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Local Ergodic Theorems

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1. Introduction and preliminaries

For T a bounded linear operator on a Banach space X and $x \in X$, the following implication is well-known,

$$\frac{1}{n}\sum_{k=0}^{n-1} T^k x \text{ converges } \Rightarrow \lim_{n\to\infty} \frac{T^n x}{n} = 0. \tag{1}$$

The operator T is said to be *ergodic* if it satisfies the converse of (1) for every $x \in X$. Dunford [5] proved that, if 1 is a pole of the resolvent operator, then

$$\frac{1}{n}\sum_{k=0}^{n-1} T^k \text{ converges in norm } \Longleftrightarrow \lim_{n\to\infty} \frac{T^n}{n} = 0.$$

Other generalizations and local versions of these results are given in [3], [4], [7] and [8].

Gelfand-Hille theorems give information about the behaviour of the operator $I-T$. In particular, these theorems give necessary and sufficient conditions for $I-T$ to be nilpotent or for the sequence $T^n(I-T)$ to be convergent to zero. The last kind of results are called Katznelson-Tzafriri theorems.

In this paper, we study some conditions implying that an operator T locally satisfies the converse of (1); i.e. it is locally ergodic at some point $x \in X$. In fact, we prove local versions of some results of [9] and [8], and a local version of the Gelfand-Hille theorem (see [10]).

Along the paper, X denote a complex Banach space and $L(X)$ the Banach algebra of all bounded linear operators defined on X. If $T \in L(X)$, we denote the kernel and the range of T by $N(T)$ and $R(T)$, respectively. Moreover, a complex number λ belongs to the resolvent set $\rho(T)$ of T if there exists $(\lambda - T)^{-1} \in L(X)$. We denote by $\sigma(T) := \mathbb{C} \setminus \rho(T)$ the spectrum of T.

We say that a complex number λ belongs to the local resolvent set of T at x, denoted $\rho(x,T)$, if there exists an analytic function $w : U \to X$, defined on a neigbourhood U of λ, which satisfies

$$(\mu - T)w(\mu) = x. \tag{2}$$

for every $\mu \in U$. The local spectrum set of T at x is the complement $\sigma(x,T) := \mathbb{C} \setminus \rho(x,T)$.

An operator $T \in L(X)$ satisfies the *Single Valued Extension Property* (hereafter referred to as SVEP) if $(\lambda - T)h(\lambda) = 0$ only has trivial analytic solutions on any open subset of the plane. If T satisfies the SVEP, then for every $x \in X$ there exists a unique analytic function $\widehat{x}_T$ on $\rho(x,T)$ satisfying (2), which is called the local resolvent function of T at x.

We say that $T \in L(X)$ satisfies *property* (C) if $X(T,H) := \{x \in X : \sigma(x,T) \subset H\}$ is closed for all closed sets $H \subset \mathbb{C}$. For $T \in L(X)$ we consider the following subsets of X:

$$E_T := \{x \in X : \lim_{n\to\infty} \frac{T^n x}{n} = 0\}$$

$$M_T := \{x \in X : M_n(T)x := \frac{1}{n} \sum_{k=0}^{n-1} T^k x \text{ converges } \}$$

Clearly, these sets are (not necessarily closed) subspaces which are invariant for any operator commuting with T. Moreover, we have $M_T \subset E_T$ (see the Introduction).

The operator T is said to be ergodic if $E_T = M_T$.

2. Global and local ergodic operators

In this section we prove some basic results and local versions of some ergodic theorems.

PROPOSITION 1. *Let $T \in L(X)$. Then the following assertions hold:*

1. $E_T \cap N(I-T)^{n+1} = N(I-T)$, *for every* $n \in \mathbb{N}$.

2. *If* $1 \in \rho(T)$, *then* $E_T = (I - T)E_T = M_T$. *In particular,* T *is ergodic.*

Although $x \in M_T$ when $T^n x$ converges, the converse implication is not true, as the following example shows.

EXAMPLE 1. Let $T \in L(\ell_2(\mathbb{N}))$ be the weighted shift defined by

$$Te_n := \sqrt{(n+1)/n}\, e_{n+1}.$$

Taking $x := (I - T)e_1$, we obtain that $x \in M_T$ (since $e_1 \in E_T$) and $T^n x$ does not converge (since $T^n x = \sqrt{n+1}e_{n+1} - \sqrt{n+2}e_{n+2}$).

The next result will be useful to describe operators with an ergodic power.

PROPOSITION 2. *Let* $T \in L(X)$. *Then for every* $k \in \mathbb{N}$ *we have* $E_{T^k} = E_T$ *and* $M_{T^k} \subset M_T$. *In particular, if* T^k *is ergodic, then* T *is ergodic.*

DEFINITION 1. Let $T \in L(X)$ and $x \in X$. We say that T is a local ergodic operator at x if $x \notin E_T$ or $x \in M_T$.

Remark 1. (1) An operator T is ergodic if and only if T is local ergodic operator at x for all $x \in X$.

(2) If $\|T\| \leq 1$ and $x \in N(I - T) \oplus \overline{R(I - T)}$, then T is a local ergodic operator at x, by the mean ergodic theorem (see [6]). In particular, if $\|T\| \leq 1$ and $1 \in \rho(x, T)$, then T is a local ergodic operator at x.

(3) If $T^n x \to 0$ as $n \to \infty$, then $x \in M_T$; hence T is a local ergodic operator at x.

In the following proposition we prove a local ergodic result using global ergodic properties.

PROPOSITION 3. *Suppose that* $T \in L(X)$ *has property* (C) *or* 1 *is an isolated point of* $\sigma(T)$.

1. *If* $x \in X$ *and* $1 \in \rho(x, T)$, *then* T *is a local ergodic operator at* x.
2. *If* $x \in X$ *and* 1 *is a pole of* $\widehat{x}_T$ *of order* n, *then* T *is a local ergodic operator at* x.

COROLLARY 1. *Let* $T \in L(X)$. *If* 1 *is a pole of* $(\lambda - T)^{-1}$, *then* T *is ergodic.*

In general, if T is a local ergodic operator at x, then 1 is neither an isolated point of $\sigma(x,T)$ nor an essential singularity (see the examples below).

EXAMPLE 2. Let $T \in L(\ell_2(\mathbb{N}))$ and $x \in \ell_2(\mathbb{N})$ as in Example 1. We have that $x \in M_T$ and $\sigma(x,T) = \overline{\mathbb{D}}$, the closed unit disc. Then 1 is not an isolated point of $\sigma(x,T)$.

EXAMPLE 3. Let $S \in L(\ell_2(\mathbb{N}))$ be the weighted shift with weights $\{1/n\}$. Define $T := (I+S)^{-1}$. Then $\|T^n\| \leq 1$ and $\overline{R(I-T)} \neq \ell_2(\mathbb{N})$. Moreover 1 cannot be a pole of $\widehat{x}_T$, because 1 is not an eigenvalue of T [2, Corollary 3.2]. Hence $E_T = \ell_2(\mathbb{N})$ and using the mean ergodic theorem, there exists $x \in E_T \setminus M_T$.

3. LOCAL POWER BOUNDED OPERATORS

For $T \in L(X)$, the local spectral radius $r_T(x)$ of T at $x \in X$ is defined by

$$r_T(x) := \max\{\,|z| \colon z \in \sigma(x,T)\}.$$

If T has the SVEP, then $r_T(x) = \limsup_{n\to\infty} \|T^n x\|^{1/n}$. Henceforth we denote by Γ the unit circle.

Next we obtain some local results using similar ideas to that of [9, Theorem 1] and local spectral theory.

THEOREM 1. *Let $T \in L(X)$ and let $x \in X$ such that $\|T^n x\| \leq M$ for all $n \in \mathbb{N}$. If $\sigma(x,T) \cap \Gamma \subset \{1\}$, then $T^n x - T^{n+1}x \to 0$ as $n \to \infty$.*

The converse of Theorem 1 is not true, as the next example shows.

EXAMPLE 4. Let $T \in L(C([-1,1]))$ be the multiplication operator defined by $Tf(t) := tf(t)$, for $f \in C([-1,1])$. Taking $x(t) := t(1-|t|)$; $t \in [-1,1]$, we obtain that $T^n x \to 0$ as $n \to \infty$ and $\sigma(x,T) = [-1,1]$.

Remark 2. We could think that it would be possible to obtain a localization of [10, Theorem 4] similar to Theorem 1; i.e., that $\sigma(x,T) = \{1\}$ and $\|(I-T)^k M_n(T)x\| \to 0$ for some positive integer k implies $(I-T)^k x = 0$. Unfortunately, the following example shows that this is not so.

EXAMPLE 5. The Volterra operator V on the Hilbert space $H := L^2([0,1])$ is defined by

$$(Vf)(t) := \int_0^t f(s)\,\mathrm{d}s.$$

Take $T := (I+V)^{-1}$. So, $1 \notin \sigma_p(T)$ and hence 1 is not a pole of the resolvent operator. Then $M_T = H$ (since $T^n x \to 0$ as $n \to \infty$ for all $x \in H$ by [1, Theorem 5.1]), and $T \neq I$.

The following result is a localization of the Gelfand-Hille theorem (see [4] and [10]).

THEOREM 2. *Assume that $T \in L(X)$ has the SVEP and let $x \in X$ such that $\sigma(x,T) = \{1\}$. If 1 is a pole of $\widehat{x}_T$ of order k, then $T^n x/n^k \to 0$ as $n \to \infty$.*

Example 5 shows that Theorem 2 is false when 1 is an essential singularity.

An operator $T \in L(X)$ is *Riesz* if every nonzero complex number is a pole of $(zI - T)^{-1}$ with finite multiplicity. In the same way that [8, Théoréme 3], we obtain the following results.

THEOREM 3. *For a Riesz operator T, the following assertions are equivalent.*

1. *$T^n x$ is bounded.*
2. *$x \in E_T$.*
3. *$x \in M_T$.*
4. *$r_T(x) \leq 1$ and $\sigma(x,T) \cap \Gamma$ consists of poles of $\widehat{x}_T$ of order 1.*

Our final result is the converse of Theorem 1 for a certain class of operators and vectors.

THEOREM 4. *Let $T \in L(X)$ be an operator with the SVEP, and let $x \in X$ such that $r_T(x) \leq 1$ and $\sigma(x,T) \cap \Gamma$ consists of poles of $\widehat{x}_T$ of order 1. If $T^n x - T^{n+1} x \to 0$ as $n \to \infty$, then $\sigma(x,T) \cap \Gamma \subset \{1\}$.*

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