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Ganea Term for CCG-Homology of Crossed Modules

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In [2] an "internal homology" theory of crossed modules was defined (CCG-homology for short), which is very much related to the homology of the classifying spaces of crossed modules ([5]). The goal of this note is to construct a low-dimensional homology exact sequence corresponding to a central extension of crossed modules, which is quite similar to the one constructed in [3] for group homology.

A *crossed module* (T, G, ∂) is a group homomorphism $\partial : T \to G$ together with an action of G on T satisfying: $\partial({}^g t) = g\partial(t)g^{-1}$ and ${}^{\partial t}s = tst^{-1}$, for $g \in G, t, s \in T$. Let **CM** be the category of crossed modules and let Ab**CM** be the category of abelian group objects in **CM**. Abelian crossed modules are nothing, but homomorphisms of abelian groups. The inclusion $Ab\mathbf{CM} \subset \mathbf{CM}$ has a left adjoint functor $(T, G, \partial) \mapsto (T, G, \partial)_{ab}$, where $\partial : T \to G$ is a crossed module and

$$(T, G, \partial)_{ab} = (T/[G, T], G/[G, G], \partial)$$

(see for example [2]). Here $[G, T]$ is a subgroup of T generated by the elements $[x, t] =^x tt^{-1}, x \in G, t \in T$. One calls this functor the abelianization of the crossed module (T, G, ∂). Then CCG-homology of crossed modules is defined as the simplicial derived functor of the abelianization functor ([2]). We now give an alternative definition which is more handabel for many purposes.

For a crossed module (T, G, ∂) we let $B(T, G, \partial)$ be the classifying space of (T, G, ∂) (see [6] and [4]). Then $(0, 1_G) : (0, G, 0) \to (T, G, \partial)$ yields an injective map of simplicial sets $i_{(T,G,\partial)} : BG \to B(T, G, \partial)$, whose cofibre is denoted by $\beta(T, G, \partial)$. Then the homology exact sequence gives rise to a homomorphism $H_{i+1}(\beta(T, G, \partial)) \to H_i G$, which can be considered as an abelian crossed module. Thanks to the result in [5] the CCG-homology of

(T, G, ∂) is isomorphic to this particular abelian crossed module

$$H_i^{CCG}(T, G, \partial) \cong (H_{i+1}(\beta(T, G, \partial)) \to H_i G),\ i \geq 1.$$

In [7] the category of abelian crossed modules was equipped with a tensor product. Here we introduce a different tensor product which plays an important role in CCG-homology theory. Let (A, B, ∂) and (M, N, ∂) be two abelian crossed modules and let $f : B \to M$ be a homomorphism of abelian groups. Then $(\partial f, f\partial) : (A, B, \partial) \to (M, N, \partial)$ is a morphism of abelian crossed modules. Hence we have a well defined homomorphism of abelian groups (and hence an abelian crossed module)

$$\delta : Hom_{Ab}(B, M) \to Hom_{Ab\mathbf{CM}}((A, B, \partial), (M, N, \partial))$$

given by $\delta(f : B \to M) = (\partial f, f\partial)$. We let $\mathbf{Hom}((A, B, \partial), (M, N, \partial))$ be this particular crossed module

Now we define the internal tensor product in $Ab\mathbf{CM}$. For abelian crossed modules (A, B, ∂) and (M, N, ∂) we let

$$\alpha : A \otimes M \to (B \otimes M) \oplus (A \otimes N)$$

be a homomorphism of abelian groups given by $\alpha = (\partial \otimes 1, -1 \otimes \partial)$. Now we define

$$(A, B, \partial) \otimes (M, N, \partial) := (Coker\ \alpha, B \otimes N, \delta),$$

where $\delta : Coker\ \alpha \to B \otimes N$ is given by $\delta(b \otimes m, a \otimes n) = b \otimes \partial m + \partial a \otimes n$ and it is not so hard to check that the following is true:

Proposition 1. *For abelian crossed modules (A, B, ∂), (M, N, ∂) and (T, G, ∂) there exist a natural isomorphism*

$$Hom_{Ab\mathbf{CM}}((A, B, \partial) \otimes (M, N, \partial), (T, G, \partial)) \cong$$

$$Hom_{Ab\mathbf{CM}}((A, B, \partial), \mathbf{Hom}((M, N, \partial), (T, G, \partial))).$$

It is also easy to check that this tensor product makes the category $Ab\mathbf{CM}$ a symmetric monoidal category whose unit is $(0, \mathbf{Z}, 0)$, that is for any abelian crossed module (T, G, ∂) there exists a natural isomorphism

$$(0, \mathbf{Z}, 0) \otimes (T, G, \partial) \cong (T, G, \partial).$$

For an abelian crossed module (A, B, ∂) we also introduce abelian crossed modules $\Lambda^2(A, B, \partial)$, $S^2(A, B, \partial)$ and $\Gamma(A, B, \partial)$ which are crossed module analogues of the second exterior power, the second symmetric power and the Whitehead Γ functor of abelian groups. Let us recall the definition of the last functor. For any abelian group A the Whitehead group $\Gamma(A)$ is the abelian group generated by elements $\gamma(a)$, $a \in A$ modulo the relations $\gamma(-a) = \gamma(a)$ and $\gamma(a+b+c) - \gamma(a+b) - \gamma(a+c) - \gamma(b+c) + \gamma(a) + \gamma(b) + \gamma(c) = 0$. The last condition means that $\Delta\gamma(a_1, a_2) := \gamma(a_1 + a_2) - \gamma(a_1) - \gamma(a_2)$ is linear on a_1 and on a_2. Therefore one has a natural homomorphism $\Delta : A \otimes A \to \Gamma A$ given by $\Delta(a_1 \otimes a_2) = \Delta\gamma(a_1, a_2)$. Clearly Δ factors through the second symmetric power $S^2 A$.

Now we extend these quadratic functors to abelian crossed modules. For an abelian crossed module (A, B, ∂) we let $B \bar{\otimes} A$ (resp. $B \underline{\otimes} A$) be the quotient of $B \otimes A$ by the subgroup generated by the elements $\partial(a) \otimes a$, $a \in A$ (resp. $\partial(a_1) \otimes a_2 - \partial(a_2) \otimes a_1$). Let us observe that for any $a_1, a_2 \in A$ one has $\partial(a_1) \bar{\otimes} a_2 + a_1 \bar{\otimes} \partial(a_2) = 0$ in $B \bar{\otimes} A$. Now we put

$$\Lambda^2(A, B, \partial) := (B \bar{\otimes} A, \Lambda^2 B, \partial_{\Lambda^2(A,B,\partial)}),$$

and

$$S^2(A, B, \partial) := (B \underline{\otimes} A, S^2 B, \partial_{S^2(A,B,\partial)}),$$

where $\partial_{\Lambda^2(A,B,\partial)}(b \bar{\otimes} a) = b \wedge \partial(a)$ and $\partial_{S^2(A,B,\partial)}(b \underline{\otimes} a) = b\partial(a)$. Similarly for an abelian crossed module (A, B, ∂) we put

$$\Gamma(A, B, \partial) := (\bar{\Gamma}(A, B, \partial), \Gamma B, \partial_{\Gamma(A,B,\partial)}),$$

where $\bar{\Gamma}(A, B, \partial)$ is the cokernel of the homomorphism

$$f : A \otimes A \to (B \underline{\otimes} A) \oplus \Gamma A$$

and $\partial_{\Gamma(A,B,\partial)}(b \underline{\otimes} a, \gamma(a_1)) = \Delta(b \otimes \partial a) + \gamma(\partial a_1)$. Here

$$f(a_1 \otimes a_2) = (\partial a_1 \underline{\otimes} a_2, -\Delta(a_1 \otimes a_2)).$$

We let $\theta : \bar{\Gamma}(A, B, \partial) \to Coker\ \alpha$ and $\eta : Coker\ \alpha \to B \bar{\otimes} A$ be the homomorphisms given by $\theta(b \underline{\otimes} a, \gamma(a_1)) = (b \otimes a + da_1 \otimes a_1, a \otimes b)$ and $\eta(b_1 \otimes a_1, a_2 \otimes b_2) = b_1 \bar{\otimes} a_1 - b_2 \bar{\otimes} a_2$. Then we get natural homomorphisms $\theta : \Gamma(A, B, \partial) \to (A, B, \partial) \otimes (A, B, \partial)$ and $\eta : (A, B, \partial) \otimes (A, B, \partial) \to \Lambda^2(A, B, \partial)$ and it is not hard to check that for any abelian crossed module (A, B, ∂) the following $\Gamma(A, B, \partial) \to (A, B, \partial) \otimes (A, B, \partial) \to \Lambda^2(A, B, \partial) \to 0$ is an exact sequence.

THEOREM 2. *For crossed modules* (T, G, ∂) *and* (R, K, ∂) *there exists a natural isomorphism*

$$H_2^{CCG}((T, G, \partial) \times (R, K, \partial)) \cong$$

$$H_2^{CCG}(T, G, \partial) \oplus H_2^{CCG}(R, K, \partial) \oplus (T, G, \partial)_{ab} \otimes (R, K, \partial)_{ab}.$$

THEOREM 3. *For any abelian crossed module* (A, B, ∂) *there exists a natural isomorphism*

$$H_2^{CCG}(A, B, \partial) \cong \Lambda^2(A, B, \partial).$$

Let us recall that a short exact sequence of crossed modules

$$0 \to (A, B, \partial) \to (T, G, \partial) \to (R, K, \partial) \to 0$$

is called *a central extension* if (A, B, ∂) is an abelian crossed module, A and B are central subgroups in T and G respectively, the action of G on A is trivial and the action of B on T is also trivial.

THEOREM 4. *For any central extension of crossed modules*

$$0 \to (A, B, \partial) \to (T, G, \partial) \to (R, K, \partial) \to 0$$

there exists a natural homomorphism $\tau : \Gamma(A, B, \partial) \to (A, B, \partial) \otimes (T, G, \partial)_{ab}$ *and a natural exact sequence*

$$H_3^{CCG}(T, G, \partial) \to H_3^{CCG}(R, K, \partial) \to Coker\ \tau \to H_2^{CCG}(T, G, \partial)$$

$$\to H_2^{CCG}(R, K, \partial) \to (A, B, \partial) \to (T, G, \partial)_{ab} \to (R, K, \partial)_{ab} \to 0.$$

Proofs are based on [1], [5], [8], [9], and will be given in [10].

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