

On Operators T such that $f(T)$ is Hypercyclic*

TERESA BERMÚDEZ AND VIVIEN G. MILLER

*Dpto. de Análisis Matemático, Universidad de La Laguna,
38271-La Laguna (Tenerife), Spain*

Dept. of Mathematics, Mississippi State Univ., Drawer MA, Mississippi State, 39762

(Research announcement presented by M. González)

AMS Subject Class. (1991): 47B99

Received July 19, 1999

1. INTRODUCTION

An operator T on a complex infinite dimensional Banach space X is *hypercyclic* provided that there is a vector x with dense orbit $\text{Orb}(T, x) = \{x, Tx, T^2x, \dots\}$. If the set of scalar multiples of the orbit of x is dense, then T is said to be *supercyclic*. Two classical examples of hypercyclic operators are the translation on the Fréchet space of entire functions [2] and the differentiation on the same space [10]. Also, if S is the backwards shift on $\ell^2(N)$, then λS is hypercyclic for each complex number λ with $|\lambda| > 1$; in particular, S is supercyclic [12]. These examples follow from a sufficient condition, known as the Hypercyclicity Criterion, due to R. Gethner and J. Shapiro [3] and, independently, C. Kitai [9]. This condition has been an important tool in much recent work on hypercyclic operators and is the basis of our results as well.

In this note, we study conditions, on an operator $T \in L(X)$ and on a function f analytic on a neighborhood of the spectrum of T , under which the operator $f(T)$ (given by the Dunford-Taylor analytic functional calculus) is hypercyclic or supercyclic.

Throughout, X will be an complex, separable, infinite-dimensional Banach space, and $L(X)$ will be denote the collection of all bounded linear operators on X . A necessary condition for an operator to be hypercyclic is that its spectrum intersects the unit circle [9], and clearly if $\dim(X) > 1$, then $\lambda I \oplus T \in$

*This work is a research announcement of the paper entitled “On operators T such that $f(T)$ is hypercyclic” that it will be appearing in *Integral Equation and Operator Theory*.

$L(X \oplus Y)$ fails to be hypercyclic for any $T \in L(Y)$. Thus, in order for $f(T)$ to be hypercyclic, the analytic function f must be nonconstant on each non-singleton component of the spectrum of T , $\sigma(T)$, and $f(\sigma(T))$ must meet the unit circle $\partial\mathbf{D}$.

This problem has been addressed by several authors. G. Godefroy and J. Shapiro [4, Theorem 4.11] consider a generalized backward shift; that is, an operator $T \in L(X)$ that is surjective and such that $\dim \text{Ker}(T) = 1$ and $\bigcup_n \text{Ker}(T^n)$ is dense. If f is a nonconstant analytic function on a neighborhood of $\sigma(T)$, then $f(T)$ is supercyclic. G. Herzog and C. Schmoeger [8, Theorem 1] prove that if T is a surjective bounded linear operator such that $\bigcup_n \text{Ker}(T^n)$ is dense and if f is a nonconstant analytic function on a neighborhood of $\sigma(T)$ such that $|f(0)| = 1$ and $0 \notin \sigma(f(T))$, then $f(T)$ is hypercyclic. In particular, this condition does not require any hypothesis on the $\dim \text{Ker}(T)$. Using Herrero's characterizations of the closure of the classes of hypercyclic and supercyclic operators [6, Theorem 2.1, 3.3], G. Herzog and C. Schmoeger also give conditions for $f(T)$ to be the limit of hypercyclic or supercyclic operators on X . Recently, T. Miller and V. Miller [11, Theorem 1, Corollary 1] have generalized Theorems 1 and 3 of G. Herzog and C. Schmoeger. The main idea of their proof is the introduction of local spectral theory in a hypercyclicity context. See also [1, 7] for applications of these results.

The purpose of this note is to present an improvement of [11, Theorem 1, Corollary 1] without using the local spectral theory. In order to obtain the new result, Theorem 2, we will use the fact that the class of operators satisfying the Hypercyclicity Criterion is stable under direct sums [14, p. 125] (see Proposition 1). The specific version of what we refer to as the Hypercyclicity Criterion is Corollary 1.5 of [4], but also see [3, 4, 5, 9]. The Supercyclicity Criterion is located in [13, Lemma 2.5].

2. MAIN RESULTS

Given $T \in L(X)$, let $\rho_K(T)$ denote the *Kato resolvent set* of T ; $\rho_K(T) := \{\lambda \in \mathbb{C}: \text{Ran}(\lambda - T) \text{ closed, } \text{Ker}(\lambda - T) \subset \bigcap_{n \in \mathbb{N}} \text{Ran}(\lambda - T)^n\}$, where $\text{Ran}(\lambda - T)$ denotes the range of $\lambda - T$.

THEOREM 1. [11, Theorem 1] *Let X be a separable Banach space and suppose that $T \in L(X)$ is such that, for some $\lambda \in \rho_K(T)$*

$$\overline{\bigcup_{n \in \mathbb{N}} \text{Ker}(\lambda - T)^n} = X.$$

If f is a nonconstant analytic function on a neighborhood of $\sigma(T)$, then $f(T)$ is supercyclic. Moreover, if $f(G_\lambda)$ intersects the unit circle, where G_λ denotes the component of $\rho_K(T)$ such that $\lambda \in G_\lambda$, then $f(T)$ is hypercyclic.

Remark 1. If one takes into account the more general Hypercyclic Criterion found in [5, Theorem 2], then the previous theorem can be improved as follows. Suppose that T and $\lambda \in \rho_K(T)$ are as in the hypotheses of Theorem 1. Assume that $\{f_n\}$ is a sequence of analytic functions on a neighborhood of $\sigma(T)$ for which there are two sequences $\{a_n\}$ and $\{b_n\}$ converging to λ such that for each k , $f_n(a_k) \rightarrow 0$ and $f_n(b_k) \rightarrow \infty$ as $n \rightarrow \infty$. Then the sequence $\{f_n(T)\}$ is hypercyclic. Namely, there exists $x \in X$ such that $\{f_n(T)x : n \in \mathbb{N}\}$ is dense in X . The proof is similar to that of Theorem 1 using some ideas from [4, 8] instead of the local spectral theory notions of the original argument.

As an application, consider $\lambda \in \rho_K(T)$ as in Theorem 1 and f a nonconstant analytic function on a neighborhood of $\sigma(T)$ with $|f(\lambda)| = 1$. If $\{g_n\}$ is a sequence of analytic functions on $\sigma(T)$, bounded and bounded below in a neighborhood of λ , then for $f_n(z) := (f(z))^n g_n(z)$, the sequence $\{f_n(T)\}$ is hypercyclic in the sense above.

Given $T \in L(X)$, let $\rho_{su}(T) := \{\lambda \in \mathbb{C} : (\lambda - T)X = X\}$. Let $\sigma_{su}(T) = \mathbb{C} \setminus \rho_{su}(T)$ be the *surjectivity spectrum* of T and denote the *approximate point spectrum* by $\sigma_{ap}(T)$. Since $\sigma_{su}(T) = \sigma_{ap}(T^*)$, we see that $\rho_{su}(T)$ is an open set.

LEMMA 1. *Suppose $T \in L(X)$ and $\lambda \in \sigma(T)$ such that $\lambda - T$ is surjective. Let G be the component of $\rho_{su}(T)$ containing λ , let $Y := \overline{\cup_{n \in \mathbb{N}} \text{Ker}(\lambda - T)^n}$ and define $S := T|_Y$. Then*

- (1) $Y = \overline{\cup_{n \in \mathbb{N}} \text{Ker}(\mu - S)^n}$ for all μ in G ,
- (2) $G \subset \rho_{su}(S)$,
- (3) $\sigma(S) = \sigma_{ap}(S)$; in particular, $\sigma(S) \subset \sigma(T)$.

In order to state our main result we have to introduce some notation.

Let $\{X_n\}$ be a sequence of Banach spaces, and let $\ell^1(X_n)$ be the Banach space defined in a natural way. Furthermore, if $T_n \in L(X_n)$ for each n , and $\{T_n\}$ is a uniformly bounded sequence, then $T := (T_n)$ is a bounded linear operator on $\ell^1(X_n)$.

The next result is a generalization of Shapiro's result for a countable number of operators [14, p. 125].

PROPOSITION 1. Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of separable Banach spaces and let $\{T_n\}_{n \in \mathbb{N}}$ be a uniformly bounded sequence of linear operators such that $T_n \in L(X_n)$ for each $n \in \mathbb{N}$. If T_n satisfies the Hypercyclicity Criterion (Supercyclicity Criterion) for each $n \in \mathbb{N}$, then $T := (T_n)$ is a hypercyclic (supercyclic) operator on $\ell^1(X_n)$.

Now, we can extend Theorem 1 to the case that a single generalized kernel fails to be dense but the span of a countable number of them is dense. We introduce the following notation.

Let $T \in L(X)$ and f be an analytic function in a neighborhood of $\sigma(T)$. Given $r > 0$, define

$$B(r) := \{z \in \rho_{su}(T) : f(G_z) \cap r\partial\mathbf{D} \neq \emptyset\}$$

and

$$E(z) := \bigcup_{n \in \mathbb{N}} \text{Ker}(z - T)^n.$$

THEOREM 2. Let X be a separable Banach space, $T \in L(X)$ and f be an analytic function in a neighborhood of $\sigma(T)$, and nonconstant on each component of $\sigma(T)$.

1. If $\text{span}\{E(z) : z \in B(1)\}$ is dense, then $f(T)$ is hypercyclic.
2. If $\text{span}\{E(z) : z \in B(r) \text{ for some } r \geq a \geq 0\}$ is dense, then $f(T)$ is supercyclic.

REFERENCES

- [1] BETANCOR, J.J., BONILLA, A., On a universality property of certain integral operators (preprint).
- [2] BIRKHOFF, G.D., Démonstration d'un théorème élémentaire sur les fonctions entières, *C. R. Acad. Sci. Paris*, **189** (1929), 473–475.
- [3] GETHNER, R., SHAPIRO, J., Universal vectors for operators on spaces of holomorphic functions, *Proc. Amer. Math. Soc.*, **100** (1987), 281–288.
- [4] GODEFROY, G., SHAPIRO, J., Operators with dense invariant cyclic vectors manifolds, *J. Funct. Anal.*, **98** (1991), 229–269.
- [5] GROßE-ERDMANN, K.G., Universal families and hypercyclic operators, *Bull. Amer. Math. Soc.* (to appear).
- [6] HERRERO, D., Limits of hypercyclic and supercyclic operators, *J. Funct. Anal.*, **99** (1991), 179–190.
- [7] HERZOG, G., On a universality of the heat equation, *Math. Nachr.*, **188** (1997), 169–171.

- [8] HERZOG, G., SCHMOEGER, C., On operators T such that $f(T)$ is hypercyclic, *Studia Math.*, **108** (1994), 209–216.
- [9] KITAI, C., “Invariant Subsets for Linear Operators”, Dissertation, University of Toronto, 1982.
- [10] MACLANE, G.R., Sequences of derivatives and normal families, *J. Analyse Math.*, **2** (1952), 72–87.
- [11] MILLER, T., MILLER, V., Local spectral theory and orbits of operators, *Proc. Amer. Math. Soc.*, **127** (1999) 1029–1037.
- [12] ROLEWICZ, S., On orbits of elements, *Stud. Math.*, **33** (1969), 17–22.
- [13] SALAS, H., Supercyclicity and weighted shifts (preprint).
- [14] SHAPIRO, J., “Composition Operators and Classical Function Theory”, Springer, New York, 1993.