

## Towards a Theory of Second Grade Thermoelasticity

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### 1. MOTIVATIONS

1.1. HOMOGENIZATION AND SECOND GRADE MEDIA. The aim of homogenization methods is to replace a heterogeneous material by a homogeneous equivalent one endowed with effective properties. The now well-established results of classical homogenization theory [15] are generally based on the assumption of constant mean fields [1]. It means that the typical size of the heterogeneities is much smaller than the wave-length  $L_w$  of the applied loading conditions. If this hypothesis is dropped or, equivalently, when the heterogeneous material is subjected to strong overall deformation gradients, homogeneous substitution media can also be constructed but they usually are generalized continua [6, 5, 14]. In particular, it can be adequately described by a second grade medium as proposed in [8, 16], which requires the use of the first and second gradients of the displacement field  $\underline{u}$  [12, 7]. Strong overall deformation gradients can also be induced by strongly non-homogeneous temperature fields. Let us consider for instance a periodically perforated thermoelastic plate under plane strain conditions subjected to a prescribed bilinear temperature field (figure 1) and to stringent boundary conditions at the bottom.

The variation of the mean deformation from cell to cell can be seen on figure 1. However figure 3a shows that the perforated plate under such conditions can be successfully replaced by a homogeneous thermoelastic medium endowed with the effective thermoelastic properties according to classical periodic homogenization. If the wave length of the temperature variation is not much bigger than the cell size (like for the sinusoidal temperature field of fig-

ure 2), the solution predicted by the classical homogeneous medium is seen to depart from the actual response (figure 3b).

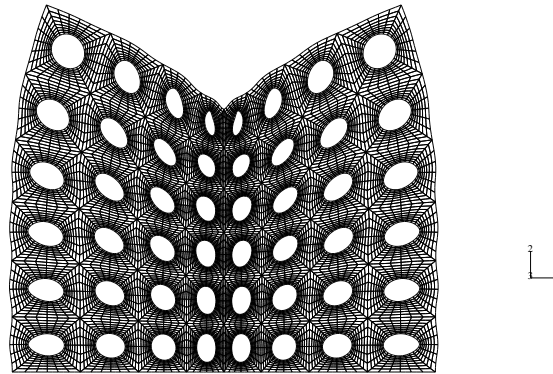


Figure 1. Bilinear temperature field  $T(x) = |x/L|.200 \text{ } ^\circ\text{C}$  where the size of one cell is  $L = 200$ .

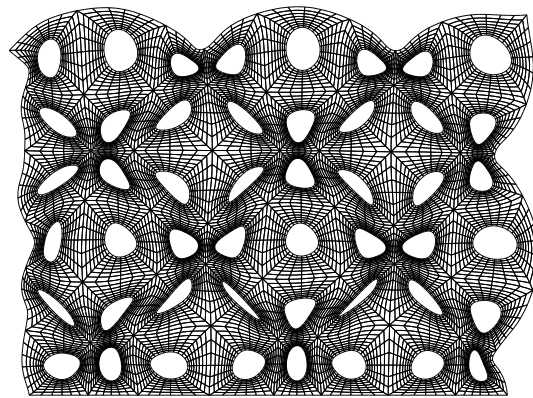


Figure 2. Sinusoidal temperature field  $T(x) = 600 + [\sin \frac{2x}{L} + \cos \frac{2y}{L}].200 \text{ } ^\circ\text{C}$  spatial period of  $T = 3L$  with  $L$  the cell size.

If the wave length of the temperature loading is not much bigger than the characteristic size of the material structure, i.e. here the cell size, then the temperature within a representative element can no more be taken as constant in principle. But this means that strain-energy is stored within the homogeneous substitution continuum due to the temperature gradient. This pleads for the use of a thermoelastic homogeneous substitution medium of grade two.

1.2. TENTATIVE FORM OF THE CONSTITUTIVE EQUATIONS. Let us consider a rectangular homogeneous Cauchy material element subjected to a linear temperature gradient  $\underline{\underline{\varepsilon}}^{th} = \alpha \frac{\Delta T}{L} x_1 \underline{\underline{1}} = \alpha \theta x_1 \underline{\underline{1}}$ . If the element can deform without hindrance (boundary free of forces), no stress arises since  $\underline{\underline{\varepsilon}}^{th}$  is a compatible field (figure 4a).

The displacement field associated with this linear temperature gradient is (in the 3D case):

$$u_1 = \frac{1}{2} \alpha \theta (x_1^2 - x_2^2 - x_3^2) ; \quad u_2 = \alpha \theta x_1 x_2 ; \quad u_3 = \alpha \theta x_1 x_3 \quad (1)$$

up to a translation term.

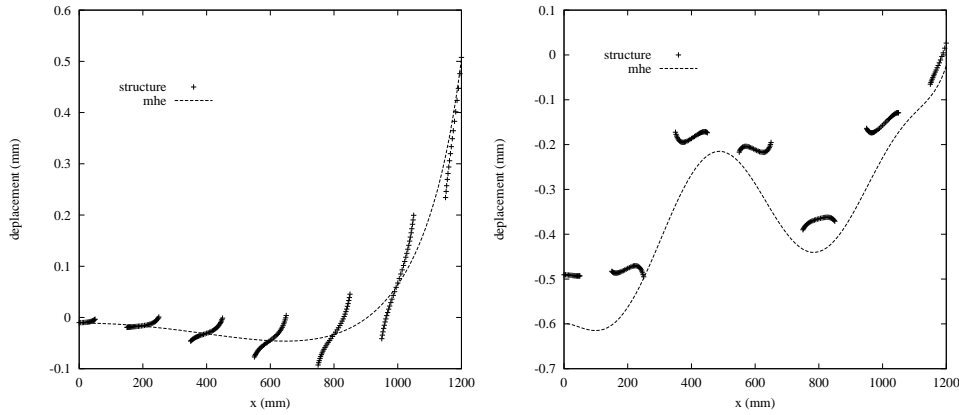


Figure 3. Comparison between the deformation of the heterogeneous and the homogeneous material under a bilinear temperature field (a) and a sinusoidal temperature field (b).

For an appropriate choice of the reference state, the following average quantities are computed:

$$\langle \underline{\underline{\varepsilon}} \rangle = 0 , \quad \langle \underline{\underline{\sigma}} \rangle = 0 , \quad \langle \underline{\underline{\varepsilon}} \otimes \underline{\underline{\nabla}} \rangle = \alpha \theta , \quad \langle \underline{\underline{\sigma}} \otimes \underline{\underline{x}} \rangle = 0 \quad (2)$$

using the same notation as in [5]. If no displacement is possible at the upper and lower boundaries in direction 2, internal stresses develop in the element. For an isotropic elastic element, they are such that:

$$u_1 = \frac{3\lambda + 2\mu}{2(\lambda + 2\mu)} \alpha \theta x_1^2 ; \quad u_2 = 0 ; \quad u_3 = 0 \quad (3)$$

and

$$\langle \underline{\underline{\varepsilon}} \rangle = 0, \quad \langle \underline{\underline{\sigma}} \rangle = 0, \quad \langle \underline{\underline{\varepsilon}} \otimes \underline{\underline{\nabla}} \rangle = 0, \quad \langle \underline{\underline{\sigma}}^{\{\otimes\}} \underline{\underline{x}} \rangle \neq 0 \quad (4)$$

except  $\varepsilon_{11,1}$  if the element is free in direction 1 (figure 4(b)). More precisely

$$\langle \sigma_{22} x_1 \rangle = \langle \sigma_{33} x_1 \rangle = -\frac{1}{6} \frac{3\lambda\mu + 2\mu^2}{\lambda + 2\mu} \alpha\theta L^2$$

The previous material element may be regarded as the homogeneous element replacing the (heterogeneous) representative volume element of the homogenization theory. As a result of (2) and (4) it appears that the situations 1 and 2 can not be distinguished from the sole point of view of classical homogenization theory. The reason is that the classical homogenization theory exclusively accounts for the first moment of stress and strain. In contrast, if the homogeneous substitution medium is taken as a second grade medium, the second moment (or derived quantities) are accounted for as shown in [5]. According to the scheme presented in [5], the situations 1 and 2 would be respectively represented at the macroscopic level by:

$$\underline{\underline{E}} = 0, \quad \underline{\underline{\Sigma}} = 0, \quad \underline{\underline{K}} = \underline{\underline{\alpha}} \otimes \underline{\underline{\nabla}} T, \quad \underline{\underline{S}} = 0 \quad (5)$$

and

$$\underline{\underline{E}} = 0, \quad \underline{\underline{\Sigma}} = 0, \quad \underline{\underline{K}} = 0, \quad \underline{\underline{S}} \neq 0 \quad (6)$$

where  $\underline{\underline{E}}$ ,  $\underline{\underline{\Sigma}}$ ,  $\underline{\underline{K}}$  and  $\underline{\underline{S}}$  respectively are the strain, stress, second gradient ( $\underline{\underline{K}} = \underline{\underline{\varepsilon}} \otimes \underline{\underline{\nabla}}$ ) and hyperstress tensors.

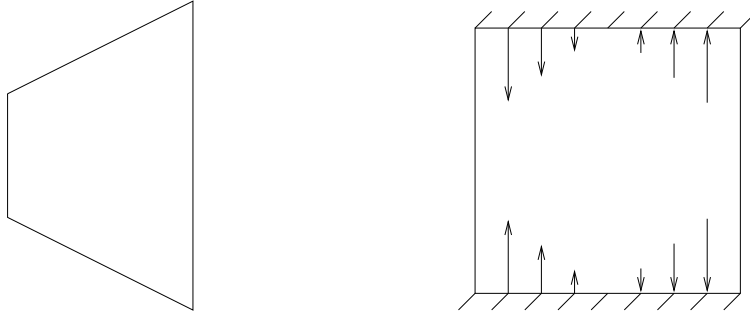


Figure 4. Boundary conditions.

The previous situations are compatible with overall constitutive equations of the form:

$$\underline{\underline{\sigma}} = \underline{\underline{C}} : (\underline{\underline{\varepsilon}} - \underline{\underline{\varepsilon}}^{th}) \quad \text{with} \quad \underline{\underline{\varepsilon}}^{th} = \Delta T \underline{\underline{\alpha}} \quad (7)$$

and

$$\underline{\underline{S}} = \underline{\underline{A}} : (\underline{\underline{K}} - \underline{\underline{K}}^{th}) \quad \text{with} \quad \underline{\underline{K}}^{th} = \underline{\underline{\alpha}} \otimes (T \underline{\underline{\nabla}}) \quad (8)$$

where  $\underline{\underline{C}}$  and  $\underline{\underline{A}}$  are the classical and generalized elastic moduli for a material of grade 2. The thermal strain  $\underline{\underline{\epsilon}}^{th}$  and the gradient of the thermal strain  $\underline{\underline{K}}^{th}$  respectively represent the parts of the total strain and strain gradient that develop without associated stress nor hyperstress as a result of homogeneous or inhomogeneous temperature changes. They can be seen as an eigenstrain and an eigenstrain gradient.

Since the formulation of such constitutive equations does not seem to have been presented hitherto, the sequel of this work deals with the derivation of thermoelastic constitutive equation for second grade media from the usual principles of mechanics and thermodynamics.

## 2. MECHANICS AND THERMODYNAMICS OF THERMOELASTIC SECOND GRADE MEDIA

We successively apply the method of virtual power, the energy and entropy principles of continuum thermodynamics to thermoelastic solids in the fully non isothermal case. For the sake of brevity, we stick to the small deformation framework.

**2.1. PRINCIPLE OF VIRTUAL POWER.** The method of virtual power has proved to be a powerful tool to derive the fields equations and the associated boundary conditions that the unknown fields must fulfill on a body  $\Omega$  [9].  $\Omega$  denotes the open body and  $\partial\Omega$  its closure. For the sake of simplicity, the surface  $\partial\Omega$  is considered twice continuously differentiable, so that it possesses at each point a normal  $\underline{n}$  and a mean curvature  $R$ . The presence of edges and vertices must be treated as shown in [7]. The degrees of freedom of a thermoelastic body of grade 2 are the displacement  $\underline{u}(\underline{x}, t)$  and the temperature  $T(\underline{x}, t)$ , from which we deduce the set of generalized virtual motions

$$\mathcal{V}^\circ = \{\dot{\underline{u}}, \dot{T}\} \quad (9)$$

The rate of temperature  $\dot{T}$  has been added to the usual velocity variable for the sake of generality and because we want to grant the transient variable  $\dot{T}$  its full status of a degree of freedom [11]. Then, according to [11], the energy flux has to be modified. The enlarged set of relevant variables for the order

of the considered theory then is

$$\mathcal{V} = \{\underline{\dot{u}}, \underline{\dot{u}} \otimes \underline{\nabla}, \underline{\dot{u}} \otimes \underline{\nabla} \otimes \underline{\nabla}, \dot{T}, \dot{T}\underline{\nabla}\} \quad (10)$$

where the virtual velocity is supposed to be at least continuous and twice continuously differentiable, and the temperature rate  $\dot{T}$  is continuous and continuously differentiable. The set of virtual motions in [7] has therefore been complemented by the pair  $\{\dot{T}, \dot{T}\underline{\nabla}\}$ .

**VIRTUAL POWER OF GENERALIZED INTERNAL FORCES:** The density of virtual power of internal forces is taken as a linear form in all the arguments in  $\mathcal{V}$ :

$$p^{(i)} = \underline{\boldsymbol{\sigma}} : \underline{\dot{\boldsymbol{\varepsilon}}} + \underline{\underline{S}} : \underline{\underline{\dot{K}}} + a^{(i)} \dot{T} + \underline{b}^{(i)} \cdot (\dot{T}\underline{\nabla}) \quad (11)$$

The expression of the virtual power of internal forces in the domain  $\mathcal{D} \subset \Omega$  reads:

$$\mathcal{P}^{(i)} = \int_{\mathcal{D}} p^{(i)} dV \quad (12)$$

$\underline{\dot{\boldsymbol{\varepsilon}}}$  the symmetric part of the velocity gradient and

$$\underline{\underline{K}} = \underline{\boldsymbol{\varepsilon}} \otimes \underline{\nabla} \quad (K_{ijk} = \frac{1}{2}(u_{i,jk} + u_{j,ik})) \quad (13)$$

The axiom of the virtual power of internal forces states that  $\mathcal{P}^{(i)}$  vanishes for any isothermal rigid-body motion. It has already been applied to expression (11) in which only objective quantities intervene. The quantities dual to the strain rate and strain rate gradient are the symmetric stress tensor  $\underline{\boldsymbol{\sigma}}$  and the hyperstress tensor  $\underline{\underline{S}}$  ( $S_{ijk} = S_{jik}$ ). Two additional dual quantities  $a^{(i)}$  and  $\underline{b}^{(i)}$  have been introduced. The purely mechanical part of the density of virtual power of internal forces has therefore been complemented by thermal contributions that represent the thermal part of the power of work, which is the convectively performed power (in opposite to the caloric power). The use of Gauss' theorem in (11) leads to the alternative expression

$$\begin{aligned} \mathcal{P}^{(i)}(\underline{\dot{u}}, \dot{T}) &= \int_{\mathcal{D}} \left( - \left( (\underline{\boldsymbol{\sigma}} - \underline{\underline{S}} \cdot \underline{\nabla}) \cdot \underline{\nabla} \right) \cdot \underline{\dot{u}} + (a^{(i)} - \underline{b}^{(i)} \cdot \underline{\nabla}) \dot{T} \right) dV \\ &\quad + \int_{\partial \mathcal{D}} \left( \left( (\underline{\boldsymbol{\sigma}} - \underline{\underline{S}} \cdot \underline{\nabla}) \cdot \underline{n} \right) \cdot \underline{\dot{u}} + (\underline{\underline{S}} \cdot \underline{n}) : (\underline{\dot{u}} \otimes \underline{\nabla}) + \underline{b}^{(i)} \cdot \underline{n} \dot{T} \right) dS \end{aligned} \quad (14)$$

The application of the divergence theorem for surfaces to the last term gives the final expression [7]:

$$\mathcal{P}^{(i)}(\underline{\dot{u}}, \dot{T}) = \int_{\mathcal{D}} \left( - \left( (\underline{\boldsymbol{\sigma}} - \underline{\underline{S}} \cdot \underline{\nabla}) \cdot \underline{\nabla} \right) \cdot \underline{\dot{u}} + (a^{(i)} - \underline{b}^{(i)} \cdot \underline{\nabla}) \dot{T} \right) dV$$

$$\begin{aligned}
& + \int_{\partial\mathcal{D}} \left( \left( (\underline{\sigma} - \underline{S} \cdot \underline{\nabla}) \cdot \underline{n} + 2R\underline{S} : (\underline{n} \otimes \underline{n}) - D_t(\underline{S} \cdot \underline{n}) \right) \cdot \dot{\underline{u}} \right. \\
& \left. + \left( \underline{S} : (\underline{n} \otimes \underline{n}) \right) \cdot D_n \dot{\underline{u}} + \underline{b}^{(i)} \cdot \underline{n} \dot{T} \right) dS \quad (15)
\end{aligned}$$

where  $D_n$  and  $D_t$  respectively denote the normal and tangent gradient operator

$$\underline{u} \otimes \underline{\nabla} = D_n \underline{u} + D_t \underline{u} \quad \text{with} \quad D_n \underline{u} = (\underline{u} \otimes \underline{\nabla}) \cdot \underline{n} \quad (16)$$

**VIRTUAL POWER OF EXTERNAL FORCES:** The usual distinction is made between distance and contact applied forces. In its most general form, the virtual power of external forces is a linear form on  $\mathcal{V}$ :

$$\mathcal{P}^{(e)} = \int_{\mathcal{D}} \left( \underline{f} \cdot \dot{\underline{u}} + \underline{\mathcal{C}} : \dot{\underline{\omega}} + \underline{F} : \dot{\underline{\xi}} + \underline{P} : \dot{\underline{K}} + a^{(e)} \dot{T} + \underline{b}^{(e)} \cdot \dot{T} \underline{\nabla} \right) dV \quad (17)$$

where  $\dot{\underline{\omega}}$  is the skew-symmetric part of the velocity gradient. The introduced dual quantities are the volume forces  $\underline{f}$ , the volume couples  $\underline{\mathcal{C}}$  and the volume double forces  $\underline{F}$ , that can exist within the classical framework. Volume triple forces  $\underline{P}$  can be added for a medium of grade 2. For completeness, the quantities  $a^{(e)}$  and  $\underline{b}^{(e)}$  must be introduced. The expression of  $\mathcal{P}^{(e)}$  can then be transformed into volume and surface parts as in (15).

**VIRTUAL POWER OF CONTACT FORCES:** The appropriate form of the virtual power of contact forces is dictated by the observation of the surface term in (15):

$$\mathcal{P}^{(c)} = \int_{\partial\mathcal{D}} \left( \underline{T} \cdot \dot{\underline{u}} + \underline{M} \cdot (D_n \dot{\underline{u}}) + a^{(c)} \dot{T} \right) dV \quad (18)$$

where  $\underline{T}$  is the traction vector,  $\underline{M}$  a normal surface double force and  $a^{(c)}$  the dual quantity of  $\dot{T}$ .

**APPLICATION OF THE PRINCIPLE OF VIRTUAL POWER:** According to the principle of virtual power, in the static case [7]

$$\forall \mathcal{D} \quad \forall \{\dot{\underline{u}}, \dot{T}\} \in \mathcal{V}^o \quad \mathcal{P}^{(i)} = \mathcal{P}^{(e)} + \mathcal{P}^{(c)} \quad (19)$$

from which the following balance equations are deduced:

$$\underline{\tau} \cdot \underline{\nabla} + \underline{f} = 0 \quad \text{with} \quad \underline{\tau} = \underline{\sigma} - \underline{F} - \underline{\mathcal{C}} - \underline{S} \cdot \underline{\nabla} + \underline{P} \cdot \underline{\nabla} \quad (20)$$

$$\underline{b} \cdot \underline{\nabla} - a = 0 \quad \text{with} \quad a = a^{(i)} - a^{(e)} \quad \text{and} \quad \underline{b} = \underline{b}^{(i)} - \underline{b}^{(e)} \quad (21)$$

The associated boundary conditions are also obtained:

$$\begin{aligned}\underline{T} &= \underline{\tau} \cdot \underline{n} + 2R\underline{\underline{S}} : (\underline{n} \otimes \underline{n}) - D_t (\underline{\underline{S}} \cdot \underline{n}) \\ \underline{M} &= \underline{\underline{S}} : (\underline{n} \otimes \underline{n}) \\ a^{(c)} &= \underline{b} \cdot \underline{n}\end{aligned}\tag{22}$$

Note that in the classical case,  $\underline{\underline{S}}$  and  $\underline{b}^{(i)}$  are not introduced in (11). The term  $a^{(i)}\dot{T}$  can still be introduced in principle but, since the external forces  $a^{(e)}$  and  $b^{(e)}$  are usually not introduced in the classical case,  $a^{(i)}$ , as it turns out, remains without counterpart. The classical framework therefore is retrieved. This advocates for the plausible introduction of  $\dot{T}$  as a generalized virtual motion even in the classical case, at least for the sake of generality and for the general fully anisothermal expression of the principle of virtual power.

**2.2. ENERGY PRINCIPLE.** The adopted global form of the energy balance on system  $\Omega$  reads:

$$\dot{E} + \dot{K} = \mathcal{P}^{(e)} + \mathcal{P}^{(c)} + Q\tag{23}$$

where  $E$  is the internal energy of the system,  $K = \int_{\Omega} \frac{1}{2} \rho \dot{\underline{u}} \cdot \dot{\underline{u}} dV$  its kinetic energy and  $Q$  the total heat supply. The application of the kinetic energy theorem easily derived from the principle of virtual power yields:

$$\dot{E} = \mathcal{P}^{(i)} + Q\tag{24}$$

Introducing the specific internal energy  $e$  and the heat flux vector  $\underline{q}$ , and an inner heat production rate  $r$ ,

$$\dot{E} = \int_{\Omega} \rho \dot{e} dV ; Q = - \int_{\partial\Omega} \underline{q} \cdot \underline{n} dS + \int_{\Omega} r dV\tag{25}$$

a local form of the energy balance is obtained:

$$\rho \dot{e} = p^{(i)} - \underline{q} \cdot \underline{\nabla} + r\tag{26}$$

and, taking (11) into account,

$$\rho \dot{e} = \underline{\sigma} : \dot{\underline{\underline{\epsilon}}} + \underline{\underline{S}} : \dot{\underline{\underline{K}}} + a^{(i)} \dot{T} + \underline{b}^{(i)} \cdot \dot{T} \underline{\nabla} - \underline{q} \cdot \underline{\nabla} + r\tag{27}$$



2.3. ENTROPY PRINCIPLE. The second principle is formulated as follows:  $\dot{\mathcal{S}} \geq N$ , where  $\mathcal{S}$  is the global entropy of the system and  $N$  is the total flux of entropy. Introducing the local specific entropy  $\eta(\underline{\varepsilon}, \underline{K}, T, T\underline{\nabla})$ , we will assume that:

$$\mathcal{S} = \int_{\Omega} \rho \eta \, dV \quad \text{and} \quad N = - \int_{\partial\Omega} \underline{\Phi} \cdot \underline{n} \, dS + \int_{\Omega} \frac{r}{T} \, dV \quad (28)$$

The entropy flux vector  $\underline{\Phi}$  is taken as:  $\underline{\Phi} = \frac{q}{T}$ . The local form of the entropy inequality follows:

$$\rho \dot{\eta} + \left(\frac{q}{T}\right) \cdot \underline{\nabla} - \frac{r}{T} \geq 0 \quad (29)$$

Combining (27) and (29), we get the Clausius-Duhem inequality

$$\rho(T\dot{\eta} - \dot{e}) + \underline{\sigma} : \dot{\underline{\varepsilon}} + \underline{S} : \dot{\underline{K}} + a^{(i)} \dot{T} + \underline{b}^{(i)} \cdot (\dot{T}\underline{\nabla}) - \left(\frac{q}{T}\right) \cdot (T\underline{\nabla}) \geq 0 \quad (30)$$

The Helmholtz free energy  $\psi = e - T\eta$  is now introduced and is a function of  $(\underline{\varepsilon}, \underline{K}, T, T\underline{\nabla})$ . The Clausius-Duhem inequality can then be written:

$$\begin{aligned} \left(\underline{\sigma} - \rho \frac{\partial \psi}{\partial \underline{\varepsilon}}\right) : \dot{\underline{\varepsilon}} + \left(\underline{S} - \rho \frac{\partial \psi}{\partial \underline{K}}\right) : \dot{\underline{K}} + (a^{(i)} - \rho \eta - \rho \frac{\partial \psi}{\partial T}) \dot{T} \\ + \left(\underline{b}^{(i)} - \rho \frac{\partial \psi}{\partial T\underline{\nabla}}\right) \cdot (\dot{T}\underline{\nabla}) - \frac{q}{T} \cdot (T\underline{\nabla}) \geq 0 \end{aligned} \quad (31)$$

The variables  $\underline{\varepsilon}$ ,  $\underline{K}$ ,  $T$  and  $T\underline{\nabla}$  being independent and assuming sufficient regularity of the introduced quantities, Coleman's argument applies and we get the laws of state:

$$\underline{\sigma} = \rho \frac{\partial \psi}{\partial \underline{\varepsilon}}, \quad \underline{S} = \rho \frac{\partial \psi}{\partial \underline{K}}, \quad \underline{b}^{(i)} = \rho \frac{\partial \psi}{\partial (T\underline{\nabla})}, \quad \rho \eta = -\rho \frac{\partial \psi}{\partial T} + a^{(i)} \quad (32)$$

This ends the construction of the proposed thermodynamical framework that is required for the development of constitutive equations for a thermoelastic medium of grade 2. It can be compared to the classical formulation of the theory of second grade media by Mindlin [12] and Germain [7]: the variable  $T\underline{\nabla}$  has been added to develop a general framework for anisothermal constitutive modelling. The classical expression of the entropy (32) is modified by the term  $a^{(i)}$  which, according to the balance equation (21), is nothing but the divergence of the generalized thermodynamical force  $\underline{b}$  associated with the temperature gradient. As a result dissipation is reduced to its thermal part:

$$D = -\frac{q}{T} \cdot (T\underline{\nabla}) \quad (33)$$

2.4. HEAT EQUATION. The purpose of this section is to show that the type and order of the partial differential equation in  $\mathbb{T}$ , so-called heat equation, is not altered by the previous results. Taking the laws of state (32) into account, the energy balance (27) can be written:

$$-\underline{q} \cdot \underline{\nabla} + r = \rho T \dot{\eta} \quad (34)$$

in which we can substitute the relation

$$\eta = -\frac{\partial \psi}{\partial T} + \frac{1}{\rho} a^{(i)} \quad (35)$$

The expanded expression of the heat equation then reads:

$$\begin{aligned} -\underline{q} \cdot \underline{\nabla} + r = & \rho \left( -T \frac{\partial^2 \psi}{\partial T^2} + \frac{T}{\rho} \frac{\partial a^{(i)}}{\partial T} \right) \dot{T} + \left( -T \frac{\partial b^{(i)}}{\partial T} + T \frac{\partial a^{(i)}}{\partial (T \underline{\nabla})} \right) \cdot \dot{T} \underline{\nabla} \\ & + T \left( -\frac{\partial \underline{\sigma}}{\partial T} + \frac{\partial a^{(i)}}{\partial \underline{\varepsilon}} \right) : \dot{\underline{\varepsilon}} + T \left( -\frac{\partial \underline{S}}{\partial T} + \frac{\partial a^{(i)}}{\partial \underline{K}} \right) : \dot{\underline{K}} \end{aligned} \quad (36)$$

where a generalized specific heat

$$C = -T \frac{\partial^2 \psi}{\partial T^2} + \frac{T}{\rho} \frac{\partial a^{(i)}}{\partial T} \quad (37)$$

can be defined. Fourier's law  $\underline{q} = -\underline{\kappa} \cdot T \underline{\nabla}$  can still be applied, thus identically satisfying the dissipation inequality. This leads to a partial differential equation of order 2, with a new term in  $\dot{T} \underline{\nabla}$ .

### 3. ALTERNATIVE FORMULATIONS

Several non-classical thermodynamical frameworks are liable to incorporate the dependence on  $T \underline{\nabla}$ . Three of them are now briefly described.

3.1. A CONCEPT AKIN TO "INTERSTITIAL WORKING". The non-classical thermodynamics, proposed in [3], has been primarily invoked to introduce higher order strain gradients into the constitutive framework, with no reference to Mindlin's nor Germain's derivation. These authors [3] modify the energy balance only formally. The weakness of this method when compared to Mindlin, Germain or Trostel's foregoing [17], is that it does not explicitly provide the adequate additional boundary condition. In fact, the aim of this method is to leave the classical boundary value problem unchanged, and some

modern work follows this track [4]. From our point of view, such an approach is not adequate in the case of second grade materials in the isothermal case. That is why we stick to Mindlin and Germain's derivation [12, 7] (see also [2]). However we can try to apply this technique for the introduction of  $T\underline{\nabla}$ , which has not been done in [3]. The power density of internal forces is taken as:

$$p^{(i)} = \underline{\sigma} : \underline{\dot{\xi}} + \underline{S} : \underline{\dot{K}} \quad (38)$$

Like in [3], an additional contribution  $\underline{w}$  is introduced in the energy balance:

$$\rho \dot{e} = p^{(i)} - \underline{q} \cdot \underline{\nabla} + \underline{w} \cdot \underline{\nabla} + r \quad (39)$$

Taking  $\underline{w} = \dot{T}\underline{b}$ , we get

$$\rho \dot{e} = p^{(i)} - \underline{q} \cdot \underline{\nabla} + \dot{T} \underline{b} \cdot \underline{\nabla} + \underline{b} \cdot \dot{T} \underline{\nabla} + r \quad (40)$$

which is exactly the expression (27) where it is recalled that  $a^{(i)} = \underline{b} \cdot \underline{\nabla}$ . The associated boundary condition  $\underline{b} \underline{n} = a^{(c)}$  can be introduced. Accordingly, this method provides the same final form as ours. Three differences must be noted: the form  $\underline{w} = \dot{T} \underline{b}$  is not well motivated, the additional boundary conditions are not explicitly derived and the power of internal forces remains unchanged.

3.2.  $T$ , DEGREE OF FREEDOM OR INTERNAL VARIABLE? Maugin [10] proposed a general formalism for the introduction of internal variables and their gradients into the constitutive framework. We demonstrate in this subsection the strict treatment of  $T$  as an "internal variable" according to that scheme. Within this framework, the relation between the entropy flux and the heat flux vectors is regarded as a constitutive relation [13] of the form

$$\underline{\Phi} = \frac{\underline{q}}{T} + \underline{k} \quad (41)$$

where  $\underline{k}$  is an extra entropy flux density to be determined. Here the entropy is introduced axiomatically, and thus without a clear characterisation by an already known flux (as  $\underline{q}/T$  is but  $\underline{k}$  not). The local form of the entropy inequality then reads:

$$\rho \dot{\eta} + \underline{\Phi} \cdot \underline{\nabla} \geq 0 \quad (42)$$

The energy balance remaining unchanged with a free energy defined now by a not clearly characterized entropy, the inequality becomes

$$-\rho (\dot{\psi} + \eta \dot{T}) + \underline{\sigma} : \underline{\dot{\xi}} + \underline{S} : \underline{\dot{K}} + (\underline{k}T) \cdot \underline{\nabla} - \underline{\Phi} \cdot (T\underline{\nabla}) \geq 0 \quad (43)$$

Assuming that the free energy is a function of  $(\underline{\varepsilon}, \underline{K}, T, T\underline{\nabla})$ , its time derivative can be written

$$\dot{\psi} = \frac{\partial\psi}{\partial\underline{\varepsilon}} : \dot{\underline{\varepsilon}} + \frac{\partial\psi}{\partial\underline{K}} : \dot{\underline{K}} + \frac{\partial\psi}{\partial T} \dot{T} - \left( \frac{\partial\psi}{\partial(T\underline{\nabla})} \cdot \underline{\nabla} \right) \dot{T} + \left( \dot{T} \frac{\partial\psi}{\partial(T\underline{\nabla})} \right) \cdot \underline{\nabla} \quad (44)$$

so that

$$\begin{aligned} (\underline{\varrho} - \rho \frac{\partial\psi}{\partial\underline{\varepsilon}}) : \dot{\underline{\varepsilon}} + (\underline{S} - \rho \frac{\partial\psi}{\partial\underline{K}}) : \dot{\underline{K}} + (\underline{b} \cdot \underline{\nabla} - \rho\eta - \rho \frac{\partial\psi}{\partial T}) \dot{T} \\ + (\underline{k}T - \dot{T} \underline{b}) \cdot \underline{\nabla} - \underline{\Phi} \cdot (T\underline{\nabla}) \geq 0 \end{aligned} \quad (45)$$

where  $\underline{b} = \rho \partial\psi / \partial(T\underline{\nabla})$ . At this point, we chose the extra entropy flux

$$\underline{k} = \frac{\dot{T}}{T} \underline{b} \quad (46)$$

so that the laws of state are:

$$\underline{\varrho} = \rho \frac{\partial\psi}{\partial\underline{\varepsilon}}, \quad \underline{S} = \rho \frac{\partial\psi}{\partial\underline{K}}, \quad \rho\eta = -\rho \frac{\partial\psi}{\partial T} + \underline{b} \cdot \underline{\nabla} \quad (47)$$

which are identical to (32). The heat equation then takes the form

$$- \underline{q} \cdot \underline{\nabla} = (\dot{T} \underline{b}) \cdot \underline{\nabla} + \rho T \dot{\eta} \quad \text{i.e.} \quad - (T \underline{\Phi}) \cdot \underline{\nabla} = \rho T \dot{\eta} \quad (48)$$

In order to identically fulfill the non-classical heat conduction inequality, the generalized Fourier's law takes the form:

$$\underline{\Phi} = -\frac{1}{T} \underline{\kappa} \cdot (T\underline{\nabla}) \quad (49)$$

which leaves the classical heat equation unchanged:  $(\underline{\kappa} \cdot T\underline{\nabla}) \cdot \underline{\nabla} = \rho T \dot{\eta}$ .

**3.3. FINAL REMARKS.** No special extension of the original form (29) of the entropy principle seems to be justified when the independent variable  $T\underline{\nabla}$  is introduced. The local entropy density  $\eta$  is simply regarded as a function of  $(\underline{\varepsilon}, T, \underline{K}, T\underline{\nabla})$ . This relies on Caratheodory's argument and its "rational" out-of-equilibrium version by Trostel [18] which involves only so-called contact variables like  $T$  but excludes  $T\underline{\nabla}$  in the (Carnot) function for the integrating factor. In all formulations, the direct physical meaning of the necessary additional terms associated with  $\dot{T}$  and  $T\underline{\nabla}$  remains unclear. In particular, it is

difficult to assess the pertinence of the introduction of an extra entropy term  $\underline{k}$  in (46).

In the case of classical thermoelasticity, the constitutive relation  $\underline{\Phi} = \underline{q}/T$  can be explicitly derived using a formulation of the entropy principle with Lagrange multipliers [13]. This procedure, which requires a tremendous computational effort, must be carried out for each set of specific constitutive equations. This therefore is a strongly different strategy from the rational one used here which aims at providing a framework for the further development of constitutive equations. However this method remains to be applied to the introduction of  $T\underline{\nabla}$  in the thermodynamical setting.

Extended thermodynamics [13] may provide an adequate framework for the construction of a theory of second grade thermoelasticity. But it was the purpose of this work to show that the more classical thermodynamics of irreversible thermodynamics can account for it, with only slight extensions.

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