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Almost Regular Operators are Regular

Teresa Bermúdez and Manuel González

Departamento de Análisis Matemático, Universidad de La Laguna,
38271 *La Laguna (Tenerife), Spain*

Departamento de Matemáticas, Universidad de Cantabria, 39071 *Santander, Spain*

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Recently, Lee and Choi [2] introduced a concept of almost regular operators, following a suggestion in [1, Preface]. They proved that if X and Y are Hilbert spaces, then $T \in L(X,Y)$ is almost regular if and only if T is regular. However, for X and Y non-complete normed spaces, they gave an example of an almost regular operator which is not regular. In the case that X and Y are Banach spaces they propose as an open problem whether almost regular operators and regular operators coincide. Here we give a positive answer to this problem.

Along this research announcement, X and Y are real or complex Banach spaces and $L(X,Y)$ denotes the set of all (bounded linear) operators acting from X into Y. For every $T \in L(X,Y)$ we denote by $R(T)$ and $N(T)$ the range and the kernel of T, respectively.

Definition 1. An operator $T \in L(X,Y)$ is called almost regular if there exists a bounded sequence $\{A_n\} \subset L(Y,X)$ such that

$$\|TA_nT - T\| \to 0 \quad \text{as } n \to \infty.$$

The operator $T \in L(X,Y)$ is called regular if there exists $A \in L(Y,X)$ such that $TAT = T$.

It is clear that regular operators are almost regular and that regular operators have close range. Moreover, $T \in L(X,Y)$ is regular if and only if $N(T)$ and $R(T)$ are (closed) complemented subspaces of X and Y, respectively.

Proposition 1. *Let $T \in L(X,Y)$ be an almost regular operator. Then $R(T)$ is closed.*

Remark 1. (1) If X and Y are Hilbert spaces, then Proposition 1 implies that almost regular operators are regular.

(2) If X is reflexive, then we can give a direct proof of the fact that every almost regular $T \in L(X,Y)$ is regular, using ultrafilter techniques.

The following characterization of regular operators is the key to prove that almost regular operators are regular, and may have some interest in its own (see [1, Theorem 3.82]).

THEOREM 1. *An operator $T \in L(X,Y)$ is regular if and only if there exists $A \in L(Y,X)$ so that $R(TAT) = R(T)$ and $N(TAT) = N(T)$. In this case,*

$$X = N(T) \oplus R(AT) \quad \text{and} \quad Y = N(TA) \oplus R(T).$$

THEOREM 2. *Every almost regular operator $T \in L(X,Y)$ is regular.*

Remark 2. (1) In the definition of almost regular operator, the condition $\{A_n\}$ bounded is not superfluous. For instance, the operator $T : \ell_2 \longrightarrow \ell_2$ given by $T(x_n) := (x_n/n)$ is not regular. However, the operators $A_n : \ell_2 \longrightarrow \ell_2$, given by

$$A_n(x_1, x_2, \ldots) := (x_1, 2x_2, \ldots, nx_n, 0, 0, \ldots)$$

satisfy $\|TA_nT - T\| \to 0$ and $\{A_n\}$ is not bounded.

(2) An operator $T \in L(X,Y)$ is regular if and only if it has closed range and there exists a (not necessarily bounded) sequence $\{A_n\}$ in $L(Y,X)$ so that $\|TA_nT - T\| \to 0$.

(3) If in the definition of almost regular operator T the operators A_n can be taken to be bijective, then $\dim N(T) = \dim Y/R(T)$. This can be seen as a "zero index" condition, although sometimes $\dim N(T) = \dim Y/R(T) = \infty$.

Finally, we give a result for operators in the closure of the set of all regular operators.

THEOREM 3. *Let $\{T_n\} \subset L(X,Y)$ be a sequence of regular operators. Assume that $T_n \to T$ as $n \to \infty$ and there exists a bounded sequence $\{U_n\} \subset L(Y,X)$ such that $T_nU_nT_n = T_n$ for all $n \in \mathbb{N}$. Then T is regular.*

Remark 3. (1) The condition of existence of a bounded sequence $\{U_n\}$ in Theorem 3 is not necessary in order that the limit of a sequence of regular operators be regular. The sequence of operators $\{T_n\}$ in $L(X\times X, X\times X)$ defined by $T_n(x,y) := (x, y/n)$ converges to a regular operator, but there is no bounded sequence $\{U_n\}$ so that $T_n U_n T_n = T_n$ for every n.

(2) If the sequence $\{U_n\}$ in Theorem 3 is unbounded, then there is a sequence $\{V_n\}$ of norm one operators in $L(Y,X)$ such that $TV_nT \to 0$ as $n \to \infty$.

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References

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[2] Woo Young Lee, Chun In Choi, Almost regular operators, *J. Korean Math. Soc.,* **29** (1992), 401 – 407.