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Variational Theory of Non-Perfect Relativistic Fluids

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A basic question in General Relativity from the point of view of the general field theory is to obtain the Einstein equations coupled with the stress-energy-momentum tensor of a dissipative fluid from a variational principle. We believe that this problem, whose solution for perfect fluids is well known, has not been faced in a systematic way, maybe by the thought of a possible nonsense, for the concept of "dissipation" is believed to be incompatible with the essentially "conservative" character of the calculus of variations. In this talk, we shall deal with this topic by discussing a variational problem that generalizes in a natural way to this kind of fluids the Einstein-Taub formalism for the perfect fluids. We shall start by summarizing this formalism looking ahead to this generalization. Next, we shall show the generalization restricting ourselves, for simplicity, to relativistic fluids with just "bulk viscosity". Finally, we shall see some simple examples that illustrate very well the essential points of the theory that has just been displayed.

1. Perfect relativistic fluids

1.1. Eulerian theory This is essentially a field theory over the space-time that can be described as follows:

Let X_4 be an oriented 4-dimensional manifold (space-time), $\mathcal{M} \to X_4$ be the bundle of Lorentz metrics over X_4 (gravitational fields), and $\xi^i \to X_4$ be the trivial bundle $X_4 \times \mathbb{R}^i$.

Let us consider the subbundle:

$$p\colon E \subset \mathcal{M} \times_{X_4} T(X_4) \times_{X_4} \xi^1 \longrightarrow X_4,$$

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where $E = \{(g_x, D_x, e_x) : \; g_x(D_x, D_x) < 0\}$.

If $\sigma = (g, D, e) \in \Gamma(X_4, E)$, we shall interpret the section σ as a fluid in the gravitational field g, with density of mass $\rho = \sqrt{-g(D, D)}$, field of velocities $V = D/\rho$ and specific energy e.

The basic datum to be considered in this theory is a 1-form $\mathcal{S}$ on E, p-horizontal and invariant under the natural lifting of the pseudogroup Diff X_4 of local diffeomorphisms of X_4. It shall be called entropy 1-form. If $\sigma \in \Gamma(X_4, E)$, the restriction 1-form $\mathcal{S}|_\sigma$ is understood as the entropy flux of the fluid defined by the section σ.

Let $\mathcal{C}^\infty_{\mathrm{Diff}\, X_4}(E)$ be the ring of the Diff X_4-invariant differentiable functions on E. If ρ and e are the functions on E assigning to each point $\zeta = (g_x, D_x, e_x) \in E$ the real numbers $\rho(\zeta) = \sqrt{-g_x(D_x, D_x)}$ and $e(\zeta) = e_x$ respectively, it is easy to see that $\mathcal{C}^\infty_{\mathrm{Diff}\, X_4}(E) = \mathcal{C}^\infty(\rho, e)$. On the other hand, if ω is the 1-form on E assigning to each point $\zeta = (g_x, D_x, e_x) \in E$ the 1-form $\omega_\zeta \in T^*_\zeta(E)$ given by $\omega_\zeta \colon \bar{\zeta}_\zeta \in T_\zeta(E) \mapsto \omega_{D_x}(p_\zeta \bar{\zeta}_\zeta)$ where $\omega_{D_x} = i_{D_x} g_x$, it is also easy to see that $\mathcal{S} = S\omega$ where $S \in \mathcal{C}^\infty_{\mathrm{Diff}\, X_4}(E)$. That is, giving the entropy 1-form $\mathcal{S}$ is equivalent to giving the Diff X_4-invariant function S. The functions ρ, $v = 1/\rho$, e and S are called density of mass, volume, energy and specific entropy, respectively.

On the ring $\mathcal{C}^\infty_{\mathrm{Diff}\, X_4}(E)$, that shall be called ring of state functions, a thermodynamic structure can be introduced by giving two 1-forms ω_Q and ω_W (1-forms of heat and work) defined as follows:

PROPOSITION. (GIBBS' EQUATION) *There exist unique functions T and p in $\mathcal{C}^\infty_{\mathrm{Diff}\, X_4}(E)$ such that:*

$$TdS = de + pdv. \tag{1.1}$$

The functions T and p are called temperature and pressure respectively, being $\omega_Q = TdS$ and $\omega_W = pdv$ the 1-forms of heat and work that we wanted to define.

From here, the following basic concept can be introduced:

DEFINITION. The stress-energy-momentum tensor is the correspondence that assigns to each section $\sigma = (g, D, e) \in \Gamma(X_4, E)$ the tensor on X_4:

$$T^2(\sigma) = \rho_\sigma e V_\sigma \otimes V_\sigma + p_\sigma (V_\sigma \otimes V_\sigma + g^{-1}), \tag{1.2}$$

where $V_\sigma = D/\rho_\sigma$ and p_σ is the restriction to σ of the pressure function $p \in \mathcal{C}^\infty_{\mathrm{Diff}\, X_4}(E)$.

Finally, the field equations for the theory described here, are the following ones:

$$\operatorname{div}_g D = 0, \qquad \operatorname{Eins}(g) = T_2(\sigma), \qquad \sigma \in \Gamma(X_4, E). \tag{1.3}$$

The first one is the continuity equation, meanwhile, the second one is the Einstein equation coupled with a source field with stress-energy-momentum tensor $T_2(\sigma)$ (= contraction of $T^2(\sigma)$ with the metric g). It is a theory of gravitation in the sense of Einstein, that, as it is well known, due to the identity $\operatorname{div}_g \operatorname{Eins}(g) = 0$, has to satisfy the compatibility condition $\operatorname{div}_g T_2(\sigma) = 0$.

For each fixed metric g, the equations:

$$\operatorname{div}_g D = 0, \qquad \operatorname{div}_g T_2(\sigma) = 0, \qquad \sigma \in \Gamma(X_4, E), \tag{1.4}$$

are interpreted as field equations of a perfect fluid (D, e) in the gravitational field defined by g.

This is a system of first order partial differential equations (parameterized by g) on (D, e), with equal number of equations as unknowns, whose solutions, substituted in the right hand term of the Einstein equation, allow us to obtain the equation on g that must be satisfied by the gravitational field.

Taking into account the decomposition of the 1-form $\operatorname{div}_g T_2(s)$:

$$\operatorname{div}_g T_2(\sigma) = (\operatorname{div}_g T_2(\sigma))^\circ + (\operatorname{div}_g T_2(\sigma))^{||}\, \omega_V$$

into its component incident to V and the one proportional to $\omega_V = i_V g$, the equation $\operatorname{div}_g T_2(\sigma) = 0$ is equivalent to the pair of equations:

$$(\operatorname{div}_g T_2(\sigma))^\circ = 0, \quad (\operatorname{div}_g T_2(\sigma))^{||} = 0, \tag{1.5}$$

that are called, equation of "balance of momentum" and equation of "balance of energy" respectively.

Bearing in mind equation (1.2) and the continuity equation, we have:

$$\operatorname{div}_g T_2(\sigma) = (De)\omega_V + eD^\nabla \omega_V + V(p)\omega_V + dp + p(\operatorname{div}_g V)\omega_V + pV^\nabla \omega_V,$$

and hence, taking into account Gibbs' equation, we have:

$$\begin{aligned}
(\operatorname{div}_g T_2(\sigma))^{||} &= (\operatorname{div}_g T_2(\sigma))\,(V) = -De + \frac{p}{\rho^2} D\rho = -\,(de + pdv)\,(D) \\
&= -\,(TdS)\,(D) = -TDS = -T \operatorname{div}_g \mathcal{S}, \\
(\operatorname{div}_g T_2(\sigma))^\circ &= \operatorname{div}_g T_2(\sigma) - \operatorname{div}_g T_2(\sigma)^{||} \omega_V = V(p)\omega_V + dp + (\rho e + p)V^\nabla \omega_V.
\end{aligned}$$

Thus, the equation of "balance of momentum" coincides with the well known Euler equation of the perfect relativistic fluids:

$$V(p)V + \operatorname{grad} p + (\rho e + p)V^{\nabla}V = 0, \tag{1.6}$$

whereas the equation of "balance of energy" for $T \neq 0$ takes the form :

$$\operatorname{div}_g \mathcal{S} = 0, \tag{1.7}$$

that is the conservation law of the entropy flux for this kind of fluids.

1.2. Lagrangian theory. The variational theory for the perfect relativistic fluids is well known and has been widely treated in the last twenty-five years, both as a variational problem with the constraint defined by the "continuity equation", as well as under a free formulation by means of an adequate choice of potentials. A novel aspect of our presentation for this theory is dealing with both descriptions from a common point of view that proves to be very adequate for the generalization we are looking for. The starting point is the following construction, introduced originally by J. Kijowski, B. Pawlik and W. Tulczyjew in [5] in order to deal with this kind of fluids as a free variational problem (see also [6]).

Let (X_4, g) be an oriented Lorentz manifold with Lorentzian volume element ω_g. The aforementioned construction is based on the observation that the Hodge isomorphism $D \in \mathfrak{X}(X_4) \mapsto \omega_3 = i_D\omega_g$ establishes a one-one correspondence between divergence-free vector fields and closed 3-forms, that, in turn, can be (locally) reduced to the form $\omega_3 = df_1 \wedge df_2 \wedge df_3$, $f_i \in \mathcal{C}^\infty(X_4)$. Identifying the 3-tuples of functions (f_1, f_2, f_3) with the maps $f\colon X_4 \to \mathbb{R}^3$ (which are identified to the sections of the trivial bundle $\xi^3 = X_4 \times \mathbb{R}^3$) a locally surjective map can be established from the space of sections $\Gamma(X_4, \xi^3)$ to the space of divergence-free vector fields in an obvious way.

Considering the fibered products $Y = \mathcal{M} \times_{X_4} \xi^3 \times_{X_4} \xi^1$ and $\overline{Y} = \mathcal{M} \times_{X_4} T(X_4) \times_{X_4} \xi^1$, the former observation suggests to establish the surjective morphism $\varphi\colon J^1(Y) \to \overline{Y}$ of bundles over X_4 given by: $j^1(g, f, z) \mapsto (g_x, D_x, z_x)$, where D_x is the unique tangent vector such that $i_{D_x}\omega_g(x) = (f^*\nu)_x$ (ν= standard volume 3-form on $\mathbb{R}^3$). If $s = (g, f, z) \in \Gamma(X_4, Y)$ and $\overline{s} = \varphi \circ j^1 s = (g, D, z)$, then the vector field D satisfies the continuity equation $\operatorname{div}_g D = 0$, and conversely in a local way. In the following, we will constrain our considerations to the dense open sets of the previous fiber bundles whose sections satisfy the condition $g(D, D) < 0$. From this situation, the variational formulation of the relativistic perfect fluids is defined as follows:

a) *As a free variational problem*: Defined by a Lagrangian density $R(g)\omega_g + \varphi_1^*\mathcal{L}\omega_g$ on $J^2(Y)$, where $R(g)$ is the scalar curvature of the metric g, $\mathcal{L}$ is a Diff X_4-invariant function on $J^1(\overline{Y})$ of the form $\mathcal{L}(\rho,\sigma)$ with $\sigma = V(z)$, and $\varphi_1\colon J^2(Y) \to J^1(\overline{Y})$ is the 1-jet prolongation of the just introduced morphism $\varphi\colon J^1(Y) \to \overline{Y}$.
b) *As a constrained variational problem*: Defined by a Lagrangian density $R(g)\omega_g + \mathcal{L}\omega_g$ on $J^1(\overline{Y})$, where the sections (g, D, z) have to satisfy the constraint condition $\operatorname{div}_g D = 0$, and where the notion of "stationariness" is taken with respect to the constraint-preserving "variations":

$$\begin{aligned} &\delta_t g = g + tg', \quad \delta_t D = D + t\{[D, D'] - (\operatorname{div}_g D' + g^{-1} \cdot g')D\}, \\ &\delta_t z = z + tz', \end{aligned} \tag{1.8}$$

being g', D', z', a metric, a vector field and a function (arbitrary ones) on X_4 respectively.

If $s = (g, f, z) \in \Gamma(X_4, Y)$ and $\overline{s} = \varphi \circ j^1 s = (g, D, z) \in \Gamma(X_4, \overline{Y})$, the variations (1.8) of $\overline{s}$ are, actually, the image of the ordinary variations of s: $\delta_t g = g + tg'$, $\delta_t f = f + tf'$, $\delta_t z = z + tz'$ (g', f', z' a metric on X_4, a map from X_4 to $\mathbb{R}^3$ and a function on X_4, all of them arbitrary) by the differential along s of the map $\varphi \circ j^1\colon \Gamma(X_4, Y) \to \Gamma(X_4, \overline{Y})$. Thus, both variational problems are φ-related in the sense that a section $s \in \Gamma(X_4, Y)$ is critical with respect to the variational problem a) if and only if $\overline{s} = \varphi \circ j^1 s$ is critical with respect to the variational problem b). Moreover, we shall see that this fact can be extended to the rest of variational concepts and results associated to both problems in a well posed way.

Let $s = (g, f, z) \in \Gamma(X_4, Y)$. By means of the identification $s^*V^*(Y) = S^2(T(X_4)) \oplus f^*T^*(\mathbb{R}^3) \oplus z^*T^*(\mathbb{R})$, the Euler-Lagrange operator $\mathcal{E}$ of the Lagrangian density $\varphi_1^*\mathcal{L}\eta_g$ can be easily decomposed in the following form: $\mathcal{E}\colon s \mapsto \left[\mathcal{E}_{\mathcal{M}}(s), \mathcal{E}_{\xi^3}(s), \mathcal{E}_{\xi^1}(s)\right]$. Analogously, if $\overline{s} = (g, D, z) \in \Gamma(X_4, \overline{Y})$, the Euler-Lagrange operator $\hat{\mathcal{E}}$ of the Lagrangian density $\mathcal{L}\eta_g$ (as a free variational problem) can be decomposed by means of the identification $\overline{s}^*V^*(\overline{Y}) = S^2(T(X_4)) \oplus T^*(X_4) \oplus z^*T^*(\mathbb{R})$, in the form $\hat{\mathcal{E}}\colon \overline{s} \mapsto \left[\hat{\mathcal{E}}_{\mathcal{M}}(\overline{s}), \hat{\mathcal{E}}_{T(X_4)}(\overline{s}), \hat{\mathcal{E}}_{\xi^1}(\overline{s})\right]$. If $T^2(s)$ is the stress-energy-momentum tensor of $\varphi_1^*\mathcal{L}\eta_g$ along the section s [2], a following fundamental result holds:

THEOREM 1. *If* $s = (g, f, z) \in \Gamma(X_4, Y)$ *and* $\overline{s} = \varphi \circ j^1 s = (g, D, z) \in \Gamma(X_4, \overline{Y})$, *then:*

$$T^2(s) = \mathcal{E}_{\mathcal{M}}(s) = \hat{\mathcal{E}}_{\mathcal{M}}(\overline{s}) - \left[i_D \hat{\mathcal{E}}_{T(X_4)}(\overline{s}) \right] g^{-1}, \tag{1.9}$$

$$f^* \mathcal{E}_{\xi^3}(s) = -i_D d\hat{\mathcal{E}}_{T(X_4)}(\overline{s}), \quad \mathcal{E}_{\xi^1}(s) = \hat{\mathcal{E}}_{\xi^1}(\overline{s}), \tag{1.10}$$

$$\operatorname{div}_g T_2(s) = -f^* \mathcal{E}_{\xi^3}(s) + z^* \mathcal{E}_{\xi^1}(s). \tag{1.11}$$

The tensor $\overline{T}^2(\overline{s}) = \hat{\mathcal{E}}_{\mathcal{M}}(\overline{s}) - \left[i_D \hat{\mathcal{E}}_{T(X_4)}(\overline{s}) \right] g^{-1}$ is nothing else but the stress-energy-momentum tensor of the Lagrangian density $\mathcal{L}\eta_g$ on $\overline{Y}$ along the section $\overline{s}$ as a constrained variational problem. So, formula (1.9) expresses the fact that the stress-energy-momentum tensors associated to the Lagrangian densities $\varphi^* \mathcal{L}\eta_g$ and $\mathcal{L}\eta_g$ of the variational problems a) and b) are φ-related. Regarding the characterization of the critical sections of both problems, we have:

THEOREM 2. $s = (g, f, z) \in \Gamma(X_4, Y)$ *is a critical section of the variational problem a) if and only if:*

$$\operatorname{Eins} g = T_2(s), \qquad \mathcal{E}_{\xi^3}(s) = 0, \qquad \mathcal{E}_{\xi^1}(s) = 0. \tag{1.12}$$

$\overline{s} = (g, D, z) \in \Gamma(X_4, \overline{Y})$ *is a critical section of the variational problem b) if and only if:*

$$\operatorname{div}_g D = 0, \quad \operatorname{Eins} g = T_2(\overline{s}), \quad i_D d\hat{\mathcal{E}}_{T(X_4)}(\overline{s}) = 0, \quad \hat{\mathcal{E}}_{\xi^1}(\overline{s}) = 0. \tag{1.13}$$

So, by formula (1.10), the Euler-Lagrange equations of the variational problems a) and b) are also φ-related.

On the other hand, according to formula (1.11), the last two equations of (1.12) are equivalent to the equation $\operatorname{div}_g T_2(s) = 0$ which is, in turn, a consequence of the equation $\operatorname{Eins} g = T_2(s)$, and analogously for the group of equations in (1.13).

Finally, for a fixed metric g and with the obvious identifications, the following result holds:

THEOREM 3. *A section* $s = (g, f, z)$ *(*f *and* z *variable) is critical for the Lagrangian density* $\varphi_1^* \mathcal{L}\eta_g$ *if and only if:*

$$\mathcal{E}_{\xi^3}(s) = 0, \qquad \mathcal{E}_{\xi^1}(s) = 0. \tag{1.14}$$

A section $\overline{s} = (g, D, z)$ *(*D *and* z *variable) is critical for the Lagrangian density* $\mathcal{L}\eta_g$ *with the constraint defined by the continuity equation* $\operatorname{div}_g D = 0$ *and the constraint preserving "variations":* $\delta_t D = D + t\{[D, D'] - (\operatorname{div}_g D')D\}$, $\delta_t z = z + tz'$ *(*D', z' *arbitrary vector field and function on* X_4 *) if and only if :*

$$\operatorname{div}_g D = 0, \qquad i_D d\hat{\mathcal{E}}_{T(X_4)}(\overline{s}) = 0, \qquad \hat{\mathcal{E}}_{\xi^1}(\overline{s}) = 0. \tag{1.15}$$

Again, by formula (1.11), the equations (1.14) of the first case are equivalent to the equation:

$$\operatorname{div}_g T_2(s) = 0, \tag{1.16}$$

meanwhile the equations for the second one are equivalent to the pair of equations:

$$\operatorname{div}_g D = 0, \qquad \operatorname{div}_g \overline{T}_2(\overline{s}) = 0. \tag{1.17}$$

Explicit computations: Bearing in mind the expression $\mathcal{L}(\rho,\sigma)\sqrt{-\det g}\, dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$, $\sigma = V(z)$, of the Lagrangian density $\mathcal{L}\eta_g$, we have:

$$\begin{aligned}
\hat{\mathcal{E}}_{T(X_4)}(\overline{s}) &= \sum_j \hat{\mathcal{E}}_j dx_j = \sum \frac{1}{\sqrt{-\det g}} \frac{\partial(\mathcal{L}(\rho,\sigma)\sqrt{-\det g})}{\partial y^j} dx_j \\
&= -\frac{\partial(\mathcal{L}(\rho,\sigma))}{\partial \rho}\omega_V + \frac{1}{\rho}\frac{\partial \mathcal{L}(\rho,\sigma)}{\partial \sigma}(\sigma\omega_V + dz), \\
\hat{\mathcal{E}}_{\mathcal{M}}(\overline{s}) &= \sum_{i\le j} \hat{\mathcal{E}}^{ij} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} = \sum_{i\le j} \frac{1}{\sqrt{-\det g}} \frac{\partial(\mathcal{L}(\rho,\sigma))\sqrt{-\det g}}{\partial g_{ij}} \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} \\
&= \left(-\rho\frac{\partial(\mathcal{L}(\rho,\sigma))}{\partial\rho} + \sigma\frac{\partial(\mathcal{L}(\rho,\sigma))}{\partial\sigma}\right) V\otimes V + \mathcal{L}(\rho,\sigma)g^{-1},
\end{aligned}$$

and then:

$$\begin{aligned}
\overline{T}^2(\overline{s}) &= \hat{\mathcal{E}}_{\mathcal{M}}(\overline{s}) - \left[i_D\hat{\mathcal{E}}_{T(X_4)}(\overline{s})\right] g^{-1} \\
&= \left(-\rho\frac{\partial(\mathcal{L}(\rho,\sigma))}{\partial\rho} + \sigma\frac{\partial(\mathcal{L}(\rho,\sigma))}{\partial\sigma}\right) V\otimes V \\
&\quad + \left(\mathcal{L}(\rho,\sigma) - \rho\frac{\partial(\mathcal{L}(\rho,\sigma))}{\partial\sigma}\right) g^{-1} \\
&= \left(-\mathcal{L}(\rho,\sigma) + \sigma\frac{\partial(\mathcal{L}(\rho,\sigma))}{\partial\sigma}\right) V\otimes V \\
&\quad + \left(\mathcal{L}(\rho,\sigma) - \rho\frac{\partial(\mathcal{L}(\rho,\sigma))}{\partial\rho}\right) \left(V\otimes V + g^{-1}\right)
\end{aligned}$$

Writing down $\mathcal{L}(\rho,\sigma) = -\rho[1+\epsilon(\rho,\sigma)]$ as usual, we have:

$$\overline{T}^2(\overline{s}) = \rho e V \otimes V + p\left(V \otimes V + g^{-1}\right), \tag{1.18}$$

where

$$e = 1 + \epsilon(\rho,\sigma) - \sigma\frac{\partial\epsilon(\rho,\sigma)}{\partial\sigma}, \qquad p = \rho^2\partial\epsilon(\rho,\sigma)\rho. \tag{1.19}$$

On the other hand, the last two equations of (1.15) can be expressed, respectively, as follows:

$$V(p)\omega_V + dp + (\rho e + p)V^{\nabla}\omega_V = 0, \qquad D\left(\frac{\partial\epsilon(\rho,\sigma)}{\partial\sigma}\right) = 0. \tag{1.20}$$

The first one of them reproduces the Euler equation for perfect fluids taking as "energy" and "pressure" the functions defined by formulas (1.19), and the second one will be shown to express the conservation law of the entropy flux for such fluids.

1.3. Relation between both theories: Hydrodynamical Legendre's transformation. Generalization. The relation between the Eulerian and Lagrangian theories that we have just described can be now established from the following basic concept:

Definition. The hydrodynamical Legendre's transformation is the map Leg: $J^1(\overline{Y}) \to E$ of bundles over X_4:

$$\mathrm{Leg}[j^1_x(g,D,z)] = (g_x, D_x, e_x), \quad j^1_x(g,D,z) \in J^1(\overline{Y}), \tag{1.21}$$

where e_x is the value at x of the energy function e defined by formula (1.19).

If $S_{\mathcal{L}\omega_g}$, p and T are the functions on $J^1(\overline{Y})$ defined by $-\frac{\partial\epsilon(\rho,\sigma)}{\partial\sigma}$, $\rho^2\frac{\partial\epsilon(\rho,\sigma)}{\partial\rho}$ and σ respectively, we have the following:

Proposition. (Gibbs' equation) *$S_{\mathcal{L}\omega_g}$, p and T are functions on $J^1(\overline{Y})$ satisfying the equation:*

$$TdS_{\mathcal{L}\omega_g} = de + pdv.$$

Hence, taking as specific entropy $S_{\mathcal{L}\omega_g}$, the stress-energy-momentum tensor $\overline{T}^2$ of the Lagrangian theory can be obtained from the stress-energy-momentum tensor T^2 of the Eulerian theory by the formula:

$$\overline{T}^2(\overline{s}) = T^2(\mathrm{Leg} \circ j^1\overline{s}), \quad \overline{s} \in \Gamma(X_4, \overline{Y}), \tag{1.22}$$

thus leading to the following result that establishes the precise relation between both theories:

THEOREM 4. *$\overline{s} = (g, D, z) \in \Gamma(X_4, \overline{Y})$ is a critical section of the variational problem b) of Lagrangian density $R(g)\omega_g + \mathcal{L}\omega_g$ if and only if the section $\sigma = \mathrm{Leg} \circ j^1\overline{s} \in \Gamma(X_4, E)$ satisfies the field equations (1.3) of the Eulerian theory defined by the entropy 1-form $\mathcal{S} = S_{\mathcal{L}\omega_g}\omega$. Fixing a metric g, $\overline{s} = (g, D, z)$ (D and z variable) is a critical section of the constrained variational problem of Lagrangian density $\mathcal{L}\omega_g$ if and only if $\sigma = \mathrm{Leg} \circ j^1\overline{s}$ satisfies the equations (1.4) of the corresponding fluid.*

Remark. By composing the map $\varphi_1 \colon J^2(Y) \to J^1(\overline{Y})$ with the Legendre's transformation, this can be extended to the bundle $J^2(Y)$, and then, the relation between the Eulerian theory and the variational problem a) with Lagrangian density $R(g)\omega_g + \varphi_1^*\mathcal{L}\omega_g$ can be established in the same terms as in the preceding Theorem.

The variational theory displayed here can be generalized, with no changes in its setting and results (Theorems 1, 2 and 3) to arbitrary Lagrangian densities $\mathcal{L}\omega_g$ on $J^r(\overline{Y})$. Such generalization can be established comparing the formulas of variation of the Lagrangian densities $R(g)\omega_g + \varphi_r^*\mathcal{L}\omega_g$ and $R(g)\omega_g + \mathcal{L}\omega_g$ of the variational problems a) and b), taking into account the new concept of stress-energy-momentum tensor of a higher order natural variational problem introduced in [2]. In particular, this allows us to characterize perfect fluids in this more general framework by the following conditions:
i) The stress-energy-momentum-tensor associated to the Lagrangian density $\mathcal{L}\omega_g$ should be as follows:

$$\overline{T}^2(\overline{s}) = \rho e V \otimes V + p\left(V \otimes V + g^{-1}\right), \quad \overline{s} = (g, D, z) \in \Gamma(X_4, \overline{Y}). \tag{1.23}$$

Being $\mathcal{L}\omega_g$ of r order, the functions e and p depend on the section $\overline{s}$ up to the $2r-1$ order, and then define functions $e, p \in \mathcal{C}^\infty(J^{2r-1}\overline{Y})$. Using the first of them, e= energy, the Legendre's transformation is defined as:

$$\mathrm{Leg} \colon j_x^{2r-1}(g, D, z) \in J^{2r-1}(\overline{Y}) \mapsto (g_x, D_x, e_x) \in E. \tag{1.24}$$

The second condition for being a perfect fluid is then:
ii) The function p=pressure is Leg-projectable

With these conditions $de + pdv$ defines a 1-form on $\mathcal{C}^\infty_{\text{Diff}\, X_4}(E) = \mathcal{C}^\infty(e, v)$ that can be expressed (locally) in the form TdS. S is the specific entropy and T the temperature.

Or, alternatively, the following stronger global condition can be imposed:
iii) There exist functions $S, T \in \mathcal{C}^\infty(E)$ such that

$$TdS = de + pdv. \tag{1.25}$$

This characterization of perfect fluids allows, as we shall see in the examples, a wide margin in the election of Lagrangians in order to formulate the theory variationally. All the formulations so obtained lead, through their corresponding Legendre's transformations, to the same Eulerian theory. But what must be remarked here is that if no condition is imposed on the Lagrangian density $\mathcal{L}\omega_g$, a more general Legendre's transformation can be associated to the resulting variational problem by means of which the theory of dissipative relativistic fluids can be obtained, in a new sense we are going to deal with in the following.

2. Π-dissipative relativistic fluids

In its most recent version, dissipative relativistic hydrodynamics was formulated for the first time in 1976 by W. Israel. Since then its interest, theoretical as well as applied, is great as the existing abundant literature on this topic (see, p.e. [1, 3, 4, 7] and references therein) proves. Of special interest for our purposes is the formalism proposed by D. Pavón, D. Jou and J. Casas-Vázquez in [7], which, in its simpler expression (fluids with just bulk viscosity), can be summarized as follows:

A Π-dissipative relativistic fluid is defined giving a horizontal and Diff X_4-invariant 1-form $\mathcal{S}$ on the bundle:

$$p \colon E \subset \mathcal{M} \times_{X_4} T(X_4) \times_{X_4} \xi^1 \times_{X_4} \xi^1 \longrightarrow X_4,$$

where $E = \{(g_x, D_x, e_x, \Pi_x) : g_x(D_x, D_x) < 0\}$.

The sections $\sigma = (g, D, e, \Pi) \in \Gamma(X_4, E)$ are interpreted as a fluid in the gravitational field g, with mass density $\rho = \sqrt{-g(D, D)}$, velocity field $V = D/\rho$, specific energy e and bulk viscosity Π. $\mathcal{S}$ is the entropy 1-form and its restriction $\mathcal{S}|_\sigma$ represents the entropy flux of the fluid given by the section σ.

In this case, the ring $\mathcal{C}^\infty_{\mathrm{Diff}\,X_4}(E)$ of the state functions is identified with $\mathcal{C}^\infty(\rho, e, \Pi)$ (being $\Pi\colon \xi = (g_x, D_x, e_x, \Pi_x) \mapsto \Pi_x \in \mathbb{R}$), and $\mathcal{S} = S\omega$ where $S \in \mathcal{C}^\infty_{\mathrm{Diff}\,X_4}(E)$ represents the specific entropy.

PROPOSITION. (GIBBS' EQUATION) *There exist unique functions T, ρ and f_Π in $\mathcal{C}^\infty_{\mathrm{Diff}\,X_4}(E)$ such that:*

$$TdS = de + pdv + f_\Pi d\Pi. \tag{2.1}$$

The functions T and p are called temperature and pressure, and the differential forms $\omega_Q = TdS$ and $\omega_W = pdv + f_\Pi d\Pi$, heat and work 1-forms, respectively.

DEFINITION. The stress-energy-momentum tensor is the map that assigns to each section $\sigma = (g, D, e, \Pi) \in \Gamma(X_4, E)$ the tensor over X_4:

$$T^2(\sigma) = \rho_\sigma e V \otimes V + (p_\sigma + \Pi)(V \otimes V + g^{-1}), \tag{2.2}$$

where p_σ is the restriction to σ of the pressure function.

The field equations for this theory, as well as the equations of the fluid for a fixed metric g, keep on being (1.3) and (1.4),respectively, just replacing T^2 by (2.2) at them.

The essential difference with regard to the case of perfect fluids is that the equations (1.4) of the fluid are a system of 5 scalar equations in the 6 components of $\sigma = (D, e, \Pi)$, still allowing the possibility to impose another scalar condition (phenomenological equation) in order to lay down the system.

Bearing in mind (2.2) and the continuity equation, we get:

$$\begin{aligned} \mathrm{div}_g\, T_2(\sigma) =& (De)\omega_V + eD^\nabla \omega_V + V(p+\Pi)\omega_V + d(p+\Pi) + \\ &+ (p+\Pi)(\mathrm{div}\, V)\omega_V + (p+\Pi)V^\nabla \omega_V, \end{aligned} \tag{2.3}$$

therefore, considering also Gibbs' equation, we have:

$$\begin{aligned} (\mathrm{div}_g\, T_2(\sigma))^{\|} &= (\mathrm{div}_g\, T_2(\sigma))(V) = -De + (p+\Pi)\frac{D\rho}{\rho^2} = \\ &= -TDS + \Pi\frac{D\rho}{\rho^2} + f_\Pi D\Pi. \end{aligned} \tag{2.4}$$

Hence, the equation of balance of energy can be stated as follows:

$$\mathrm{div}_g\, \mathcal{S} = \frac{\Pi}{T\rho^2} D\rho + \frac{f_\Pi}{T} D\Pi. \tag{2.5}$$

The additional condition we have previously mentioned is now chosen in such a way that the fundamental thermodynamic inequality is satisfied:

$$\operatorname{div}_g \mathcal{S} \geq 0. \tag{2.6}$$

The two options more frequently adopted in the literature are the following:

Eckart-Landau's Theory: It is established for specific entropies of kind of the perfect fluids: $S(\rho, e)$. Then $\operatorname{div}_g \mathcal{S} = -\frac{\Pi}{T}\Theta$, being $\Theta = \operatorname{div} V = -\frac{D\rho}{\rho^2}$. In this case, the simplest condition assuring (2.6) is taking $\Pi = -\xi(\rho, T)\Theta$, where $\xi(\rho, T)$ is a positive function called bulk viscosity coefficient. By replacing this new condition in the equation of balance of momentum, we obtain the well known relativistic Navier-Stokes equation. In particular, for $\xi(\rho, T) = 0$, the phenomenological equation is $\Pi = 0$, hence following the theory for perfect fluids.

Causal Theory: In this theory, specific entropies of the form $S(\rho, e, \Pi) = S_0(\rho, e) + \alpha(\rho, e)\frac{\Pi^2}{2}$ are considered. It follows therefore $\operatorname{div}_g \mathcal{S} = -\frac{\Pi}{T}\Theta + T\alpha(\rho, e)\Pi D\Pi$. And, following the same simplicity criterion as before, the phenomenological equation $\Pi = -\xi(\rho, e)[\Theta - T^2\alpha(\rho, e)D\Pi]$ is adopted.

The Eulerian Theory just stated can be now obtained defined from the variational principle given at the end of §1.3 proceeding as follows:

The stress-energy-momentum tensor associated to the Lagrangian density $\mathcal{L}\omega_g$ over $J^r(\overline{Y})$ admits a canonical decomposition as follows:

$$\begin{aligned} \overline{T}^2(\overline{s}) =& \rho e V \otimes V + P(V \otimes V + g^{-1}) + \\ &+ q \otimes V + V \otimes q + \pi, \quad \overline{s} = (g, D, z) \in \Gamma(X_4, \overline{Y}), \end{aligned} \tag{2.7}$$

where q is a vector field such that $g(V, q) = 0$ and π a symmetric 2-contravariant tensor such that $i_{\omega_V}\pi = 0$ and whose associated endomorphism by g has null trace.

Choosing Lagrangian densities whose stress-energy-momentum tensors have the form:

$$\overline{T}^2(\overline{s}) = \rho e V \otimes V + P(V \otimes V + g^{-1}), \quad \overline{s} = (g, D, z) \in \Gamma(X_4, \overline{Y}), \tag{2.8}$$

we may introduce a new notion of Legendre's transformation proceeding as follows:

Let $p(\rho, e, \Pi) \in \mathcal{C}^\infty(E)$ be the pressure function of the Eulerian theory with entropy function $\mathcal{S} = S\omega$.

Let us suppose that the equation $P = p(\rho, e, \Pi) + \Pi$ in the unknown $\Pi \in \mathcal{C}^\infty(J^{2r-1}\overline{Y})$, where P and e are the functions on $J^{2r-1}(\overline{Y})$ defined by the tensor, has a single solution. This is an essential hypothesis and raises a previous question that must be answered in each particular case (for example, in Eckart-Landau's theory, where $p = p(\rho, e)$, the solution is trivial: $\Pi = P - p(\rho, e)$). In these conditions we can give the following:

DEFINITION. We shall call Legendre's transformation the vector bundle morphism Leg: $J^{2r-1}(\overline{Y}) \to E$ over X_4:

$$\begin{array}{rcl} \mathrm{Leg}\colon J^{2r-1}(\overline{Y}) & \to & E \\ j_x^{2r-1}(g, D, z) & \mapsto & \left[g_x, D_x, e(j_x^{2r-1}(g, D, z)), \Pi(j_x^{2r-1}(g, D, z))\right] \end{array} \tag{2.9}$$

where $e, \Pi \in \mathcal{C}^\infty(J^{2r-1}(\overline{Y}))$ are the already introduced energy and bulk viscosity functions.

As constructed, the stress-energy-momentum tensor $\overline{T}^2$ associated to the Lagrangian density $\mathcal{L}\omega_g$ is obtained from the stress-energy-momentum tensor T^2 of the Eulerian theory using the formula:

$$\overline{T}^2(\overline{s}) = T^2(\mathrm{Leg} \circ j^{2r-1}\overline{s}), \quad \overline{s} \in \Gamma(X_4, \overline{Y}),$$

therefore, in analogy with the case of perfect fluids (Theorem 4), we have:

THEOREM 4'. *The section $\overline{s} = (g, D, z) \in \Gamma(X_4, \overline{Y})$ is a critical section of the constrained variational problem with Lagrangian density $R(g)\omega_g + \mathcal{L}\omega_g$ if and only if the section $\sigma = \mathrm{Leg} \circ j^{2r-1}\overline{s} \in \Gamma(X_4, E)$ satisfies both field equations* (1.3) *of the Eulerian theory with entropy* 1*-form $\mathcal{S} = S\omega$. For a fixed metric g, $\overline{s} = (g, D, z)$ (D and z variable) is a critical section of the constrained variational problem with Lagrangian density $\mathcal{L}\omega_g$ if and only if $\sigma = \mathrm{Leg} \circ j^1\overline{s}$ satisfies the equations* (1.4) *of the corresponding fluid.*

This variational characterization is used for the sections $\sigma = (g, D, e, \Pi) \in \Gamma(X_4, E)$ such that: $\sigma = \mathrm{Leg} \circ j^{2r-1}\overline{s}$, where $\overline{s} = (g, D, z) \in \Gamma(X_4, \overline{Y})$. If we could eliminate z from the pair of equations $e = e(j^{2r-1}(g, D, z))$, $\Pi = \Pi(j^{2r-1}(g, D, z))$ that define the Legendre's transformation, a differential invariant $\Phi \in \mathcal{C}^\infty_{\mathrm{Diff}\, X_4}(J^k E)$ would be obtained (for k great enough), whose annihilation along the sections $j^k\sigma$ would characterize the previous condition $\sigma = \mathrm{Leg} \circ j^{2r-1}\overline{s}$. In other words, the system of Eulerian equations of the fluid obtained from this variational principle would be:

$$\mathrm{div}_g(D) = 0, \qquad \mathrm{div}_g\, T^2(\sigma) = 0, \qquad \Phi(j^k\sigma) = 0. \tag{2.10}$$

Hence, that principle would have its own associated phenomenological equation, $\Phi(j^k\sigma) = 0$, which could not now be imposed following any other criterion.

The inequality $\mathrm{div}_g\, \mathcal{S} \geq 0$ should then be directly imposed, restricting the whole theory to the "locus" of J^kE where the differential invariant defined by the expression of $\mathrm{div}_g\, \mathcal{S}$ has positive values.

3. Examples

3.1. The Lagrangian $\mathcal{L}(\rho,\sigma) = -\rho[1+\epsilon(\rho,\sigma)]$, $\sigma = Vz$, chosen in §1.2 to describe perfect fluids is an example of 0-order differential invariant in g and D, with order 1 in z. As we have already seen, this choice implies for the specific entropy S and temperature T to be $-\frac{\partial\epsilon(\rho,\sigma)}{\partial\sigma}$ and σ, respectively. To illustrate the characterization given in §1.3 for the perfect fluids, we are going to show in this example which 0-order differential invariants in g and D of order 1 in z can be taken as Lagrangians for this kind of fluids.

First of all, it is easy to see that the most general expression of these invariants is $\mathcal{L}(\rho, z, \sigma, \theta)$, where $\sigma = Vz$ and $\theta = g^{-1}(dz, dz)$. In order that the condition i) of our characterization holds, it is easy to see that $\mathcal{L}$ must be independent from θ. Giving $\mathcal{L}(\rho, z, \sigma) = -\rho[1+\epsilon(\rho, z, \sigma)]$, its stress-energy-momentum tensor is:

$$\overline{T}^2(\overline{\sigma}) = \rho e V \otimes V + P(V\otimes V + g^{-1}), \quad \overline{s} = (g, D, z) \in \Gamma(X_4, \overline{Y}),$$

where:

$$e = 1 + \epsilon(\rho, z, \sigma) - \sigma\frac{\partial\epsilon(\rho, z, \sigma)}{\partial\sigma}, \quad P = \rho^2\frac{\partial\epsilon(\rho, z, \sigma)}{\partial\rho}. \tag{3.1}$$

The distribution tangent to the fibers of Leg: $J^1(\overline{Y}) \to E$, the Legendre's transformation, is generated by the vector field:

$$X = \left(\sigma\frac{\partial^2\epsilon}{\partial\sigma^2}\right)\frac{\partial}{\partial z} + \left(\frac{\partial\epsilon}{\partial z} - \sigma\frac{\partial^2\epsilon}{\partial\sigma\partial z}\right)\frac{\partial}{\partial\sigma}, \tag{3.2}$$

hence, the condition ii), Leg-projectability of the function P, can be expressed infinitesimally as $XP = 0$, implying the following condition for the function $\epsilon(\rho, z, \sigma)$:

$$\frac{\partial^2\epsilon}{\partial\rho\partial\sigma}\left(\sigma\frac{\partial^2\epsilon}{\partial\sigma\partial z} - \frac{\partial\epsilon}{\partial z}\right) - \sigma\frac{\partial^2\epsilon}{\partial\rho\partial z}\frac{\partial^2\epsilon}{\partial\sigma^2} = 0. \tag{3.3}$$

In particular, as one would expect, this condition is fulfilled by the functions $\epsilon(\rho,\sigma)$ that define the Langrangians chosen in §1.3.

The calculation of the specific entropy, S, and of the temperature, T, must be done for each case once the particular solution of (3.3) one wants to study has been chosen.

As an illustration of this, let us consider the Lagrangian

$$\mathcal{L} = -\rho\,[1+\epsilon(\rho,z,\sigma)]\,,$$

where:

$$\epsilon(\rho,z,\sigma) = \frac{\sigma^2}{2}\exp\rho + z\exp(-\rho), \tag{3.4}$$

which, clearly, verifies the condition (3.3).

In this case we have:

$$e = 1-\frac{\sigma^2}{2}\exp\rho + z\exp(-\rho), \quad P=\rho^2\left(\frac{\sigma^2}{2}\exp\rho - z\exp(-\rho)\right). \tag{3.5}$$

Thus, $P=\rho^2(1-e)$.

The integration of the second member, $de+(e-1)d\rho$, of the Gibbs' equation leads to the following expressions for the specific entropy and temperature:

$$S=(e-1)\exp\rho, \quad T=\exp(-\rho). \tag{3.6}$$

3.2. Let us consider, as a second example, the Lagrangian density $\mathcal{L}\omega_g$ on $J^1(\overline{Y})$, where:

$$\mathcal{L} = \mathcal{L}_0 + F(\rho,\sigma,\Theta), \tag{3.7}$$

and $\mathcal{L}_0 = -\rho\,[1+\epsilon(\rho,\sigma)]$, $\sigma = V(z)$ and $\Theta = -\rho D(\rho)$.

This is a "perturbation" for the perfect fluid with Lagrangian density $\mathcal{L}_0\omega_g$ defined by the term $F(\rho,\sigma,\Theta)$. Its stress-energy-momentum tensor has again the form:

$$\overline{T}^2(\overline{s}) = \rho e V\otimes V + P\left(V\otimes V + g^{-1}\right), \quad \overline{s}=(g,D,z)\in\Gamma(X_4,\overline{Y}), \tag{3.8}$$

being now:

$$\begin{aligned} e &= e_0 + \frac{1}{\rho}\left(-F+\sigma\frac{\partial F}{\partial\sigma}+\Theta\frac{\partial F}{\partial\Theta}\right),\\ P &= p_0 + F - \rho\frac{\partial F}{\partial\rho} - \Theta\frac{\partial F}{\partial\Theta} - \rho^2 D\left(\frac{\partial F}{\partial\Theta}\right), \end{aligned} \tag{3.9}$$

where e_0 and P_0 are the specific energy and pressure of the perfect fluid with Lagrangian density $\mathcal{L}_o\omega_g$, specific entropy $S_0 = -\frac{\partial\epsilon(\rho,\sigma)}{\partial\sigma}$ and temperature $T_0 = \sigma$.

In particular, if the additional condition $e = e_0$ is imposed, $F(\rho, \sigma, \Theta)$ must be a homogeneous function of degree 1 in the variables σ and Θ.

Let us be a bit more concrete. We are interested in considering this case as Lagrangian model for a Π-dissipative relativistic fluid, taking the same specific entropy $S = S_0$ as the initial system's one.

Gibbs' equation (2.1) leads to $T = T_0 = \sigma$ and $P = p_0$, and thus to the following expression for bulk viscosity:

$$\begin{aligned}\Pi = P - p_0 &= F - \rho\frac{\partial F}{\partial\rho} - \Theta\frac{\partial F}{\partial\Theta} - \rho^2 D\left(\frac{\partial F}{\partial\Theta}\right) \\ &= \sigma\frac{\partial F}{\partial\sigma} - \rho\frac{\partial F}{\partial\rho} - \rho^2 D\left(\frac{\partial F}{\partial\Theta}\right).\end{aligned} \tag{3.10}$$

Formula (3.10), being expressed as it is in terms of the Eulerian variables, is already the phenomenological equation associated to this problem.

For example, if $F(\rho, z, \Theta) = -\frac{\rho}{T}\lambda(\rho)\Theta^2$, then it holds:

$$\Pi = -\frac{\rho\lambda'(\rho)}{T}\Theta^2 - 2\rho^3\frac{\lambda(\rho)}{T^2}D(T)\Theta + 2\rho^3\frac{\lambda(\rho)}{T}D(\Theta). \tag{3.11}$$

3.3. Let us follow with the previous example, but now taking the following function as specific entropy:

$$S(\rho, e, \Pi) = S_0(\rho, e) + \alpha(\rho, e)\frac{\Pi^2}{2}.$$

Making use of Gibbs' equation for the perfect fluid $T_0 dS_0 = de - \frac{p_0}{\rho^2}d\rho$, we obtain:

$$\frac{T_0}{1 + T_0\Pi^2\frac{\partial\alpha}{\partial e}}dS = de - \frac{1}{\rho^2}\frac{p_0 - \rho^2 T_0\Pi^2\frac{\partial\alpha}{\partial\rho}}{1 + T_0\Pi^2\frac{\partial\alpha}{\partial e}}d\rho + \frac{T_0\Pi\alpha}{1 + T_0\Pi^2\frac{\partial\alpha}{\partial e}}d\Pi,$$

which is Gibbs' equation for the specific entropy S with pressure and temperature given by:

$$T = \frac{T_0}{1 + T_0\frac{\partial\alpha}{\partial e}\Pi^2}, \qquad p = \frac{p_0 - \rho^2 T_0\frac{\partial\alpha}{\partial\rho}\Pi^2}{1 + T_0\frac{\partial\alpha}{\partial e}\Pi^2}.$$

Thus, the fenomenological equation is:

$$\Pi = P - p = P - \frac{p_0 - \rho^2 T_0 \frac{\partial \alpha}{\partial \rho} \Pi^2}{1 + T_0 \frac{\partial \alpha}{\partial e} \Pi^2},$$

where

$$P = p_0 + F - \rho \frac{\partial F}{\partial \rho} - \Theta \frac{\partial F}{\partial \Theta} - \rho^2 D \left(\frac{\partial F}{\partial \Theta} \right), \quad p_0 = \rho^2 \frac{\partial \epsilon}{\partial \rho}, \quad T_0 = \sigma,$$

that, in general, is a equation of third degree on Π.

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