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On the Associated Locally Connected Space

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In Functional Analysis the concept of certain associated topologies (such as e.g. the associated barrelled or bornological or nuclear topology) has been a standard tool for the last thirty years. In fact, such an associated topology will exist in an abstract functorial sense if a property $\mathcal{P}$ of topological vector spaces is either stable under the formation of initial topologies w.r. to linear maps or stable under the formation of final topologies in the category of topological vector spaces; in that case, the supremum (infimum) of all vector space topologies with $\mathcal{P}$, coarser (finer) than a given one, fulfills the demands on an associated topology.

In General Topology the properties of topological spaces with such a stability behaviour as described above do not often occur; therefore associated topologies are less frequent in set theoretic topology. Of course, "completely regular" (without T_2) is stable under initial topologies and the associated (from below) completely regular topology exists.

Fortunately, the property of being locally connected is stable under the formation of arbitrary final topologies and thus gives rise to an associated topology from above, which is a useful tool in the theory of covering spaces (e.g. for the construction of universal coverings) and which will be studied in this note.

1. The associated locally connected space

A topological space X is called locally connected (lc), if every $x \in X$ has a neighbourhood basis consisting of connected sets.

We first observe that the property (lc) is stable under the formation of arbitrary final topologies. In fact, as (lc) is trivially stable under topological direct sums and as discrete spaces are (lc), it suffices to show that quotients of locally connected spaces are again (lc), which is a result of G.T. Whyburn [2], see also R. Engelking [1, 6.3.3.d)].

Now let $X = (X, \mathfrak{T})$ be a topological space. Then the infimum of all topologies $\mathfrak{R}$ on X such that $\mathfrak{R} \supset \mathfrak{T}$ and such that $(X, \mathfrak{R})$ is (lc), is the coarsest locally connected topology on X finer than $\mathfrak{T}$; it will be denoted by $\mathfrak{T}^{\mathrm{lc}}$ and will be called the associated locally connected topology with $(X, \mathfrak{T})$. This formation obviously possesses the following functorial property: whenever $(X, \mathfrak{T})$, $(Y, \mathfrak{S})$ are topological spaces and $f : (X, \mathfrak{T}) \to (Y, \mathfrak{S})$ is a continuous map, then $f : (X, \mathfrak{T}^{\mathrm{lc}}) \to (Y, \mathfrak{S}^{\mathrm{lc}})$ is again continuous. (In fact, the final topology on Y w.r. to $f : (X, \mathfrak{T}^{\mathrm{lc}}) \to Y$ is (lc) and finer than $\mathfrak{S}$, hence finer than $\mathfrak{S}^{\mathrm{lc}}$). Consequently, we obtain for the relative topology $\mathfrak{T} \cap Z$ induced by $(X, \mathfrak{T})$ on a subset Z that $(\mathfrak{T} \cap Z)^{\mathrm{lc}} \supset \mathfrak{T}^{\mathrm{lc}} \cap Z$. As local connectedness is stable under finite products, we obtain for product topologies $\mathfrak{T} \times \mathfrak{S}$ that $(\mathfrak{T} \times \mathfrak{S})^{\mathrm{lc}} = \mathfrak{T}^{\mathrm{lc}} \times \mathfrak{S}^{\mathrm{lc}}$, whereas for general product topologies $\prod\limits_{i \in I} \mathfrak{T}_i$ we only have $(\prod\limits_{i \in I} \mathfrak{T}_i)^{\mathrm{lc}} \supset \prod\limits_{i \in I} \mathfrak{T}_i^{\mathrm{lc}}$.

2. The topology of connected components of open sets

It would of course be quite desirable to have an intrinsic description of the associated locally connected topology. For that purpose let $(X, \mathfrak{T})$ be a topological space. For every $U \in \mathfrak{T}$ and $x \in U$ let $Z_U(x)$ denote the connected component of $(U, \mathfrak{T} \cap U)$ containing x.

It is easy to see that

$$\mathfrak{L} := \{Z_U(x) : U \in \mathfrak{T},\ x \in U\}$$

is the basis of a topology $\mathfrak{T}'$ on X. In fact, given $U, V \in \mathfrak{T},\ x \in U,\ y \in V$, then

$$Z_U(x) \cap Z_V(y) = \bigcup \{Z_{U \cap V}(z) \colon z \in Z_U(x) \cap Z_V(y)\}.$$

As $U = \bigcup_{x \in U} Z_U(x)$ for all $U \in \mathfrak{T}$, we obtain that $\mathfrak{T} \subset \mathfrak{T}'$, and, clearly, $\mathfrak{T} = \mathfrak{T}'$ if and only if $(X, \mathfrak{T})$ is (lc).

Moreover, $\mathfrak{T}' \subset \mathfrak{T}^{\mathrm{lc}} =: \mathfrak{S}$. In fact, let $U \in \mathfrak{T},\ x \in U$. Then

$$Z := Z_U(x) = \bigcup_{y \in Z} Z_{(U, \mathfrak{S} \cap U)}(y) \in \mathfrak{S},$$

since in particular $U \in \mathfrak{S}$, whence its components w.r. to $\mathfrak{S} \cap U$ again belong to the (lc)–topology $\mathfrak{S}$.

Consequently, we obtain that

$$\mathfrak{T}' = \mathfrak{T}^{\mathrm{lc}} \Longleftrightarrow \mathfrak{T}' \text{ is (lc)}.$$

The following example shows that for a given topological space $(X, \mathfrak{T})$ the topology $\mathfrak{T}'$ need not be locally connected.

3. Example

We start with a Hausdorff connected topological space $(X, \mathfrak{T})$ containing $U \in \mathfrak{T}$ and $x \in U$ such that $Z_U(x) = \{x\}$. For the construction of such a space, choose an irrational number $\alpha \in [0,1]$, put

$$X := ([0,1] \times \{0\}) \cup (([0,1] \cap \mathbb{Q}) \times (0,1)) \cup \{(\alpha, 1)\}$$

provided with the relative topology induced by $\mathbb{R}^2$,

$$U := \{(\beta, \gamma) \in X : \gamma > \frac{1}{2}\}, \quad x := (\alpha, 1).$$

Next we form the countable product $Y := X^{\mathbb{N}}$ with product topology $\mathfrak{S} := \mathfrak{T}^{\mathbb{N}}$ and show that $(Y, \mathfrak{S}')$ is not locally connected:

Let $y := (x)_{n \in \mathbb{N}} \in Y$. We first prove that

$$\prod_{k \leq n} \{x\} \times \prod_{h > n} X, \quad n \in \mathbb{N}$$

is a basis of the neighbourhood filter of y in $(Y, \mathfrak{S}')$ consisting of $\mathfrak{S}'$–open sets. In fact, for all n,

$$V_n := \prod_{k \leq n} U \times \prod_{h > n} X \in \mathfrak{S}$$

and it is easy to see that

$$W_n := \prod_{k \leq n} \{x\} \times \prod_{k > n} X$$

is the connected component of $(V_n, \mathfrak{S} \cap V_n)$ containing $y = (x)_{n \in \mathbb{N}}$. Thus, $W_n \in \mathfrak{S}'$. On the other hand, let $V \in \mathfrak{S}$ such that $y \in V$. Then there is $n \in \mathbb{N}$ such that $W_n \subset V$; as $(W_n,\ \mathfrak{S} \cap W_n)$ is connected, we obtain that $W_n \subset Z_{(V, \mathfrak{S} \cap V)}(y)$. In particular, $\{y\}$ is not open in $(Y, \mathfrak{S}')$.

On the other hand, assume that there is a connected set Z in $(Y, \mathfrak{S}')$ such that $\{y\} \underset{\neq}{\subset} Z$. Choose $z = (x_k)_{k\in\mathbb{N}} \in Z \setminus \{y\}$. Then there is $n \in \mathbb{N}$ such that $x_n \neq x$, hence $z \notin W_n$. W_n is $\mathfrak{S}$–closed, therefore $\mathfrak{S}'$–closed and Z is contained in the union of the two $\mathfrak{S}'$–open sets W_n and $Y \setminus W_n$ which have both nonempty intersection with Z, a contradiction. Now, in $(Y, \mathfrak{S}')$ the one–point–set $\{y\}$ is a connected component, which is not open, hence $(Y, \mathfrak{S})$ is not (lc).

4. Transfinite iteration

The previous example shows that the associated locally connected topology cannot be described by the connected components of open sets in the original topology. Nevertheless, the associated (lc)–topology is obtained by iterating this procedure. In fact, let $(X, \mathfrak{T})$ be a topological space and let α be an ordinal. Define by transfinite recursion

$$\begin{aligned} \alpha = 1 \quad &\Rightarrow \quad \mathfrak{T}^{(\alpha)} := \mathfrak{T}' \\ \alpha = \beta + 1 \quad &\Rightarrow \quad \mathfrak{T}^{(\alpha)} := \mathfrak{T}^{(\beta)'} \\ \alpha \text{ a limit ordinal} \quad &\Rightarrow \quad \mathfrak{T}^{(\alpha)} := \bigvee_{\beta<\alpha} \mathfrak{T}^{(\beta)}. \end{aligned}$$

Then clearly, $\mathfrak{T} \subset \mathfrak{T}^{(\alpha)} \subset \mathfrak{T}^{(\mathrm{lc})}$ for all ordinals α. Since there are at most Pot(Pot(X))–many topologies on X, there is a smallest ordinal α such that $\mathfrak{T}^{(\alpha)} = \mathfrak{T}^{(\alpha+1)} = \mathfrak{T}^{(\alpha)'}$, which implies that $\mathfrak{T}^{(\alpha)}$ is (lc) and thus equal to $\mathfrak{T}^{\mathrm{lc}}$.

Remark. This interaction between $\mathfrak{T}^{\mathrm{lc}}$ and $\mathfrak{T}'$ is analogous to the interaction of the formation of the associated barrelled and the strong topology, respectively, on a locally convex space.

References

[1] Engelking, R., "General Topology", PWN – Polish Scientific Publishers, Warszawa, 1977.

[2] Whyburn, G.T., On quasi–compact mappings, *Duke Math. Journ.*, **19** (1952), 445–446.