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Derivations of crossed modules

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Let us start by considering (T,G,∂) a crossed module. We denote by $\mathrm{Der}(G,T)$ the set of all derivations from G to T. Each derivation d defines endomorphisms ϑ_d and Θ_d of G and T respectively, given by

$$\vartheta_d(g) = \partial\big(d(g)\big)g, \quad \Theta_d(t) = d\big(\partial(t)\big)t, \qquad g \in G, \quad t \in T.$$

If d_1, $d_2 \in \mathrm{Der}(G,T)$ then the formula $d_1 \cdot d_2 = d$ with

$$d(g) = d_1\big(\vartheta_{d_2}(g)\big)d_2(g) = \theta_{d_1}\big(d_2(g)\big)d_1(g)$$

defines on $\mathrm{Der}(G,T)$ a monoid structure. The *group of Whitehead* $\mathrm{D}(G,T)$ is defined as the group of units of $\mathrm{Der}(G,T)$, [5].

If (M,P,μ) is also a crossed module, an *action* of (M,P,μ) on (T,G,∂) is a morphism of crossed modules (α,β) from (M,P,μ) to $\mathrm{Act}(T,G,\partial)$, [5].

Let $(f_1,f_2) : (T,G,\partial) \to \big(T',G',\partial'\big)$ be a morphism of crossed modules. The crossed module $\mathrm{Ker}(f_1,f_2) = (\mathrm{Ker}\, f_1, \mathrm{Ker}\, f_2, \partial_{|\,\mathrm{Ker}\, f_1})$ is the *kernel* and the crossed module $\mathrm{Im}(f_1,f_2) = \big(f_1(T), f_2(G), \partial'_{|f_1(T)}\big)$ is the *image* of (f_1,f_2). If, moreover, there are actions, (α,β) and (α',β'), of (M,P,μ) on (T,G,∂) and (T',G',∂') respectively, then we say that $(f_1,f_2) : \big((T,G,\partial),(\alpha,\beta)\big) \to \big((T',G',\partial'),(\alpha',\beta')\big)$ *preserves* the action of (M,P,μ) if it satisfies:

(a) $f_1\left({}^p t\right) = {}^p f_1(t)$,

(b) $f_2\left({}^p g\right) = {}^p f_2(g)$,

(c) $f_1\big(\alpha(m)(g)\big) = \alpha'(m)\big(f_2(g)\big)$, for all $p \in P$, $t \in T$ and $m \in M$. The set of these morphisms will be denoted by $\mathrm{Hom}_{(M,P,\mu)}\big((T,G,\partial),(T',G',\partial')\big)$.

A sequence of morphisms of crossed modules of the form

$$(T',G',\partial') \xrightarrow{(f_1,f_2)} (T,G,\partial) \xrightarrow{(f_1',f_2')} (T'',G'',\partial'')$$

is *exact* if $\mathrm{Im}(f_1,f_2) = \mathrm{Ker}(f_1',f_2')$, [5], [4].

Definition 1. A *derivation* from (M,P,μ) to $((T,G,\partial),(\alpha,\beta))$ is a pair of groups derivations (d_1,d_2) that make the diagram

$$\begin{array}{ccc} M & \xrightarrow{\mu} & P \\ {\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle d_2} \\ T & \xrightarrow[\partial]{} & G \end{array}$$

commutative and $d_1\left({}^{p}m\right) = {}^{d_2(p)}\left({}^{p}d_1(m)\right)\left(\alpha\left({}^{p}m\right)(d_2(p))\right)^{-1}$, for all $p \in P$, $m \in M$.

Let $\mathrm{Der}\big((M,P,\mu),(T,G,\partial),(\alpha,\beta)\big)$ be the set of all derivations from (M,P,μ) to (T,G,∂). This set may be given an obvious abelian group structure when (T,G,∂) is an (M,P,μ)-module, that is if (T,G,∂) is abelian ($\mathrm{Z}(T,G,\partial) = (T,G,\partial)$, [5]) and (M,P,μ) acts over it.

Example. If A is a G-module then there exists actions of $(1,G,i)$ on $(0,A,i)$ and of (G,G,id) on (A,A,id). In this case the groups $\mathrm{Der}\big((1,G,i),(0,A,i)\big)$ and $\mathrm{Der}\big((G,G,id),(A,A,id)\big)$ are $\mathrm{Der}(G,A)$.

Proposition 1. *Let us to consider* $(f_1,f_2) : (M',P',\mu') \to (M,P,\mu)$ *a morphism,* $(d_1,d_2)\colon (M,P,\mu) \to (T,G,\partial)$ *a derivation and* $(g_1,g_2)\colon (T,G,\partial) \to (T',G',\partial')$ *a morphism that preserves the action of* (M,P,μ)*, then*

$$(d_1,d_2)(f_1,f_2) \in \mathrm{Der}\left(\left(M',P',\mu'\right),(T,G,\partial),(\alpha,\beta)(f_1,f_2)\right)$$

$$(g_1,g_2)(d_1,d_2) \in \mathrm{Der}\left((M,P,\mu),\left(T',G',\partial'\right),(\alpha',\beta')\right).$$

We now remember Norrie's definition of the *semidirect-product* [5]: let $(\alpha,\beta) : (M,P,\mu) \to \mathrm{Act}(T,G,\partial)$ be an action of (M,P,μ) on (T,G,∂). We can consider the semi-direct products $T \rtimes M$ and $G \rtimes P$ and there exists an action of $G \rtimes P$ on $T \rtimes M$ defined as follows:

$${}^{(g,p)}(t,m) = \left({}^{g}\left({}^{p}t\right)\left(\alpha\left({}^{p}m\right)(g)\right)^{-1}, {}^{p}m\right),$$

for $(g,p) \in G \rtimes P$ and $(t,m) \in T \rtimes M$. Then $(T,G,\partial) \rtimes_{(\alpha,\beta)} (M,P,\mu) = (T \rtimes M, G \rtimes P, \partial \times \mu)$. We can prove the following

THEOREM 1. *Suppose given two crossed modules* (M,P,μ) *and* (T,G,∂) *and an action* (α,β) *of* (M,P,μ) *over* (T,G,∂). *To every morphism* $(f_1,f_2) : (N,R,\nu) \to (M,P,\mu)$ *and to every derivation* $(d_1,d_2) \in \mathrm{Der}\big((N,R,\nu), (T,G,\partial),(\alpha,\beta)(f_1,f_2)\big)$, *there exists a unique morphism of crossed modules* $(h_1,h_2) : (N,R,\nu) \to (T,G,\partial) \rtimes_{(\alpha,\beta)} (M,P,\mu)$ *such that the following diagram is commutative*

$$\begin{array}{ccccc} & & (N,R,\nu) & & \\ & \swarrow^{(d_1,d_2)} & \big\downarrow{\scriptstyle (h_1,h_2)} & \searrow^{(f_1,f_2)} & \\ (T,G,\partial) & \underset{(q_1,q_2)}{\overset{(i_1,i_2)}{\rightleftarrows}} & (T,G,\partial) \rtimes_{(\alpha,\beta)} (M,P,\mu) & \xrightarrow[(p_1,p_2)]{} & (M,P,\mu) \end{array}$$

Conversely, every morphism $(h_1,h_2) : (N,R,\nu) \to (T,G,\partial) \rtimes_{(\alpha,\beta)} (M,P,\mu)$ *determines a morphism* $(f_1,f_2) = (p_1,p_2)(h_1,h_2) : (N,R,\nu) \to (M,P,\mu)$ *and a derivation* $(d_1,d_2) = (q_1,q_2)(h_1,h_2) : (N,R,\nu) \to (T,G,\partial)$.

By taking $(N,R,\nu) = (M,P,\mu)$ and $(f_1,f_2) = id_{(M,P,\mu)}$ we obtain

COROLLARY 1. *The set of derivations from* (M,P,μ) *to* (T,G,∂) *is in one-to-one correspondence with the set of morphisms of crossed modules* $(h_1,h_2) : (M,P,\mu) \to (T,G,\partial) \rtimes_{(\alpha,\beta)} (M,P,\mu)$ *for which* $(p_1,p_2)(h_1,h_2) = id_{(M,P,\mu)}$.

As an application we shall prove the following result

THEOREM 2. *Let*

$$(N,R,\nu) \xrightarrow{(\chi_1,\chi_2)} (T,G,\partial) \xrightarrow{(\pi_1,\pi_2)} (M,P,\mu)$$

be an exact sequence of crossed modules and let (A,H,σ) *be an* (M,P,μ)*-module,* $(\alpha,\beta) : (M,P,\mu) \to \mathrm{Act}(A,H,\sigma)$. *Then*

$$\begin{aligned} 0 \to \mathrm{Der}\big((M,P,\mu),(A,H,\sigma),(\alpha,\beta)\big) &\xrightarrow{\mathrm{Der}(\pi_1,\pi_2)} \\ \mathrm{Der}\big((T,G,\partial),(A,H,\sigma),(\alpha,\beta)(\pi_1,\pi_2)\big) &\xrightarrow{\ \rho\ } \\ \mathrm{Hom}_{(M,P,\mu)}\big((N,R,\nu)_{ab},(A,H,\sigma)\big) & \end{aligned}$$

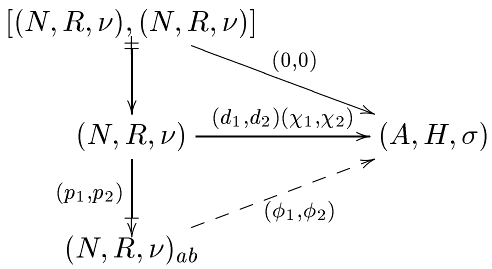
is a natural exact sequence of abelian groups.

Proof. For $(d_1, d_2) \in \operatorname{Der}\big((M,P,\mu),(A,H,\sigma),(\alpha,\beta)\big)$ we define

$$\big(\operatorname{Der}(\pi_1,\pi_2)\big)(d_1,d_2) := (d_1,d_2)(\pi_1,\pi_2).$$

It is easy to see that $\operatorname{Der}(\pi_1,\pi_2)$ is injective.

Let (d_1,d_2) be in $\operatorname{Der}\big((T,G,\partial),(A,H,\sigma),(\alpha,\beta)(\pi_1,\pi_2)\big)$, then the composition $(d_1,d_2)(\chi_1,\chi_2)$ belongs to $\operatorname{Hom}\big((N,R,\nu),(A,H,\sigma)\big)$. So we can construct the diagram



where $[(N,R,\nu),(N,R,\nu)] = \big(\operatorname{D}_R(N), [R,R], \nu_{|\operatorname{D}_R(N)}\big)$, $(N,R,\nu)_{ab} = (N/\operatorname{D}_R(N), R/[R,R], \overline{\nu})$, [5] and $(\phi_1,\phi_2) : (N,R,\nu)_{ab} \to (A,H,\sigma)$ is the unique morphism of crossed modules which satisfies

$$(\phi_1,\phi_2)(p_1,p_2) = (d_1,d_2)(\chi_1,\chi_2).$$

We define $\rho(d_1,d_2) = (\phi_1,\phi_2)$ being the action of (M,P,μ) over $(N,R,\nu)_{ab}$ given by

$$\begin{aligned} \overline{H}(m)(r[R,R]) &= t\,{}^{r}t^{-1}\operatorname{D}_R(N) && \text{if} \quad \pi_1(t)=m,\\ \overline{\theta}_1(p)(n\operatorname{D}_R(N)) &= {}^{g}n\operatorname{D}_R(N) && \text{if} \quad \pi_2(g)=p,\\ \overline{\theta}_2(p)(r[R,R]) &= grg^{-1}[R,R] && \text{if} \quad \pi_2(g)=p. \end{aligned}$$

Obviously $\operatorname{Im}\operatorname{Der}(\pi_1,\pi_2) \subseteq \operatorname{Ker}\rho$ since to a given derivation $(d_1,d_2) \in \operatorname{Der}\big((M,P,\mu),(A,H,\sigma),(\alpha,\beta)\big)$ we have

$$\big(\rho\operatorname{Der}(\pi_1,\pi_2)\big)(d_1,d_2) = \rho\big((d_1,d_2)(\pi_1,\pi_2)\big) = (0,0).$$

Conversely, if $(d_1,d_2) \in \operatorname{Der}\big((T,G,\partial),(A,H,\sigma),(\alpha,\beta)(\pi_1,\pi_2)\big)$ satisfies $\rho(d_1,d_2) = (0,0)$, (we mean, (d_1,d_2) is the trivial derivation restricted to (N,R,ν)) then the pair (γ_1,γ_2) given by

$$\begin{aligned} \gamma_1(m) &= d_1(t) \quad \text{if} \quad t\in T, m=\pi_1(t),\\ \gamma_2(p) &= d_2(g) \quad \text{if} \quad g\in G, p=\pi_2(g), \end{aligned}$$

for all $m \in M$, $p \in P$, is a derivation from (M, P, μ) to (A, H, σ) that satisfies $(\mathrm{Der}(\pi_1, \pi_2))(\gamma_1, \gamma_2) = (\gamma_1, \gamma_2)(\pi_1, \pi_2) = (d_1, d_2)$. Hence $\mathrm{Ker}\,\rho \subseteq \mathrm{Im}\,\mathrm{Der}(\pi_1, \pi_2)$. ∎

This exact and natural sequence is easily obtained by keeping in mind the natural isomorphism

$$\mathcal{CM}/(M,P,\mu) \; \underset{\mathrm{Hom}_{\mathcal{CM}/(M,P,\mu)}\left(-,\, \begin{smallmatrix}(A,H,\sigma)\times(M,P,\mu)\\ \downarrow\\ (M,P,\mu)\end{smallmatrix}\right)}{\overset{\mathrm{Der}(-,(A,H,\sigma))}{\Downarrow}} \; \mathcal{A}b$$

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