EXTRACTA MATHEMATICAE Vol. 13, Núm. 2, 249 – 253 (1998)

On the Structure of the Dual Complexity Space: The General Case

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(Research announcement presented by F. Montalvo)

AMS *Subject Class.* (1991): 54E15, 54E50, 46A15 *Received January* 27, 1998

The purpose of this note is to report the main results obtained by the authors in [8] and [9], respectively.

The notion of a Smyth completable quasi-uniform space provides an efficient tool to give a topological foundation for many kinds of spaces which arise naturally in Theoretical Computer Science; in particular in Domain Theory (e.g. [5], [13] and [14]) and Complexity Theory (e.g. [9] and [10]). In fact, Smyth presented in [13] and [14] a topological framework for denotational semantic based on the theory of complete (and totally bounded) quasi-uniform and quasi-metric spaces. Later on, Matthews introduced in [5] the weightable quasi-metric spaces, or the equivalent partial metric spaces, as a part of the study of denotational semantics of dataflow networks. It was proved by Künzi [4] that, in fact, every weigthable quasi-metric space is Smyth completable. Recently, Schellekens [10] introduced the complexity (quasi-metric) space to the study of complexity analysis of programs and proved, among other results, that every complexity space is weightable and, thus, Smyth completable.

Our basic references for quasi-metric spaces are [3] and [4].

In our context by a quasi-metric on a (nonempty) set X we mean a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x,y) = d(y,x) = 0 \Leftrightarrow x = y$, and (ii) $d(x,y) \leq d(x,z) + d(z,y)$.

If d is a quasi-metric on X, then the function d^s defined on $X \times X$ by $d^s(x,y) = \max\{d(x,y), d(y,x)\}$ is a metric on X.

*The first author acknowledges the support of the DGES, grant PB95-0737.

A quasi-metric d on X is called *bicomplete* [3], [4], if d^s is a complete metric on X.

A quasi-metric space is a pair (X, d) such that X is a (nonempty) set and d is a quasi-metric on X.

Künzi characterized in [4] both Smyth completable and Smyth complete quasi-uniform spaces in terms of left K-Cauchy filters as discussed in [7]. By using [12, Proposition 4], it is possible to formulate these characterizations, for quasi-metric spaces, in terms of sequences:

A quasi-metric space (X, d) is Smyth completable if and only if every left K-Cauchy sequence in (X, d) is a Cauchy sequence in the metric space (X, d^s), where a sequence $(x_n)_{n\in\mathbb{N}}$ in (X, d) is left K-Cauchy provided that for each $\varepsilon > 0$ there is an $n_\varepsilon \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ whenever $m \geq n \geq n_\varepsilon$ [6].

Let us recall [10] that the complexity space (with range $(0, +\infty]$) is the pair (C, d_C), where

$$C = \{f : \omega \to (0, +\infty] \mid \textstyle\sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} < +\infty\},$$

and d_C is the quasi-metric defined on C by

$$d_C(f, g) = \sum_{n=0}^{\infty} 2^{-n} [(\frac{1}{g(n)} - \frac{1}{f(n)}) \vee 0], \quad f, g \in C.$$

d_C is called in [10] "the complexity distance", and intuitively it measures relative improvements in the complexity of programs.

The dual complexity (quasi-metric) space (C^*, d_{C^*}) is introduced in [8]. A motivation for the use of the dual instead of the original complexity space is the fact that the dual is mathematically somewhat more appealing, since d_{C^*} is "derived" from the restriction to $[0, +\infty)$ of the quasi-metric u defined on $\mathbb{R} \times \mathbb{R}$ by $u(x, y) = \max\{y - x, 0\}$. Consequently, the presentation of the proofs becomes somewhat more elegant. Furthermore, it is possible to carry out the complexity analysis of algorithms based on the dual complexity space and from a Computer Science point of view the dual complexity space respects the interpretation usually given to the minimum $\perp$ in semantic domains [8, Section 4].

The dual complexity space (with range $[0, +\infty)$) is the pair (C^*, d_{C^*}), where

$$C^* = \{f : \omega \to [0, +\infty) \mid \textstyle\sum_{n=0}^{\infty} 2^{-n} f(n) < +\infty\},$$

and d_{C^*} is the quasi-metric defined on C^* by

$$d_{C^*}(f, g) = \sum_{n=0}^{\infty} 2^{-n} [(g(n) - f(n)) \vee 0], \quad f, g \in C^*.$$

In [8], via the analysis of the dual, several quasi-metric properties of the complexity space are obtained. In particular, by using that every weightable bicomplete quasi-metric space is Smyth complete [4], we prove the following

THEOREM 1. *The complexity space (C, d_C) is Smyth complete.*

If $\mathcal{F} \subseteq C$, we say that $(\mathcal{F}, d_C)$ is a complexity space. A complexity space $(\mathcal{F}, d_C)$ has a lower bound $L \in C$ provided that $L(n) \leq f(n)$ for all $f \in \mathcal{F}$ and $n \in \omega$. Given $L \in C$, we define

$$C_L = \{f \in C \mid L(n) \leq f(n),\ \forall n \in \omega\}.$$

By using that every weightable hereditarily precompact quasi-metric space is totally bounded [4], we also show in [8] the following results.

THEOREM 2. *Every complexity space with a lower bound is totally bounded.*

THEOREM 3. *Let $L \in C$. Then, $(C_L, (d_C)^s)$ is a compact metric space.*

Since the complexity of a given program is frequently obtained by a summation of complexity functions or by a product of a complexity function by a constant, where these operations intuitively correspond to operations carried out by the program on data structures, it is interesting and useful to obtain an appropriate context both for realizing these operations and for developing a consistent theory for the analysis of the dual complexity space. In [9] it is shown that the notion of a quasi-normed semilinear space provides a suitable setting to carry out the analysis claimed above, even in the general case that the range space is any structural biBanach semilinear space.

A quasi-normed space $(E, \|\cdot\|)$ (in the sense of [1] and [2]) whose induced quasi-metric is bicomplete is called a *biBanach space* [9].

DEFINITION 1. A biBanach semilinear space is a pair $(F, \|\cdot\|_F)$ such that F is a nonempty subset of a biBanach space $(E, \|\cdot\|)$, F is closed in the Banach space $(E, \|\cdot\|^s)$ and $(F, \|\cdot\|_F)$ is a quasi-normed semilinear space. If in addition, the quasi-metric space $(F, d_{\|\cdot\|_F})$ is an optimal order-convex quasi-metric join semilattice and it has a maximum (in the sense of [11]), then $(F, \|\cdot\|_F)$ is called a structural biBanach semilinear space.

Structural biBanach semilinear spaces are called *biBanach norm-weightable spaces* in [9].

Let $(F, \|\cdot\|_F)$ be a biBanach semilinear space. Then, there exists a biBanach space $(E, \|\cdot\|)$ for which F satisfies the conditions in Definition 1. Now put

$$\mathcal{C}_F^* = \{f : \omega \to F \mid \sum_{n=0}^{\infty} 2^{-n}(\|f(n)\|_F)^s < +\infty\},$$

and, for each $f \in \mathcal{C}_F^*$,

$$\|f\|_{\mathcal{C}^*} = \sum_{n=0}^{\infty} 2^{-n}\|f(n)\|_F.$$

It is shown in [9] that the central (quasi-metric) theorems in [8], cited above, still hold for the general theory of structural biBanach semilinear spaces. Furthermore, we prove the following results.

PROPOSITION 1. *Let $(F, \|\cdot\|_F)$ be a biBanach semilinear space. Then, $(\mathcal{C}_F^*, \|\cdot\|_{\mathcal{C}^*})$ is a biBanach semilinear space.*

It follows from Proposition 1 that the quasi-metric $d_{\|\cdot\|_{\mathcal{C}^*}}$ defined on $\mathcal{C}_F^*$ by

$$d_{\|\cdot\|_{\mathcal{C}^*}}(f, g) = \sum_{n=0}^{\infty} 2^{-n} \|g(n) - f(n)\|$$

is bicomplete.

DEFINITION 2. Let $(F, \|\cdot\|_F)$ be a structural biBanach semilinear space. Then, the quasi-metric space $(\mathcal{C}_F^*, d_{\|\cdot\|_{\mathcal{C}^*}})$ is called the dual complexity space of $(F, \|\cdot\|_F)$.

PROPOSITION 2. *The dual complexity space $(\mathcal{C}_F^*, d_{\|\cdot\|_{\mathcal{C}^*}})$ is an optimal order-convex quasi-metric join semilattice and it has a maximum.*

From Propositions 1 and 2, we deduce the main result of [9].

THEOREM 4. *Let $(F, \|\cdot\|_F)$ be a structural biBanach semilinear space. Then, $(\mathcal{C}_F^*, \|\cdot\|_{\mathcal{C}^*})$ is a structural biBanach semilinear space.*

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