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Riesz's Lattices and Almost Positive Operators †

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1. Introduction

The classic Korovkin-Bohman's theorem (1950), [13], [14], [7], states that for a sequence (L_n), $n \in N$, of linear positive operators from $C([a,b])$ into $C([a,b])$ the following statements are equivalent:

(i) $L_n f$ converges uniformly to f in $[a,b]$ for all $f \in C([a,b])$;

(ii) $L_n f_i$ converges uniformly to f_i in $[a,b]$ for $f_i(x) = x^i$ and $i = 0,1,2$;

(iii) $L_n \mathbb{1}$ converges uniformly to $\mathbb{1}$ in $[a,b]$; H_n converges uniformly to 0 in $[a,b]$, where $H_n(t) = (L_n \Phi_t)(t)$ and $\Phi_t(x) = (x-t)^2$.

(i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iii) are the results of Korovkin [13, 14] and Bohman [7]. Korovkin result is based in Bersntein's proof of the Weierstrass theorem. Now, the proof of the Weierstrass theorem is an elegant consequence of the Korovkin-Bohman theorem, using the sequence of linear positive operators $(B_n)_{n \in N}$, $B_n : C([0,1]) \to C([0,1])$, such that for all $f \in C([a,b])$ $B_n(f)$ is the Bernstein polynomial of order n.

An abstract formulation of the results of Korovkin is as follows:

Let E, F be two real or complex normed spaces. Let S be a fixed operator from E to F. Let M be a fixed subclass of operators, usually linear, from E to F, and let H be a subset of E.

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The Korovkin closure, or shadow, of H with respect to M and S is the set of all $x \in E$ satisfying the following condition:

For every net of operators (L_i) in M such that $\lim L_i h = Sh$ holds for all $h \in H$ then $\lim L_i x = Sx$.

The Korovkin Closure of H with respect to M and S is denoted by $\mathrm{Kor}_{M,S}(H)$.

Some classic examples:

(1) Classic Korovkin theorem.

In the classic Korovkin Theorem $E = F = C_R[a,b]$ are the real continuos functions in $[a,b]$, M is the set of positive and linear operators in $C_R[a,b]$, S is the identity operator and H the subspace generated by $\{1, x, x^2\}$. Then $\mathrm{Kor}_M(H) = C_R(X)$.

(2) Y.A. Shaskin.

In the work of Shaskin [16],[17],[18] $E = F = C_R(X)$ are the spaces of real and continuous functions on the compact metric space X. M is the set of positive and linear operators in $C_R(X)$, S is the identity operator and H a subsbspace of $C_R(X)$, containing the constants, that separates the points of X. Then $\mathrm{Kor}_M(H) = C_R(X)$ if and only if the Choquet boundary of H (see [16]) coincide with X.

(3) H. Berents and G.G. Lorentz.

The paper [5] is concerned with the geometric aproach to Korovkin's theorems and has its origin in the paper [17] of Shaskin. In this case $E = F = C_R(X)$, S is the identity operator and M is one of the following sets: either the set T^+: positive and linear operators in $C_R(X)$; or the set T_1 of linear contractions in $C_R(X)$; or else their union. $H = \mathrm{lin}\, M$, is the linear hull of M in $C_R(X)$. Then $\mathrm{Kor}_H(G) = C_R(X)$ if and only if the unique representing measure is the evaluation in x.

(4) H. Bauer.

In [2] H. Bauer shows that Korovkin type theorems can be obtained on arbitrary topological spaces without any use of measure theory.

He shows that the methods developed in a previous paper [4] lead in a natural way to generalizations of Korovkin's theorems for increasing, possibly non-linear maps $L_i : C_R(X) \to B_R(X)$, (the space of real bounded functions on X). He shows that all those generalizations can be obtained for locally compact Hausdorff spaces.

In other paper of H. Bauer and K. Donner, [3] they work in $C_0(X)$ the space of real continuous functions vanishing at infinity on a locally compact topological space X. Here the definition of the Korovkin closure has to be modified by admitting only equicontinuous nets.

Our work begins with the paper of G.I. Kudriasev [15]. In that paper it is proved the classic Korovkin theorem for sequences of linear almost-positive operators.

In [9] X is a topological compact Hausdorff space and $F_C(X)$, (respectively $B_C(X)$, $C_C(X)$) the linear spaces of complex functions on X (respectively bounded, continuous). Analogously $F(X)$, $B(X)$ and $C(X)$ denote the linear subspaces of real functions.

In [9], the results of [15] are expanded to the space of continuous complex functions defined in a compact topological space, working with sequences of almost positive linear operators and using uniform convergence. Subsequently, in [10], [11], [12], other results with other sorts of convergence were obtained.

In this work we extend some of these results to Riesz's lattices.

Let $(E, \leq)$ be a Riesz's lattice. We write $(u_\alpha) \downarrow 0$ to denote a net $(u_\alpha)_{\alpha\in\Lambda}$ in E decreasing to zero; we write $\text{o-lim}\, u_\alpha = u$ to indicate order convergence of $(u_\alpha)_{\alpha\in\Lambda}$ to u. Given $u \in E_+$ (the set of positive elements of E) we set $E_u = \{x \in E\colon |x| \leq \lambda u \text{ for some } \lambda > 0\}$. An element $u \in E_+$ is called a strong unit if $E_u = E$.

2. Basic Results

DEFINITION 2.1. Let E be a *Riesz's lattice*, with u as strong unit and let $(L_\alpha)_{\alpha\in\Lambda}$ be a family of linear operators from E into E. We shall say that *the family* $(L_\alpha)_{\alpha\in\Lambda}$ *is of class* $\widetilde{R}$, if for every $x \in E$ with $x \geq 0$, there exists $(u_\alpha) \downarrow 0$ such that $L_\alpha x \geq -u_\alpha$, $\alpha \in \Lambda$.

THEOREM 2.2. *Let E be a Riesz's lattice with u as strong unit and $(L_\alpha)_{\alpha\in\Lambda}$, a family of linear operators from E into E. Then* $\text{o-lim}\, L_\alpha x = 0$, *for all* $x \in E$ *if and only if* $(L_\alpha)_{\alpha\in\Lambda}$, *is of class* $\widetilde{R}$ *and* $\text{o-lim}\, L_\alpha u = 0$.

Proof. If $x \in E$ with $x \geq 0$, then $| L_\alpha x | \leq w_\alpha$, for some net $(w_\alpha) \downarrow 0$; thus $L_\alpha x \vee -L_\alpha x \leq w_\alpha$, and then $L_\alpha x \geq -w_\alpha$ and $(L_\alpha)_{\alpha\in\Lambda}$ is of class $\widetilde{R}$.

On the other hand if $x \in E$, there exists $\lambda > 0$ such that $| x | \leq \lambda u$. Let's define $\rho_1 = \lambda u - x$; and $\rho_2 = \lambda u + x$.

We have that $\rho_1 \geq 0$ and $\rho_2 \geq 0$. Since $(L_\alpha)_{\alpha\in\Lambda}$ is of class $\widetilde{R}$, there exists $w_\alpha \downarrow 0$ and $v_\alpha \downarrow 0$ such that $L_\alpha \rho_1 \geq -w_\alpha$ and $L_\alpha \rho_2 \geq -v_\alpha$ for all $\alpha \in \Lambda$.

If $\alpha \in \Lambda$, we define $h_\alpha = w_\alpha \vee v_\alpha$ then $h_\alpha \geq w_\alpha$ and $h_\alpha \geq v_\alpha$; by the linearity of L_α we have that $\lambda L_\alpha u - L_\alpha x \geq -w_\alpha \geq -h_\alpha$ and $\lambda L_\alpha u + L_\alpha x \geq -v_\alpha \geq -h_\alpha$; then, $|L_\alpha x| \leq \lambda |L_\alpha u| + h_\alpha$.

Since o-lim $L_\alpha u = 0$, there exists $r_\alpha \downarrow 0$ such that $|L_\alpha u| \leq r_\alpha$; therefore

$$|L_\alpha x| \leq \lambda r_\alpha + h_\alpha = t_\alpha, \text{ where } t_\alpha \downarrow 0,$$

so that o-lim $L_\alpha x = 0$. ∎

THEOREM 2.3. *Let E be a Riesz's lattice with u as strong unit and $(L_\alpha)_{\alpha\in\Lambda}$ a family of linear operators from E into E. Let $(\beta_\alpha)_{\alpha\in\Lambda}$ be a family of linear operators from E in E such that* o-lim $\beta_\alpha x = x$ *for all $x \in E$. Then* o-lim $L_\alpha x = x$, *for all $x \in E$, if and only if $(L_\alpha - \beta_\alpha)_{\alpha\in\Lambda}$ is of class $\widetilde{R}$ and* o-lim $L_\alpha u = u$.

Proof. If $x \in E$ with $x \geq 0$, and o-lim $L_\alpha x = x$ since o-lim $\beta_\alpha x = x$, then o-lim$(L_\alpha - \beta_\alpha)(x) = 0$, and then $(L_\alpha - \beta_\alpha)_{\alpha\in\Lambda}$ is of class $\widetilde{R}$, and the necessity is proved.

On the other hand if o-lim$(L_\alpha - \beta_\alpha)u =$ (o-lim $L_\alpha u$) $-$ (o-lim $\beta_\alpha u$) $= u - u = 0$, then by Theorem 2.2, if $x \in E$ then o-lim$(L_\alpha - \beta_\alpha)x = 0$ and since o-lim $\beta_\alpha x = x$, then o-lim $L_\alpha x = x$. The sufficiency is proved. ∎

Remark 2.4. If there is a net of operators $(\beta_\alpha)_{\alpha\in\Lambda}$ such that $(L_\alpha - \beta_\alpha)_{\alpha\in\Lambda}$ is of class $\widetilde{R}$, then for every net $(H_\alpha)_{\alpha\in\Lambda}$ such that o-lim $H_\alpha t = t$, $t \in E$, we have that $(L_\alpha - H_\alpha)_{\alpha\in\Lambda}$ is of class $\widetilde{R}$. In effect, there are $w_\alpha \downarrow 0$ and $v_\alpha \downarrow 0$ such that $(L_\alpha - \beta_\alpha)(x) \geq -w_\alpha$, and $|\beta_\alpha(x) - H_\alpha(x)| \leq v_\alpha$, $t \geq 0$, $\alpha \in \Lambda$. Then $(\beta_\alpha - H_\alpha)(x) \geq -v_\alpha$.

Therefore,

$$(L_\alpha - H_\alpha)(x) \geq -w_\alpha - v_\alpha$$

and $(L_\alpha - H_\alpha)$ is of class $\widetilde{R}$.

3. MAIN RESULTS

DEFINITION 3.1. Let E and F be Riesz's lattices. A function $T : E \to F$ is o-continuous in $v \in E$, if for every net $(v_\alpha)_{\alpha\in\Lambda}$ o-convergent to $v \in E$, then $(T(v_\alpha))_{\alpha\in\Lambda}$ is o-convergent to $T(v) \in F$.

From now on X shall denote a compact Hausdorff topological space, and $C(X)$ the Riesz lattice of real continuous functions on X with pointwise order and 1 as strong unit. K shall denote a subspace of $C(X)$ containing the constant functions.

DEFINITION 3.2. We shall say that $\{f_x\}_{x\in X}$, is a test family of functions in K, if the following conditions hold:

a) For every $x \in X$, $f_x \in K$ and the function $(x,t) \rightsquigarrow f_x(t)$ is continuous in $X \times X$.
b) For every $x \in X$, $f_x(x) = 0$.
c) For every $t \in X$, $x \neq t$, $f_x(t) > 0$.

THEOREM 3.3. *Let $\{f_x\}_{x\in X}$ be a test family of functions in K. Let $(L_\alpha)_{\alpha\in\Lambda}$ be a family of operators from K in $C(X)$. Then* o-lim $L_\alpha f = f$ *for all $f \in K$, if and only if:*

1.- $(L_\alpha)_{\alpha\in\Lambda}$*, of class* $\widetilde{R}$;

2.- o-lim $L_\alpha \mathbb{1} = \mathbb{1}$;

3.- o-lim $L_\alpha f_x = f_x$, $x \in X$.

Proof. The necessity is immediate. For the sufficiency, let $f \in K$, $f \geq 0$, and let $(\epsilon_\alpha)_{\alpha\in\Lambda}$, be a net of real numbers decreasing to zero. We define the auxiliary function $X \to \mathbb{R}$ given by $\Phi_x^\alpha = f - f(x) + \epsilon_\alpha + hf_x$.

We are going to prove first that there exists $h > 0$ that only depends on α and f, such that $\Phi_x^\alpha(t) \geq 0$ for all $(x,t) \in X \times X$.

The function $(x,t) \rightsquigarrow F(x,t) = \Phi_x^\alpha(t)$ is continuous in the compact topological space $X \times X$ and $F(x,x) = \Phi_x(x) = \epsilon_\alpha > 0$; thus there exists an open neighborhood V_x of (x,x) such that $F(x,t) > 0$ for all $(x,t) \in V_x$.

The set $A = \cup_{x\in X} V_x$, is open. If $B = (X \times X) \setminus A$, then B is compact.

If $B \neq \emptyset$ then the continuous function $(x,t) \rightsquigarrow f_x(t)$ satisfies that $f_x(t) > 0$, for all $(x,t) \in B$. Hence there exist $m > 0$ such that $f_x(t) \geq m$ for all $(x,t) \in B$. Thus

$$\frac{f(x) - f(t) - \epsilon_\alpha}{f_x(t)} \leq \frac{f(x) - f(t)}{f_x(t)} \leq \frac{2\|f\|}{m}.$$

Take $h \in R$ such that $h \geq \frac{2\|f\|}{m}$; then

$$\frac{f(x) - f(t)}{f_x(t)} \leq h \Longrightarrow f(x) - f(t) \leq hf_x(t),$$

and then $f(x) - f(t) - \epsilon_\alpha \leq hf_x(t)$; that is $\Phi_x^\alpha(t) \geq 0$ for all $(x,t) \in X \times X$.

Now let $x \in X$ be fixed; since $(L_\alpha)_{\alpha\in\Lambda}$, is of class $\widetilde{R}$ there exists $w_\alpha^x \downarrow 0$ such that $L_\alpha \Phi_x^\alpha \geq -w_\alpha^x$. By the linearity of the operator L_α one has

$$L_\alpha f - f(x)L_\alpha u + \epsilon_\alpha L_\alpha u + hL_\alpha f_x \geq -w_\alpha^x,$$

and then

$$\begin{aligned}
L_\alpha f &\geq f(x)L_\alpha u - \epsilon_\alpha L_\alpha u - hL_\alpha f_x - w_\alpha^x \\
&= f(x)[L_\alpha u - u] - \epsilon_\alpha[L_\alpha u - u] - h[L_\alpha f_x - f_x] + \\
&\quad + f(x)u - \epsilon_\alpha u - hf_x - w_\alpha^x
\end{aligned} \tag{1}$$

Since o-lim $L_\alpha u = u$ and o-lim $L_\alpha f_x = f_x$, there exists $r_\alpha \downarrow 0$ and $t_\alpha^x \downarrow 0$ such that $|L_\alpha u - u| \leq r_\alpha$ and $|L_\alpha f_x - f_x| \leq t_\alpha^x$; that is: $(L_\alpha u - u) \vee (-L_\alpha u + u) \leq r_\alpha$; $(L_\alpha f_x - f_x) \vee (-L_\alpha f_x + f_x) \leq t_\alpha^x$.

Therefore

$$\begin{aligned}
L_\alpha u - u &\geq -r_\alpha; \\
-L_\alpha u + u &\geq -r_\alpha; \\
-L_\alpha f_x + f_x &\geq -t_\alpha^x,
\end{aligned} \tag{2}$$

substituting (2) in (1) and simplifying

$$L_\alpha f - f(x)u \geq -f(x)r_\alpha - \epsilon_\alpha r_\alpha - ht_\alpha^x - \epsilon_\alpha u - hf_x - w_\alpha^x.$$

Evaluating in x one obtain $L_\alpha(f,x) - f(x)u \geq -f(x)r_\alpha(x) - \epsilon_\alpha r_\alpha(x) - ht_\alpha^x(x) - \epsilon_\alpha u - w_\alpha(x)^x = -p_\alpha(x)$; finally writting $p_\alpha = fr_\alpha + \epsilon_\alpha r_\alpha + ht_\alpha^x + \epsilon_\alpha u + w_\alpha^x$, one obtain

$$L_\alpha f - f \geq -p_\alpha, \text{ where } p_\alpha \downarrow 0,$$

and thus $(L_\alpha - \mathrm{Id})_{\alpha\in\Lambda}$ is of class $\widetilde{R}$. Applying now the theorem 2.3 the proof is complete. ∎

THEOREM 3.4. *Let $\{f_x\}_{x\in X}$ be a test family of functions in K and $(L_\alpha)_{\alpha\in\Lambda}$ a net of linear operators from K into $C(X)$. Then* o-lim $L_\alpha f = f$, *for all $f \in H$, if and only if:*

(i) o-lim $L_\alpha \mathbb{1} = \mathbb{1}$;

(ii) o-lim $L_\alpha f_x = f_x$, $x \in X$;

(iii) o-$\lim_\alpha \sup |L_\alpha f| \leq |f|$, $(f \in K)$.

Proof. For the sufficiency we just have to prove that $(L_\alpha)_{\alpha\in\Lambda}$ is of class $\widetilde{R}$ in K, and then apply theorem 3.3.

Otherwise, there exists $f_0 \in K$, $f_0 \geq 0$, such that for every net $(w_\alpha)_{\alpha\in\Lambda}$, $w_\alpha \downarrow 0$, and $\alpha \in \Lambda$, there exists $\beta \geq \alpha$, such that

$$L_\beta f_0 \text{ is not greater than } -w_\alpha. \tag{3}$$

If $M = \vee_x f_0(x)$, then

$$L_\alpha(M - f_0) = ML_\alpha \mathbb{1} - L_\alpha f_0. \tag{4}$$

Applying the hypothesis (i), there exists $(v_\alpha)_{\alpha\in\Lambda}$, $v_\alpha \downarrow 0$, such that

$$|L_\alpha \mathbb{1} - 1| \leq v_\alpha, \quad (\alpha \in \Lambda). \tag{5}$$

Taking the net $(Mv_\alpha)_{\alpha\in\Lambda}$ and by (3), for every $\alpha \in \Lambda$ there exists $\beta \geq \alpha$ and $x_\beta \in X$ such that

$$L_\beta(f_0, x_\beta) < -Mv_\beta(x_\beta), \tag{6}$$

from (5) we obtain:

$$L_\beta \mathbb{1} \geq 1 - v_\beta, \quad (\beta \in \Lambda). \tag{7}$$

Evaluating in (4) , for $\beta \geq \alpha$,

$$L_\beta((M - f_0), x_\beta) = ML_\beta(1, x_\beta) - L_\beta(f_0, x_\beta). \tag{8}$$

Substituting (6) and (7) in (8):

$$L_\beta((M - f_0), x_\beta) \geq M - Mv_\beta(x_\beta) + Mv_\beta(x_\beta) = M. \tag{9}$$

On the other hand,

$$|M - f_0| = (M - f_0) \vee (f_0 - M) = M - f_0 \leq M. \tag{10}$$

From (9) and (10) we have that

$$L_\beta((M - f_0), x_\beta) \geq |M - f_0|,$$

then $\limsup_{\alpha\in\Lambda} |L_\alpha((M - f_0), x_\beta)| \geq |M - f_0|$, which contradicts (iii).

On the other hand if $\text{o-lim}\, L_\alpha f = f$, then $\text{o-lim}_\alpha \sup L_\alpha f = \text{o-lim}_\alpha \inf L_\alpha f = f$ and by the continuity of limit regarding the sup, we have that $\text{o-lim}_\alpha \sup |L_\alpha f| \leq |f|$. ∎

COROLLARY 3.5. *Let $\{f_1, f_2, \ldots, f_n, \ldots\}$ be a countable family of real functions in K such that for every $i \in N$, f_i and f_i^2 belongs to K. Assume that the set $\{f_i\}_{i\in\mathbb{N}}$ separates the points of X. Let $\{L_n\}_{n\in\mathbb{N}}$, be a sequence of linear operators $H \to C(X)$ that are continuous with the uniform norm. Then, $\text{o-lim}\, L_n f = f$ for every $f \in K$ if and only if:*

(i) $\text{o-lim}\, L_n \mathbb{1} = \mathbb{1}$;

(ii) $\text{o-lim}\, f_i = f_i$, $i \in \mathbb{N}$;

(iii) $\text{o-lim}\, f_i^2 = f_i^2$, $i \in \mathbb{N}$;

(iv) $\{L_n\}_{n\in\mathbb{N}}$ *is of class* $\widetilde{R}$.

Proof. The necessity is immediate. For the sufficiency we build a test family of functions in K defining

$$f_x := \sum_{i=1}^{\infty} \alpha_i (f_i - f_i(x))^2,$$

where $\alpha_i = \frac{1}{2^i M_i^2}$ and $M_i = \|f_i\|$. The function f_x is continuous, because every f_i is continuous and since

$$|\alpha_i (f_i - f_i(x))^2| \leq \alpha_i 4M_i^2 = \frac{1}{2^i M_i^2} 4M_i^2 = \frac{1}{2^{i-2}},$$

then the series converges uniformly on X.

Now, we check that $(f_x)_{x\in X}$ is a test family of functions in K:

Conditions (a) and (b) are immediate; and so is (c) since the family (f_i) separates the points of X. Given $m \in \mathbb{N}$, let

$$f_x^m = \sum_{i=1}^{m} \alpha_i (f_i - f_i(x))^2.$$

We have that $(f_x^m)_{m\in\mathrm{N}}$ is uniformly convergente to f_x, and thus $\lim_m L_n(f_x^m) = L_n(f_x)$. Hence,

$$L_n(f_x) = \sum_{i=1}^{\infty} \alpha_i L_n((f_i - f_i(x))^2).$$

Since o-lim $L_n((f_i - f_i(x))^2) = (f_i - f_i(x))^2$ we have that o-lim $L_n(f_x) = f_x$. The hypothesis of the theorem 3.4 are fulfilled, and therefore the proof of the corollary is complete. ∎

REFERENCES

[1] ALTOMARE, F., CAMPITI, M., "Korovkin-type Approximation Theory and its Applications", De Gruyter Studies in Mathematics 17, 1994.

[2] BAUER, H., Theorem of Korovkin type for adapted spaces, *Ann. Inst. Fourier,* **23** (4) (1973), 245–260.

[3] BAUER, H., DONNER, K., Korovkin Approximation in $C_0(X)$, *Math. Ann.,* **236** (1978), 225–237.

[4] BAUER, H., Silovscher Rand und Dirichletsches Problem, *Ann. Inst. Fourier,* **11** (1964), 89–136.

[5] BERENS, H., LORENTZ, G.G., Geometric theory of Korovkin sets, *J. Approx. Theory,* **15** (1975), 161–169.

[6] BERENS, H., LORENTZ G.G., Theorems on Korovkin type for positive linear operators on Banach lattices, *J. Approx. Theory,* **15** (1975), 161–189.

[7] BOHMAN, H., On approximation of continuous and of analytic functions, *ARK,* **2** (1952), 43–56.

[8] BOMBAL, F., "Retículos de Riesz y de Banach", Notas de curso (1986-1987), Depto. de Análisis Matemático. Universidad Complutense de Madrid, 1987.

[9] FERNÁNDEZ MUÑIZ, J.L., Teoremas cualitativos de tipo Korovkin para sucesiones de operadores de la clase $\tilde{R}$, *Cienc. Mat.,* **3** (3) (1982), 57–69.

[10] FERNÁNDEZ MUÑIZ, J.L., Qualitative Korovkin type theorems for R_F convergence, *J. of Aprox. Theory,* **80** (1995), 1–9.

[11] FERNÁNDEZ MUÑIZ, J.L., Approximation theorems with A-convergence, *Bol. Soc. Mat. Mexicana,* **3** (3) (1996), 175–191.

[12] FERNÁNDEZ MUÑIZ, J.L., Approximation in Measurable Spaces, in "Proceedings of the Second International Conference on Approximation and Optimization in the Caribbean", Peter Lang Publising House, V. 8, 1995, 256–273.

[13] KOROVKIN, P.P., On convergence of linear positive operators in the space of continuous functions, *DAN,* **90** (1959), 961–964.

[14] KOROVKIN, P.P., "Linear operators and Approximation Theory", Hindustan Pull. Corp., Delhi, India, 1960.

[15] KUDRIASEV, G.I., Convergence of a sequence of linear operators, *Mat. Anal. y Terra Funks,* **8** (1977), 102–105.

[16] SHASKIN, Y.A., On the convergence of linear positive operators in the space of continuous functions, *DAN,* **131** (1960), 525–527.

[17] SHASKIN, Y.A., Korovkin systems in spaces of continuous functions, *Inv. Akad. Nauk.,* Ser. Mat. **26** (1962), 495 - 512.
[18] SHASKIN, Y.A., On convergence of contraction operators, *Math. Cluj.,* **11** (1969), 355 - 360.
[19] YOSIDA, K., "Functional Analysis", Springer-Verlag, 1971.