R}^3 : (z, \dot{z}, t) \neq (0, \dot{z}, 0 \pmod{2\pi})\}$. That is, the motion of the restricted isosceles three-body problem is not defined in triple collision.

Now we want to show that any solution $z(t)$ which do not begin or end in triple collision is defined for all time, i.e. for all $t \in \mathbb{R}$.

Let Δ be an open set of $\mathbb{R}^n$ and let $\mathcal{X}(\mathbf{x}, t)$ be a vector field of class C^r, with $1 \leq r \leq +\infty$ or $r = \omega$ (that is, analytic), defined on the open set of $\Delta \times \mathbb{R}$. The vector field $\mathcal{X}$ has associated the differential system

$$\dot{\mathbf{x}} = \mathcal{X}(\mathbf{x}, t), \tag{3}$$

and vice versa. By a solution of the differential equations (3) we mean a differentiable function $\phi : I \to \Delta$ defined on some open interval $I \subset \mathbb{R}$ such that $d\phi(t)/dt = \mathcal{X}(\phi(t), t)$ for all $t \in I$. The image $\phi(t)$ in Δ is called a *trajectory*, *orbit* or an *integral curve*.

Let $\mathbf{x} \in \Delta$. A solution $\phi : I \to \Delta$ of (3) such that $0 \in I$ and $\phi(0) = \mathbf{x}$ is called *maximal* if for every solution $\psi : J \to \Delta$ such that $I \subset J$, $\psi(0) = \mathbf{x}$, and $\phi = \psi \mid_I$, it follows that $I = J$ and, consecuently, $\phi = \psi$. In this case $I_{\mathbf{x}} = I = (\omega_-(x), \omega_+(x))$ is called *maximal interval* of the solution $\phi(t)$ which passes through $\mathbf{x}$ when $t = 0$, and this solution $\phi(t)$ will be denoted by $\phi_t(\mathbf{x})$. We say that the solution $\phi_t(\mathbf{x})$ tends to the boundary of Δ when $t \nearrow \omega_+$ (respectively $t \searrow \omega_-$) if for every compact for every compact $K \subset \Delta$ there exists $\varepsilon = \varepsilon(K) > 0$ such that if $t \in [\omega_+(x_0) - \varepsilon, \omega_+(x_0))$ (respectively $t \in (\omega_-(x_0), \omega_-(x_0) + \varepsilon])$ then $\phi_t(x_0) \notin K$. Here the notation $t \nearrow \omega_+$ (respectively $t \searrow \omega_-$) means that we choose any monotonally increasing (respectively decreasing) sequence of the values of t tending to ω_+ (respectively ω_-).

THEOREM 3.1. *Let $\mathcal{X}(x, t)$ be a vector field of class C^r, with $1 \leq r \leq +\infty$ or $r = \omega$, defined on an open set $\Delta \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R}$.*

a) *For any $\mathbf{x} \in \Delta$ there is an open interval $I_x \subset \mathbb{R}$ where the maximal solution $\phi_t(x)$ of (3) satisfying $\phi_0(x) = x$ is defined and unique.*
b) *If $y = \phi_t(\mathbf{x})$, and $t \in I_x$, then $I_y = I_x - t = \{r - t : r \in I_x\}$ and $\phi_s(y) = \phi_{t+s}(x)$ for all $s \in I_y$.*

Proof. See [31, pp. 209-210]. ∎

PROPOSITION 3.2. *Let $\mathcal{X}(\mathbf{x}, t)$ be a vector field of class C^r, with $1 \leq r \leq +\infty$ or $r = \omega$, defined on an open set $\Delta \times \mathbb{R} \subset \mathbb{R}^n \times \mathbb{R}$. If $(x_0, t_0) \in \Delta \times \mathbb{R}$, we denote by $(\omega_-(\mathbf{x}_0), \omega_+(\mathbf{x}_0))$ the maximal interval of the solution $\phi_t(\mathbf{x}_0)$ of $\dot{\mathbf{x}} = \mathcal{X}(\mathbf{x}, t)$ such that $\phi_{t_0}(\mathbf{x}_0) = \mathbf{x}_0$. If $\omega_+ < +\infty$ (respectively $\omega_- > -\infty$), then $\phi_t(x_0)$ tends to the boundary of Δ when $t \nearrow \omega_+$ (respectively $t \searrow \omega_-$).*

Proof. See [31, p. 210]. ∎

The circular Sitnikov problem for masses $m_1 = m_2 = 1$ and radius of the circular orbit of m_1 and m_2 around their center of mass equal to 1/2 is governed by the Hamiltonian

$$H = \frac{1}{2}p^2 - \left(q^2 + \frac{1}{4}\right)^{-\frac{1}{2}}.$$

Some analytic expressions for the solutions of this problem have been given by several authors, see MacMillan [24] and Belbruno-Llibre-Ollé [4]. The next theorem is proved in [4].

THEOREM 3.3. *The solution of the circular Sitnikov problem can be written as*

$$r = \frac{\frac{1}{2}}{1 - \frac{2+h}{2} sn^2(\frac{s}{\sqrt{2}}, k)} \quad if \quad |h| < 2,$$

$$r = \frac{\frac{1}{2}}{1 - 2sn^2(\frac{k}{\sqrt{2}}s, k^{-1})} \quad if \quad |h| > 2,$$

$$3 - \frac{4(1 + \sqrt{(2r-1)r}}{2r+1} = exp(-\sqrt{2}s) \quad if \quad h = 2,$$

where $r = q^2 + \frac{1}{4}$, $k = \frac{1}{2}(2+h)^{1/2}$, $t = \int r^2 ds$ *and* $sn(u,k)$ *is the sinus amplitude Jacobi elliptic function.*

Now we are ready to prove the announced result.

PROPOSITION 3.4. *The maximal interval* (ω_-, ω_+) *of every non-triple ejection or collision solution* $(z(t), v(t))$ *of the system*

$$\begin{aligned} \frac{dz}{dt} &= v, \\ \frac{dv}{dt} &= -\frac{z}{(x^2(t) + z^2)^{3/2}}, \end{aligned} \tag{4}$$

is $(-\infty, +\infty)$.

Proof. We suppose that ω_+ is finite and we will find a contradiction. In a similar way we would show that ω_- cannot be finite. If ω_+ is finite, from

Proposition 3.2 we have that the solution $(z(t), v(t))$ tends to the boundary of the phase space where $t \nearrow \omega_+$. This boundary has three components: $z = t \pmod{2\pi} = 0$ (triple collision); $z = \pm\infty$ (infinity in position); and $v = \pm\infty$ (infinity in velocity).

Since $(z(t), v(t))$ is a non-triple collision orbit, we have that $\lim_{t\nearrow\omega_+} z(t) = \pm\infty$ or $\lim_{t\nearrow\omega_+} v(t) = \pm\infty$. Since $v(t) = \dot{z}(t)$, if $\lim_{t\nearrow\omega_+} v(t) = \pm\infty$ then $\lim_{t\nearrow\omega_+} z(t) = \pm\infty$.

Now we assume that $\lim_{t\nearrow\omega_+} z(t) = +\infty$. In a similar way we would get a contradiction for the case $\lim_{t\nearrow\omega_+} z(t) = -\infty$. Then there exist $t_0 \in (0, \omega_+)$ such that for every $t \in [t_0, \omega_+)$ we have that $z(t) > 0$. Cleary we have

$$\ddot{z} = -\frac{z}{(x^2(t) + z^2)^{3/2}} \leq -\frac{z}{(1 + z^2)^{3/2}}.$$

The differential equation

$$\ddot{s} = -\frac{s}{(1 + s^2)^{3/2}} \tag{5}$$

corresponds to the equation of motion of the circular Sitnikov problem with primaries of mass equal to 1/2 and when the radius of the circular orbit of the primaries around their center of mass is 1. From Theorem 3.3 it follows that the flow of the circular Sitnikov problem is *complete*, i.e. all its solutions are defined for all $t \in \mathbb{R}$.

Let $s(t)$ be the solution of (5) such that $s(t_0) = z(t_0)$ and $\dot{s}(t_0) = \dot{z}(t_0)$. Since the solution $s(t)$ is less braked than the solution $z(t)$ of (4) and at $t = t_0$ they coincide, we get that $s(t) \geq z(t)$ for $t \in [t_0, \omega_+)$. Hence, since $\lim_{t\nearrow\omega_+} z(t) = +\infty$, we get that $\lim_{t\nearrow\omega_+} s(t) = +\infty$, in contradiction with the fact that solution $s(t)$ is defined for all $t \in \mathbb{R}$. ∎

PROPOSITION 3.5. *The followings statments hold.*

a) *If the solution $(z(t), v(t))$ starts at triple collision then its maximal interval of definition (ω_-, ω_+) satisfies that $\omega_- > -\infty$.*

a) *If the solution $(z(t), v(t))$ tends to triple collision then its maximal interval of definition (ω_-, ω_+) satisfies that $\omega_+ < +\infty$.*

Proof. The existence of a triple collision forces the existence of a binary collision between the two primaries. Since the binary collisions take place when $t = 2k\pi$ for $k \in \mathbb{Z}$, and all those times are finite, the proposition follows. ∎

4. Boundary manifolds

Trying to extend the flow of equation (2) to the boundary manifolds of its phase space we make the following changes of coordinates.

Using the time scale transformation

$$\frac{dt}{dE} = 1 - \cos E,$$

equation (2) becomes

$$\begin{aligned} \frac{dz}{dE} &= (1 - \cos E)v, \\ \frac{dv}{dE} &= -(1 - \cos E)\frac{z}{(x^2(E) + z^2)^{3/2}}. \end{aligned} \tag{6}$$

Note that from now on the variable E will be the new time, moreover E module 2π allows us to know the position of the primaries, but E runs over all $\mathbb{R}$. Then the system (6) is 2π-periodic on the variable E.

We introduce polar coordinates through the change of variables $(z, v, E) \to (r, \varphi, E)$ defined by

$$x = r\cos\varphi, \quad z = r\sin\varphi,$$

see the geometrical meaning of r and φ in Figure 2. We note that $r = \sqrt{x^2 + z^2}$ and $\varphi = \tan^{-1} z/x$, where $r \geq 0$ and $-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}$. Then r gives the distance between m_1 and m_3, or m_2 and m_3, $r = 0$ corresponds to triple collision, and $\varphi = \pm\frac{\pi}{2}$ correspond to other double collision between the primaries, or to the escape at infinity of m_3.

The equations of motion (6) in the variables (r, φ, E) are

$$\begin{aligned} \frac{dr}{dE} &= \cos\varphi\Big(2rv\sin\varphi + \frac{1}{2}\sin E\Big), \\ \frac{d\varphi}{dE} &= 2v\cos^2\varphi - \frac{1}{2}r^{-1}\sin\varphi\sin E, \\ \frac{dv}{dE} &= -2r^{-1}\sin\varphi\cos\varphi. \end{aligned} \tag{7}$$

We remark that the variables r, φ and E are not independent because we can obtain each one of them in function of the other two:

$$r = \frac{1 - \cos E}{2\cos\varphi}, \quad \varphi = \cos^{-1}\left(\frac{1 - \cos E}{2r}\right), \quad E = \cos^{-1}(1 - 2r\cos\varphi).$$

Nevertheless, we work with the four variables r, φ, v and E because each one gives different information which allows a better description of the problem. Thus, for instance $r = 0$ (or similary $x = 0$ and $z = 0$) is associated to triple collision, $E = 0$ to double collision of the primaries, $r = \pm\infty$ to escape to infinity of the infinitesimal mass m_3, and $\varphi = 0$ to collinear configuration.

We note that in equations (7) there is the term r^{-1}, so they are not defined in the triple collision. Trying to extend the solutions to the boundary associated to the triple collision $r = 0$, we make the change $(r, \varphi, v, E; E) \to (R, \varphi, w, E; s)$ through

$$R = r^{1/2}, \qquad w = r^{1/2} v, \qquad \frac{dE}{ds} = r^{1/2}.$$

Since

$$x(t) = \frac{1}{2}(1 - \cos E) = \sin^2 \frac{E}{2}, \quad x = r\cos\varphi, \quad r^{-1/2} = \frac{\cos^{1/2}\varphi}{|\sin\frac{E}{2}|},$$

the expression

$$\frac{\sin E}{R} = \frac{\sin E}{r^{1/2}} = \frac{\cos^{1/2}\varphi}{|\sin\frac{E}{2}|}\sin E = 2\cos^{1/2}\varphi\cos\frac{E}{2}\text{sign}\left(\sin\frac{E}{2}\right),$$

is well defined in $R = 0$, and equations of motion (7) go over to

$$\begin{aligned}
\frac{dR}{ds} &= \cos\varphi\Big(Rw\sin\varphi + \frac{1}{4}\sin E\Big),\\
\frac{d\varphi}{ds} &= \cos^{1/2}\varphi\Big(2w\cos^{3/2}\varphi - \sin\varphi\cos\frac{E}{2}\text{sign}\left(\sin\frac{E}{2}\right)\Big),\\
\frac{dw}{ds} &= \cos\varphi\Big(-2\sin\varphi + w^2\sin\varphi + \frac{1}{2}w\cos^{1/2}\varphi\cos\frac{E}{2}\text{sign}\left(\sin\frac{E}{2}\right)\Big),\\
\frac{dE}{ds} &= R.
\end{aligned} \tag{8}$$

We remark that since the factors sign $(\sin\frac{E}{2})$ and $\cos\frac{E}{2}$ appear in system (8), this system is 4π-periodic in the E variable. Since $R = 0$ implies $E = 0$, it follows that $dR/ds = 0$; that is, the manifold $R = 0$ is invariant by the flow.

We must note that system (8) splits into three discontinuous systems according the values of sign $(\sin\frac{E}{2})$. Clearly, any of these two systems corresponding to sign $(\sin\frac{E}{2}) = \pm 1$ have physical meaning.

For knowing the asymtotic behavior of the orbits when $R \to \pm\infty$, the next step is to scale the position R and the velocity w through the change of variables $(R, \varphi, w, E; s) \to (\rho, \varphi, \theta, E; s)$ given by

$$\rho = \frac{R}{1+R}, \quad w = \tan\theta \quad \text{with } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.$$

In such way $\theta = \pm\frac{\pi}{2}$ corresponds to velocity of m_3 equal to $\pm\infty$; $\rho = 0$ to triple collision; and $\rho = 1$ to infinity in the position of m_3.

The equations of motion (8) can be writen down as

$$\begin{aligned}
\frac{d\rho}{ds} &= (1-\rho)^2 \cos\varphi \Big(\rho(1-\rho)^{-1} \sin\varphi \tan\theta + \frac{1}{4} \sin E \Big), \\
\frac{d\varphi}{ds} &= \cos^{1/2}\varphi \Big(2\cos^{3/2}\varphi \tan\theta - \sin\varphi \cos\frac{E}{2} \operatorname{sign}\left(\sin\frac{E}{2}\right) \Big), \\
\frac{d\theta}{ds} &= \cos\varphi \cos^2\theta \Big(-2\sin\varphi + \sin\varphi \tan^2\theta \\
&\quad + \frac{1}{2}\cos^{1/2}\varphi \tan\theta \cos\frac{E}{2} \operatorname{sign}\left(\sin\frac{E}{2}\right) \Big), \\
\frac{dE}{ds} &= \rho(1-\rho)^{-1}.
\end{aligned} \tag{9}$$

The above equations still have singularities at $\rho = 1$ and $\theta = \pm\frac{\pi}{2}$, but these singularities can be removed by a new time s defined by

$$\frac{ds}{d\tau} = (1-\rho)\cos\theta.$$

Thus, the equations (9) become

$$\begin{aligned}
\frac{d\rho}{d\tau} &= (1-\rho)^2 \cos\varphi \Big(\rho \sin\varphi \sin\theta + \frac{1}{4}(1-\rho) \cos\theta \sin E \Big), \\
\frac{d\varphi}{d\tau} &= (1-\rho)\cos^{1/2}\varphi \Big(2\cos^{3/2}\varphi \sin\theta - \sin\varphi \cos\theta \cos\frac{E}{2} \operatorname{sign}\left(\sin\frac{E}{2}\right) \Big), \\
\frac{d\theta}{d\tau} &= (1-\rho)\cos\varphi \cos\theta \Big(-2\sin\varphi \cos^2\theta + \sin\varphi \sin^2\theta \\
&\quad + \frac{1}{2}\cos^{1/2}\varphi \sin\theta \cos\theta \cos\frac{E}{2} \operatorname{sign}\left(\sin\frac{E}{2}\right) \Big), \\
\frac{dE}{d\tau} &= \rho\cos\theta.
\end{aligned} \tag{10}$$

5. Triple collision

Our goal in this section is to show that triple collision cannot be regularized. This means that we cannot read off the local behaviour of solution which pass very near of triple collision. In other words, system (10) defines analytic vector fields on the phase space $(\rho, \varphi, \theta, E) \in (0,1] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times (\mathbb{R} \setminus \{2k\pi : k \in \mathbb{Z}\})$, which cannot be extended to the phase space $[0,1] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times \mathbb{R}$.

We note that if $\rho = 0$ then $E = 0 \pmod{2\pi}$, and that ρ can tends to zero either in a region with sign $(\sin \frac{E}{2}) = 1$, or in a region with sign $(\sin \frac{E}{2}) = -1$. Since the function $\sin \frac{E}{2}$ is 4π-periodic, sign $(\sin \frac{E}{2}) = 1$ if $E \in (0, 2\pi) \mod 4\pi$, sign $(\sin \frac{E}{2}) = -1$ if $E \in (2\pi, 4\pi) \mod 4\pi$, and sign $(\sin \frac{E}{2}) = 0$ if $E = 0 \pmod{2\pi}$. In the phase space $(\rho, \varphi, \theta, E)$ we may distinguish three cases. We will denote by $\rho \searrow 0$ a monotonally decreasing sequence of the values of ρ tending to 0. In the same way $E \searrow 0, 2\pi$ (respectively $E \nearrow 2\pi, 4\pi$) means that we choose any monotonally decreasing (respectively increasing) sequence of the values of E tending $0, 2\pi$ (respectively $2\pi, 4\pi$).

Case 1: sign $(\sin \frac{E}{2}) = 1$. We start by studying the limit flow of system (10) when $\rho \searrow 0$ and $E \searrow 0 \pmod{4\pi}$; that is, when the primaries are leaving the triple collision and the corresponding flow is given by

$$\begin{aligned} \frac{d\varphi}{d\tau} &= \cos^{1/2}\varphi\Big(2\cos^{3/2}\varphi \sin\theta - \sin\varphi\cos\theta\Big), \\ \frac{d\theta}{d\tau} &= \cos\varphi\cos\theta\Big(-2\sin\varphi\cos^2\theta + \sin\varphi\sin^2\theta + \frac{1}{2}\cos^{1/2}\varphi\sin\theta\cos\theta\Big). \end{aligned} \tag{11}$$

The system (11) is defined on the square $[-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ and has the following equilibrium points:

$$\Big(\pm\frac{\pi}{2}, \theta\Big) \text{ for all } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

$$(0,\, 0), \quad \left(\frac{\pi}{3}, \tan^{-1}\sqrt{\frac{3}{2}}\right), \quad \left(-\frac{\pi}{3}, -\tan^{-1}\sqrt{\frac{3}{2}}\right).$$

Now, we compute the linear part of system (11) at these last three isolated equilibrium points. At $(0, 0)$ the linear part is given by the matrix

$$\begin{pmatrix} -1 & 2 \\ -2 & 0.5 \end{pmatrix}.$$

Since its eigenvalues are $\lambda_{1,2} = -0.25 \pm 1.85404962 \cdots i$, $(0,0)$ is a stable focus [3]. The linear matrix at the equlibrium points

$$\left(\frac{\pi}{3}, \tan^{-1}\sqrt{\frac{3}{2}}\right), \quad \left(-\frac{\pi}{3}, -\tan^{-1}\sqrt{\frac{3}{2}}\right),$$

is in both cases equal to

$$\begin{pmatrix} -\frac{11}{4\sqrt{5}} & \frac{1}{2}\sqrt{\frac{5}{2}} \\ -\frac{1}{4\sqrt{10}} & \frac{7}{4\sqrt{5}} \end{pmatrix}$$

having eigenvalues $\lambda_1 = -1.1982826232 \cdots$ y $\lambda_2 = 0.7510726368 \cdots$; so, these two points are saddles [3]. By Hartman Theorem [19] since these singular points are hyperbolic, their local flow is qualitatively the same as their linear flow.

Once we know the local flow at the equilibrium points the following step is to obtain the global phase portrait. According with the nonlinear nature of system (11), it is not easy to make an analytic study for determining the flow outside a neighborhood of the equilibrium points, more especifically to determinate the global behavior of the separatrices of the saddles points.

On the other hand, close to $(0,0)$ the divergence of system (11) given by

$$\frac{d\dot{\varphi}}{d\varphi} + \frac{d\dot{\theta}}{d\theta} = \sin\varphi\cos\varphi\sin\theta(9\cos^2\theta - 5) + \frac{1}{2}\sin^2\varphi\cos^{-1/2}\varphi\cos\theta$$
$$+ \cos^{3/2}\varphi\cos\theta\left(\frac{3}{2}\cos^2\theta - 2\right),$$

and consequently it is always negative. So there are no periodic orbits near the focus $(0,0)$ (for more details see [33]).

To analyse the vector field over the zero velocity curves defined by $\dot{\varphi} = d\varphi/d\tau = 0$ and $\dot{\theta} = d\theta/d\tau = 0$ is very useful for describing the global flow in $[-\frac{\pi}{2}, \frac{\pi}{2}]^2$. Moreover, the local phase portrait at equilibrium points, together with the global behavior of the separatrices of the saddle points of the system (11) computed numerically, allows us to show that the phase portrait of the system (11) is topologically equivalent to the phase portrait of Figure 3.

Note that the system (11) is invariant under the symmetry $(\varphi, \theta, \tau) \rightarrow (-\varphi, -\theta, \tau)$.

For analyzing the nonlinear system (10) when $\rho \searrow 0$ and $E \searrow 0 \pmod{4\pi}$, we start analysing the local phase portrait of system (10) near its equilibrium

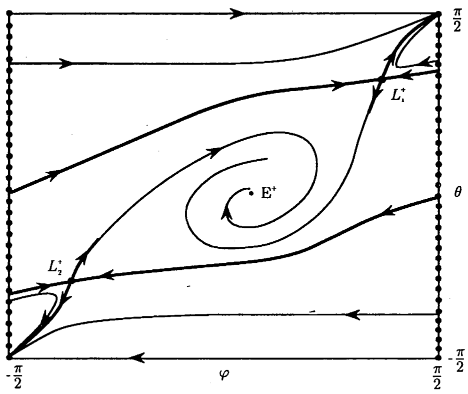

Figure 3: Phase portrait when $\rho \searrow 0$ and $E \searrow 0$ (mod 4π).

points given by

$$\mathrm{E}^+ = (0,0,0,0), \quad L_1^+ = \Big(0, \frac{\pi}{3}, \tan^{-1}\sqrt{\frac{3}{2}}, 0\Big), \quad L_2^+ = \Big(0, -\frac{\pi}{3}, -\tan^{-1}\sqrt{\frac{3}{2}}, 0\Big),$$

which is again qualitatively determined by the behavior of the linear part of system at these equilibrium points.

We remark that the equilibrium point E^+ correspond to the collinear Euler central configuration, and both $L_{1(2)}^+$ corresponds to the equilateral triangular Lagrange central configuration; see [29] for more details about Euler and Lagrange central configurations of the three-body problems.

Since the variable E may be obtained in function of ρ y φ, we study the flow in the phase space (ρ, φ, θ).

The linear part of the system (10) at E^+ is given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0.25 \\ 0 & -1 & 2 & 0 \\ 0 & -2 & 0.5 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

having eigenvalues $\lambda_1 = \frac{1}{2}$, $\lambda_2 = -\frac{1}{2}$, $\lambda_{3,4} = -0.25 \pm 1.85404962\cdots i$.

We have that the local phase portrait at $(0,0,0,0)$ restricted to the flow (φ, θ) is a sink; that is, the flow in a neighborhood tends to $(0,0)$. On the other hand, in a neighborhood of E^+ with $\rho \approx 0$, $\varphi \approx 0$, $\theta \approx 0$ and $E \gtrapprox 0$ is satisfied $d\rho/d\tau \approx (\sin E)/4 > 0$. We denote by $E \gtrsim 0$ the positive values of E near 0. Therefore in the phase space (ρ, φ, θ) the flow is leaving $\mathtt{E}^+$ along the ρ-axis. Using the Hartman Theorem we get that in the phase space (ρ, φ, θ) the equilibrium point $\mathtt{E}^+$ has an unstable manifold of dimension one, and one stable manifold of dimension two.

We compute now the linear part of the system (10) at the equilibium point L_1^+. The resulting matrix is

$$\begin{pmatrix} \frac{3}{4\sqrt{5}} & 0 & 0 & \frac{1}{4\sqrt{10}} \\ 0 & -\frac{11}{4\sqrt{5}} & \frac{1}{2}\sqrt{\frac{5}{2}} & 0 \\ 0 & -\frac{1}{4\sqrt{10}} & \frac{7}{4\sqrt{5}} & 0 \\ \sqrt{\frac{2}{5}} & 0 & 0 & 0 \end{pmatrix},$$

having eigenvalues $\lambda_1 = -0.1118033988\cdots$, $\lambda_2 = 0.4472135954\cdots$, $\lambda_3 = -1.1982\ 86232\cdots$ and $\lambda_4 = 0.7510726368\cdots$.

We have that in the phase space (φ, θ) the equilibrium point L_1^+ is a saddle. Furthermore, in a neighborhood of L_1^+ with $\rho \approx 0$, $\varphi \approx \pi/3$, $\theta \approx \tan^{-1}\sqrt{3/2}$ and $E \gtrsim 0 \pmod{4\pi}$ we have that $d\rho/d\tau \approx (\sin E)/(4\sqrt{10}) > 0$. Consequently in the phase space (ρ, φ, θ) the flow is leaving L_1^+ along the ρ-axis.

From Hartman Theorem we can say that in the phase space (ρ, φ, θ) the Lagrange point L_1^+ has one unstable manifold of dimension 2, and one stable manifold of dimension 1.

In a similar way we can study the linear part of system (10) at Lagrange point L_2^+, and we would get similar results to those of L_1^+. More precisely, we have proved the following proposition.

PROPOSITION 5.1. *For the restricted isosceles three-body problem and in a neighborhood of the triple collision in the region where sign* $(\sin \frac{E}{2}) = 1$ *for* $E \gtrsim 0 \pmod{4\pi}$, *the following statements hold.*

(a) *There exists a unique orbit which begins at triple collision in the Euler configuration* $\mathtt{E}^+$.

(b) *There exists a 2-dimensional manifold formed by orbits which begin at triple collision in the Lagrange configuration* L_1^+.

(c) *There exists a 2-dimensional manifold formed by orbits which begin at triple collision in the Lagrange configuration* L_2^+.

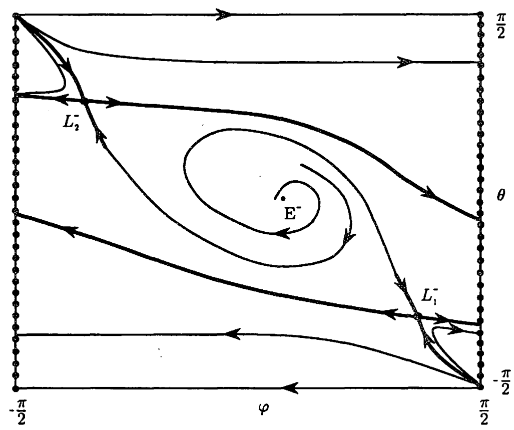

Figure 4: The phase portrait when $\rho \nearrow 0$ and $E \nearrow 2\pi \pmod{4\pi}$.

Now we consider the limit of system (10) when $\rho \searrow 0$ and $E \nearrow 2\pi \pmod{4\pi}$, this means that the primaries are going to collision with m_3 and this limit flow is giving by the system

$$\begin{aligned} \frac{d\varphi}{d\tau} &= \cos^{1/2}\varphi\Big(2\cos^{3/2}\varphi\sin\theta + \sin\varphi\cos\theta\Big), \\ \frac{d\theta}{d\tau} &= \cos\varphi\cos\theta\Big(-2\sin\varphi\cos^2\theta + \sin\varphi\sin^2\theta - \frac{1}{2}\cos^{1/2}\varphi\sin\theta\cos\theta\Big). \end{aligned} \tag{12}$$

System (12) can be obtained from system (11) through the symmetry $(\varphi, \theta, \tau) \to (-\varphi, -\theta, \tau)$. So its phase portrait is given in Figure 4.

PROPOSITION 5.2. *For the restricted isosceles three-body problem and in a neighborhood of the triple collision in the region where sign* $(\sin\frac{E}{2}) = 1$ *for* $E \lessapprox 2\pi \pmod{4\pi}$, *the following statements hold.*

(a) *There exists a unique orbit which ends at triple collision in the Euler configuration* E^-.

(b) *There exists a 2-dimensional manifold formed by orbits which end at triple collision in the Lagrange configuration* L_1^-.

(c) *There exists a 2-dimensional manifold formed by orbits which end at triple collision in the Lagrange configuration* L_2^-.

So the triple collision manifold for a component of the phase space $(0,1] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \times (\mathbb{R} \setminus \{2k\pi : k \in \mathbb{Z}\})$ with $\operatorname{sign}(\sin\frac{E}{2}) = 1$ has two components having different phase portraits.

CASE 2: $\operatorname{sign}(\sin\frac{E}{2}) = -1$. The objective is to describe the limit flow of system (10) when $\rho \searrow 0$ with $E \searrow 2\pi \pmod{4\pi}$ or $E \nearrow 4\pi \pmod{4\pi}$. Simple cumputations show that these limit flows are exactly the same than in Case 1. That is, the case $E \searrow 2\pi \pmod{4\pi}$ corresponds with $E \searrow 0 \pmod{4\pi}$, and the case $E \nearrow 4\pi \pmod{4\pi}$ corresponds with $E \nearrow 2\pi \pmod{4\pi}$. Hence, Propositions 5.1 and 5.2 are satisfied when $\operatorname{sign}(\sin\frac{E}{2}) = -1$ for $E \gtrsim 2\pi$ and $E \lesssim 4\pi$ respectively. Of course, $E \gtrsim 2\pi$ denotes values larger than and near 2π. Similary for $E \lesssim 4\pi$.

Let

$$\mathcal{C}^u = \left\{(\rho, \varphi, \theta, E) : \rho \searrow 0,\ -\frac{\pi}{2} \leq \varphi,\ \theta \leq \frac{\pi}{2},\ E \searrow 0,\ 2\pi \pmod{4\pi}\right\}$$

and

$$\mathcal{C}^s = \left\{(\rho, \varphi, \theta, E) : \rho \searrow 0,\ -\frac{\pi}{2} \leq \varphi,\ \theta \leq \frac{\pi}{2},\ E \nearrow 2\pi,\ 4\pi \pmod{4\pi}\right\},$$

be the two different components of the limit flow of the triple collision. Since the phase portraits of these two components of the limit flow at triple collision, $\mathcal{C}^u$ and $\mathcal{C}^s$ do not coincide, we cannot extend in a continuous way the flow of system (10) given in

$$(\rho, \varphi, \theta, E) \in (0,1] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \times (\mathbb{R} \setminus \{2k\pi : k \in \mathbb{Z}\})$$

to the boundary $\rho = 0$. Hence we have proved the following result.

THEOREM 5.3. *The triple collision of the restricted isosceles three-body problem is not regularizable. That is, we loss the continuous dependence on the inicial conditions for the orbits which pass near triple collision.*

On the other hand we also study the following case.

CASE 3: $\operatorname{sign}(\sin\frac{E}{2}) = 0$. Now we describe the flow of system (10) on $\rho = 0$, which is giving by

$$\begin{aligned}\frac{d\varphi}{d\tau} &= 2\cos^2\varphi \sin\theta, \\ \frac{d\theta}{d\tau} &= \sin\varphi \cos\varphi \cos\theta \left(-2\cos^2\theta + \sin^2\theta\right).\end{aligned} \tag{13}$$

Of course, this flow has no physical meaning. The $(0,0)$ is the unique isolated equilibrium point of system (13). The linear part at this point has eigenvalues $\pm 2i$. Then the origin is a linear center.

Removing the time τ in system (13) we obtain

$$\frac{d\theta}{d\varphi} = \frac{\sin\varphi\cos\theta(-2\cos^2\theta + \sin^2\theta)}{2\cos\varphi\sin\theta}.$$

Separating variables and integrating, we get that the integral curves are contained into the level curves

$$G(\varphi,\theta) = \frac{\cos^2\theta}{\cos\varphi(-3\cos^2\theta + 1)} = c, \tag{14}$$

where $c \in \mathbb{R}$, $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Since $G(\varphi,\theta)$ is a first integral of system (13), if $c_0 \in \mathbb{R}$ is a regular value of G, then there exists a neighborhood U of c_0 such that $G^{-1}(c_0)$ is a differentiable curve in $(-\frac{\pi}{2},\frac{\pi}{2}) \times (-\frac{\pi}{2},\frac{\pi}{2})$ and all $G^{-1}(c)$ with $c \in U$ are differentiable curves (for more details see [18]). So, $G^{-1}(c_0)$ is diffeomorphic either to S^1 or to an open interval (see for instance [27]). Furthermore, the topology of $G^{-1}(c)$ can only change when c is a critical value of the function $G(\varphi,\theta)$.

The unique critical point of $G(\varphi,\theta)$ on $(-\frac{\pi}{2},\frac{\pi}{2}) \times (-\frac{\pi}{2},\frac{\pi}{2})$ is $(0,0)$. Since the level curve (14) is invariant by the symmetry $(\varphi,\theta) \to (-\varphi,-\theta)$, if the point (φ,θ) satisfies $G(\varphi,\theta) = c$ then $(-\varphi,-\theta)$ also satisfies it. Moreover, if such level curve crosses the φ-axis in two points, then it is closed. From

$$G(\varphi,\theta = 0) = \frac{1}{-2\cos\varphi} = c,$$

it follows that the level curve intersects the φ-axis at point $(0,0)$ if and only if $0 \le \cos\varphi = -1/(2c) \le 1$; that is, if only if $-\infty < c \le -1/2$. For these values of c the integral curves are closed because due to the symmetry always intersect the φ-axis in two points. Therefore the curves $G(\varphi,\theta) = c$ are closed if and only if $-\infty < c \le -1/2$. In particular $c = -1/2$ at $(0,0)$.

Notice that $\theta = \pm\cos^{-1}\frac{1}{\sqrt{3}}$ are straight line solutions of system (13). At these straight lines $c = +\infty$, and the function $G(\varphi,\theta)$ is not defined over them. Moreover, from (14) we have that $c \to 0$ when $\theta \to \pm\frac{\pi}{2}$.

In short, the integral curves $G(\varphi,\theta) = c$ are homeomorphic to S^1 if $-\infty < c < -1/2$, to two open intervals if $-1/2 < c < +\infty$, and to an equilibrium point if $c = -1/2$. The phase portrait of system (13) is topologically equivalent to the phase portrait shown in the Figure 5.

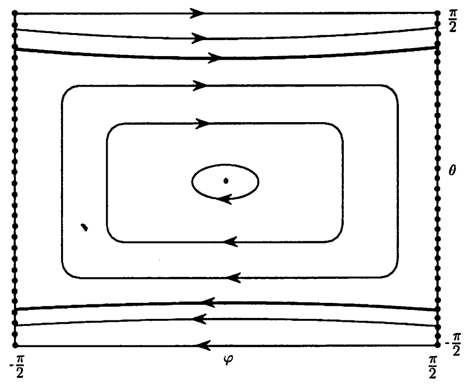

Figure 5: The phase portrait on $\rho = 0$.

REFERENCES

[1] ALEKSEEV, V.M., Quasirandom dynamical systems I, II, III, *Math. URSS Sbornik,* **5** (1968), 73-128; **6** (1968), 505-560; **7** (1969), 1-43.

[2] ALEKSEEV, V.M., Sur l'allure finale du mouvement dans le problème de trois corps, in "Actes du Congrès International des Mathématiciens", Gauthiers-Villars, Paris, 1970, Vol. **2** 1971, 893-907.

[3] ANDRONOV, A.A., LEONTOVICH, E.A., GORDON, I.I., MAINER, A.G., "Qualitative theory of second-order dynamic systems", John Wiley & Sons, New York, 1973.

[4] BELBRUNO, E., LLIBRE, J., OLLÉ, M., On the families of periodic orbits which bifurcate from the circular Sitnikov motions, *Cel. Mech. and Dyn. Astr,* **60** (1994), 99-129.

[5] BROUCKE, R., Periodic orbits in the rectilinear restricted three-body problem, *J. Mécanique,* **10** (1971), 449-465.

[6] BROUCKE, R., On the triangle configuration in the planar general three-body problem, *Astron. Astrophys.,* **73** (1979), 303-313.

[7] CHAZY, J., Sur l'allure finale du mouvement dans le problème des trois corps quand le temps croit indefiniment, *Annales de l'Ecole Norm. Sup. 3e. sér.,* **39** (1922), 22-130.

[8] CHAZY, J., Sur l'allure finale du mouvement dans le problème des trois corps, *J. Math. Pures et Appl.,* **8** (1929), 353-380.

[9] CHAZY, J., Sur l'allure finale du mouvement dans le problème des trois corps, *Bull. Astron.,* **8** (1932), 403-436.

[10] COLOMBO, G., FRANKLIN, F.A., MUNFORD, C.N., On a family of periodic orbits of the restricted three-body problem and the question of the gaps in the asteroid belt and in Saturn's rings, *Astron. J.,* **73** (1968), 111-123.

[11] CORS, J., LLIBRE, J., The global flow of the hyperbolic restricted three-body problem, *Arch. Rational Mech. Anal.,* **131** (1995), 335-358.

[12] CORS, J., LLIBRE, J., Qualitative study of the parabolic collision restricted three-body problem, *Contemporany Mathematics,* **198** (1996), 1 - 19.

[13] CORS, J., LLIBRE, J., Qualitative study of the hyperbolic collision restricted three-body problem, *Nonlinearity,* **9** (1996), 1299 - 1316.

[14] CORS, J., LLIBRE, J., The global flow of the parabolic restricted three-body problem, preprint, 1997.

[15] EASTON, R., Regularization of vector fields by surgery, *J. Diff. Equations,* **10** (1971), 92 - 99.

[16] ELBIALY, M.S., Triple collisions in the isosceles three body problem with small mass ratio, *ZAMP,* **40** (1989), 643 - 644.

[17] GÓMEZ, G., LLIBRE, J., Capture/escape boundary in the collinear restricted three-body problem, in "Dynamical trapping and evolution in the solar system", V.V. Markellos and Y. Kozai (Eds.), D. Reidel Publishing Company Dordrecht-Boston-London, 1983, 325 - 338.

[18] GUILLEMIN, V., POLLACK, A., "Differential Topology", Prentice Hall, Englewood Cliffs, N.J., 1974.

[19] HARTMAN, P., "Ordinary Differential Equations", J. Wiley and Sons, Inc., New York, 1964.

[20] LLIBRE, J., PIÑOL, C., On the elliptic restricted three-body problem, *Cel. Mech. and Dyn. Astr.,* **48** (1990), 319 - 345.

[21] LLIBRE, J., SIMÓ, C., Oscilatory solutions in the planar restricted three body problem, *Math. Ann.,* **248** (1980), 153 - 184.

[22] LLIBRE, J., SIMÓ, C., Some homoclinic phenomena in the three-body problem, *J. Diff. Equations,* **33** (1980), 444 - 465.

[23] LLIBRE, J., SIMÓ, C., Characterization of transversal homothetic solutions in the n-body problem, *Arch. Rational Mech. Anal.,* **77** (1981), 189 - 198.

[24] MACMILLAN, W.D., An integrable case in the restricted problem of three bodies, *Astron. J.,* **27** (1913), 11.

[25] MARTÍNEZ-ALFARO, J., ORELLANA, R.B., Collision orbits in the isosceles rectilinear restricted problem, in "From Newton to Chaos", A.E. Roy and B.A. Steves (Eds.), Plenum Press, New York, 1995, 387 - 393.

[26] MEYER, R.K., WANG, Q., Global phase structure of the restricted isoceles three-body problem with positive energy, *Trans. Amer. Math. Soc.,* **338** (1993), 311 - 336.

[27] MILNOR, J., "Topology from the differentiable viewpoint", The University Press of Virginia, Charlottesville, Virginia, 1965.

[28] MOSER, J., "Stable and random motions in dynamical systems", Annals Mathematics Studies, Number 77, Princeton University Press, Princeton, N.J., 1973.

[29] POLLARD, H., "Mathematical introduction to Celestial Mechanics", Prentice-Hall, Inc., Englewwod Cliffs, N.J., 1966.

[30] SITNIKOV, K.A., The existencia of oscillatory motions in the three-body problem, *Dokl. Akad. Nauk, USSR,* **2** (1960), 303 - 306.

[31] SOTOMAYOR, J., "Liçőes de equações diferenciais ordinárias", IMPA, Río de Janeiro, 1979.

[32] SZEBEGHELY, V., "Theory of orbits", Academic Press, New York, 1967.

[33] YE YAN QUIAN, "Theory of limit cycles", Translations of Mathematical Monographs of Amer. Math. Soc., Vol. 66, Providence, Rhode Island, 1986.