EXTRACTA MATHEMATICAE Vol. **15**, Núm. 1, 181 – 192 (2000)

Absolutely Summing Operators between Banach Spaces of Finite Cotype

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(Research paper presented by Jesús F. Castillo)

AMS *Subject Class.* (1991): 46M05, 46A32 *Received January* 8, 1999

1. Introduction

It is well-known that the notions of type and cotype have found interesting applications in the context of the theory of absolutely summing operators (see [6], [7], [15] or [21]). These results are often related to the inclusion of all the s-absolutely summing operators -for every s- into another ideal. As a particular example we recall the following result. Let T be an operator from a Banach space E in a Banach space F that has cotype 2. It is well-known that all s-absolutely summing operators are 2-absolutely summing for each $s \geq 1$ (see 26.4(3) in [6]). Moreover, a result due to Maurey gives more information about this case: if E also has cotype 2, T is 2-summing if and only if T is 1-summing (see also [6]). In the present paper we investigate the following questions that are closely related to these results.

1) What can we say if we require in addition that also E has finite cotype q, $q \neq 2$?

2) What can we say if F has cotype p and E has cotype q?

The main idea is to define the new class of ideals of $(2,(1,\sigma))$-mixing operators $(\mathcal{M}_{(2,(1,\sigma))}, 0 \leq \sigma < 1)$, and to use their properties in order to answer the stated questions. We will define $(\mathcal{M}_{(2,(1,\sigma))}$ as the quotient ideal $\mathcal{P}_2^{-1}\mathcal{P}_{1,\sigma}$, where $\mathcal{P}_2$ is the ideal of 2-absolutely summing operators and $\mathcal{P}_{1,\sigma}$ is the ideal of $(1,\sigma)$-absolutely continuous operators defined by Matter in [12]. If $1 \leq p \leq \infty$, the ideal $\mathcal{P}_{p,\sigma}$ of (p,σ)-absolutely continuous operators was obtained by Jarchow and Matter using the interpolative procedure defined in [8] and [12], although the ideal of absolutely continuous operators was introduced by Niculescu (see [13]). The reader can find more information about all these ideals in [10], [11], [17] and [16].

Throughout this paper we will employ standard Banach space notation. Let E be a Banach space. Then $Id_E : E \to E$ will be the identity operator, and $W(B_{E'})$ will be the set of all regular Borel probabilities on the unit ball $B_{E'}$ of E' (endowed with the weak* topology). If $x_1, \dots, x_n \in E$, we will use the following notation:

$$w_p((x_i)) := \sup_{x' \in B_{E'}} \Big(\sum_{i=1}^{n} | < x_i, x' > |^p \Big)^{\frac{1}{p}},$$

$$l_p((x_i)) := \Big(\sum_{i=1}^{n} \|x_i\|^p \Big)^{\frac{1}{p}},$$

$$\delta_{p,\sigma}((x_i)) := \sup_{x' \in B_{E'}} \Big(\sum_{i=1}^{n} \big(| < x_i, x' > |^{1-\sigma} \|x_i\|^{\sigma} \big)^{\frac{p}{1-\sigma}} \Big)^{\frac{1-\sigma}{p}},$$

$$m_{(q,p)} := \inf\{ l_r((\tau_i)) w_q((x_i^0)) : x_i = \tau_i x_i^0 \},$$

where $\frac{1}{q} + \frac{1}{r} = \frac{1}{p}$, $0 \le \sigma < 1$ and $1 \le p, q \le \infty$. Note that

$$w_{\frac{1}{1-\sigma}}((x_i)) \le \delta_{1,\sigma}((x_i)) \le w_1((x_i)).$$

Let $\mathcal{U}$ be an operator ideal and E a Banach space. We will put $E \in$ Space$(\mathcal{U})$ iff $Id_E \in \mathcal{U}$. We will denote by p' the conjugate index of p.

Let $1 \le p \le q \le \infty$ and $0 \le \sigma < 1$. An operator $T \in \mathcal{L}(E, F)$ is (q,p)-mixing if for every Banach space G and operator $S \in \mathcal{P}_q(F, G)$, the composition ST is p-absolutely summing. The operator ideal of all (q,p)-mixing operators will be denoted by $\mathcal{M}_{(q,p)}$, and the ideal norm by $M_{(q,p)}$. The reader is also expected to be familiar with the properties of (q,p)-summing operators, with the concept of (Rademacher) cotype of a Banach space and the operators of cotype p. We will denote by $(\mathcal{C}_p, C_p)$ the ideal of operators of cotype p. If M is an ideal norm, we will put $M(E)$ for $M(Id_E)$.

We will need the following results. A Banach space E has cotype p ($p > 2$) iff $E \in$ Space$(\mathcal{P}_{p,1})$, but this fails for $p = 2$ (Talagrand, [18], [19]). The (strict) inclusions between operator ideals $\mathcal{M}_{(q,p)} \subset \mathcal{P}_{(r,p)} \subset \mathcal{M}_{(s,p)}$ hold, where $\frac{1}{q} + \frac{1}{r} = \frac{1}{p}$ and $p \le s < q$ (see Chapter 20 in [14]). Thus, if $q > 2$ and $E \in$ Space$(\mathcal{M}_{(q',1)})$ then E has cotype q, but the converse does not hold (see Chapter 32 in [6]). In order to obtain the main result of our paper (theorem 12), we will use the above results, a composition formula and the characterization theorem of $\mathcal{M}_{(2,(1,\sigma))}$.

2. The quotient operator ideal of $(2,(1,\sigma))$-mixing operators

Definition 1. (Matter) [12, 3.1] Let $\mathcal{U}$ be an operator ideal and let $0 \leq \sigma < 1$. An operator $T : E \to F$ belongs to $\mathcal{U}_\sigma$ if there exist a Banach space G and an operator $S \in \mathcal{U}(E,G)$ such that

$$\|Tx\| \leq \|x\|^\sigma \|Sx\|^{1-\sigma} \qquad \forall x \in E. \tag{1}$$

If $\mathcal{U}$ is a normed operator ideal and α is its norm, $\mathcal{U}_\sigma$ is a normed operator ideal; the norm is given by $T \to \inf \alpha(S)^{1-\sigma}$, where the infimum is extended over all operators S an all Banach spaces G satisfying (1).

For the particular case $\mathcal{U} = \mathcal{P}_p$ we obtain $\mathcal{P}_{p,\sigma}$. This ideal satisfies intermediate properties between $\mathcal{P}_p$ and $\mathcal{P}_{(\frac{p}{1-\sigma},p)}$ and its description generalizes the case $\mathcal{P}_p$. In particular, Matter got that $T \in \mathcal{P}_{p,\sigma}$ iff one of the following statements holds.

a) There is a constant C and $\mu \in W(B_{E'})$ such that

$$\|Tx\| \leq C\Big(\int_{B_{E'}} |< x, x' >|^p d\mu\Big)^{\frac{1-\sigma}{p}} \|x\|^\sigma \qquad \forall x \in E.$$

b) There is a constant C such that, for each sequence $(x_i)_1^n \subset E$,

$$l_{\frac{p}{1-\sigma}}((Tx_i)) \leq C\delta_{p,\sigma}((x_i)).$$

In addition, the operator norm $\pi_{p,\sigma}(T)$ on $\mathcal{P}_{p,\sigma}(E,F)$ is the smallest number C for which a) and b) hold (see Proposition 4.1 in [12]).

Definition 2. Let $1 \leq p \leq \infty$ and $0 \leq \sigma < 1$. We say that an operator $T \in \mathcal{L}(E,F)$ is $(p,(1,\sigma))$-absolutely summing if there exists $C > 0$ such that for each finite family $x_1, \dots, x_m \in E$,

$$l_p((Tx_j)) \leq C\delta_{1,\sigma}((x_j)). \tag{2}$$

We denote the normed operator ideal of all $(p,(1,\sigma))$-absolutely summing operators by $(\mathcal{P}_{(p,(1,\sigma))}, \pi_{(p,(1,\sigma))})$, where the norm $\pi_{(p,(1,\sigma))}(T)$ is the infimum of all constants C satisfying (2).

Definition 3. Let $T \in \mathcal{L}(E,F)$ and $0 \leq \sigma < 1$. We say that T is $(2,(1,\sigma))$-mixing if it belongs to the quotient operator ideal $\mathcal{M}_{(2,(1,\sigma))} := \mathcal{P}_2^{-1}\mathcal{P}_{1,\sigma}$. We denote by $M_{(2,(1,\sigma))}$ the quotient ideal norm, that is defined by

$$M_{(2,(1,\sigma))} := \sup\{\pi_{1,\sigma}(ST) : \pi_2(S) \leq 1\},$$

where the supremum is taken over all Banach spaces G an all 2-absolutely summing operators $S : F \to G$.

THEOREM 4. *For every operator $T \in \mathcal{L}(E,F)$, the following conditions are equivalent:*

(i) $T \in \mathcal{M}_{(2,(1,\sigma))}(E,F)$.

(ii) *There is a constant $C > 0$ such that for each probability measure μ on $B_{F'}$ there is a probability measure ν on $B_{E'}$ such that for all $x \in E$*

$$\Big(\int_{B_{F'}} |<Tx,y'>|^2 d\mu(y')\Big)^{\frac{1}{2}} \leq C\Big(\int_{B_{E'}} |<x,x'>| d\nu(x')\Big)^{1-\sigma} \|x\|^{\sigma}.$$

(iii) *There is a constant $C > 0$ such that for each pair of finite collections $x_1, \dots, x_m$ in E and $y_1, \dots, y_n$ in F',*

$$\Big(\sum_{j=1}^{m} \big(\sum_{k=1}^{n} |<Tx_j, y'_k>|^2\big)^{\frac{1}{2(1-\sigma)}}\Big)^{1-\sigma} \leq C\delta_{1,\sigma}((x_j))l_2((y'_k)).$$

(iv) *There is a constant $C > 0$ such that for each finite sequence $x_1, \dots, x_m$ in E*

$$m_{(2,\frac{1}{1-\sigma})}((Tx_j)) \leq C\delta_{1,\sigma}((x_j)).$$

In this case, $M_{(2,(1,\sigma))}(T) = \inf C$, where the infimum is taken over all C satisfying (ii), or (iii), or (iv).

Proof. (i) $\Rightarrow$ (ii). If $T \in \mathcal{M}_{(2,(1,\sigma))}(E,F)$ and μ is a probability measure on $B_{F'}$, then the canonical embedding $I : F \to L_2(\mu)$ is 2-absolutely summing and hence $IT \in \mathcal{P}_{1,\sigma}(E, L_2(\mu))$. By theorem 4.1 in [12], (see Definition 1 a)), there exists a probability measure ν on $B_{E'}$ such that for every $x \in E$,

$$\begin{aligned}\|IT(x)\| &= \Big(\int_{B_{F'}} |<Tx,y'>|^2 d\mu(y')\Big)^{\frac{1}{2}} \\ &\leq \pi_{1,\sigma}(IT)\Big(\int_{B_{E'}} |<x,x'>| d\nu(x')\Big)^{1-\sigma} \|x\|^{\sigma}.\end{aligned}$$

Finally, since $\pi_2(I) \leq 1$, we have $\pi_{1,\sigma}(IT) \leq M_{(2,(1,\sigma))}(T)$, and (ii) holds.

(ii) ⇒ (iii). Let $x_1, \dots, x_m \in E$ a finite family of vectors. Let $(y'_k)_{k=1}^n \subset F'$. Consider the probability measure on $B_{F'}$ given by

$$\mu = \big(\sum_{k=1}^n \|y'\|^2 \delta_k\big)\big(\sum_{k=1}^n \|y'_k\|^2\big)^{-1},$$

where δ_k is the Dirac measure at the point $\frac{1}{\|y'_k\|} y'_k$. Then

$$\begin{aligned}
&\Big(\sum_{j=1}^m \big(\sum_{k=1}^n | < Tx_j, y'_k > |^2\big)^{\frac{1}{2(1-\sigma)}}\Big)^{1-\sigma} \\
&= l_2((y'_k))\Big(\sum_{j=1}^m \big(\int_{B_{F'}} | < Tx_j, y' > |^2 d\mu(y')\big)^{\frac{1}{2(1-\sigma)}}\Big)^{1-\sigma} \\
&\leq C l_2((y'_k))\Big(\sum_{j=1}^m \big(\int_{B_{E'}} | < x_j, x' > | d\nu(x')\big)\|x_j\|^{\frac{\sigma}{1-\sigma}}\Big)^{1-\sigma} \\
&\leq C\delta_{1,\sigma}((x_j)) l_2((y'_k)).
\end{aligned}$$

(iii) ⇒ (i). Condition (iii) means that all discrete probability measures μ on $B_{F'}$ satisfy for every finite family of vectors $x_1, \dots, x_m \in E$

$$\Big(\sum_{j=1}^m \big(\int_{B_{F'}} | < Tx_j, y' > |^2 d\mu(y')\big)^{\frac{1}{2(1-\sigma)}}\Big)^{1-\sigma} \leq C\delta_{1,\sigma}((x_j)).$$

Since the set of all discrete probabilities is dense in $W(B_{F'})$ with respect to the weak $C(B_{F'})$-topology, we can state that this inequality is true for every $\mu \in W(B_{F'})$. The domination theorem for 2-absolutely summing operators gives $\pi_{1,\sigma}(ST) \leq C\pi_2(T)$.

(iv) ⇒ (i). Let $S \in \mathcal{P}_2(F, G)$, a finite family of vectors $x_1, \dots, mx_m \in E$ and $\epsilon > 0$. Then there are $\tau_1, \dots, \tau_m$ and $y_1, \dots, y_m$ such that $Tx_j = \tau_j y_j$ and

$$l_r((\tau_j)) w_2((y_j)) \leq (1+\epsilon) m_{(2, \frac{1}{1-\sigma})}((Tx_j)),$$

where $\frac{1}{2} + \frac{1}{r} = 1 - \sigma$. Then

$$\begin{aligned}
&l_{\frac{1}{1-\sigma}}((STx_j)) \leq l_r((\tau_j)) l_2((Sy_j)) \leq \pi_2(S) l_r((\tau_j)) w_2((y_j)) \\
&\leq \pi_2(S)(1+\epsilon) m_{(2, \frac{1}{1-\sigma})}((Tx_j)) \leq \pi_2(S) C (1+\epsilon) \delta_{1,\sigma}((x_j)).
\end{aligned}$$

According to Theorem 4.1 in [12] (see Definition 1 b)) the above inequalities mean that $ST \in \mathcal{P}_{1,\sigma}$ and thus $T \in \mathcal{M}_{(2,(1,\sigma))}(E,F)$.

(ii) $\Rightarrow$ (iv). If $T \in \mathcal{M}_{(2,(1,\sigma))}(E,F)$, for every finite family of vectors $x_1, \dots, x_m \in E$, Proposition 16.4.3 in [14] gives

$$\begin{aligned} &m_{(2,\frac{1}{1-\sigma})}((Tx_j)) \\ &\leq \sup\Big\{\Big(\sum_{j=1}^m \big(\int_{B_{F'}} |<Tx_j, y'>|^2 d\mu(y')\big)^{\frac{1}{2(1-\sigma)}}\Big)^{1-\sigma} : \mu \in W(B_{F'})\Big\} \\ &\leq C \sup\Big\{\Big(\sum_{j=1}^m \big(\int_{B_{E'}} |<x_j, x'>| d\nu(x')\big) \|x_j\|^{\frac{\sigma}{1-\sigma}}\Big)^{1-\sigma} : \nu \in W(B_{E'})\Big\} \\ &\leq C\delta_{1,\sigma}((x_j)). \end{aligned}$$

∎

Corollary 5. *Let $0 \leq \sigma < \frac{1}{2}$. Then $\mathcal{M}_{(2(1-\sigma),1)} \subset \mathcal{M}_{(2,(1,\sigma))}$, and*

$$M_{(2,(1,\sigma))}(T) \leq M^{1-\sigma}_{(2(1-\sigma),1)}(T)\|T\|^\sigma$$

for each operator $T \in \mathcal{M}_{(2(1-\sigma),1)}(E,F)$.

Proof. Let $T \in \mathcal{M}_{(2(1-\sigma),1)}(E,F)$. Then for each $\mu \in W(B_{F'})$ there exists $\nu \in W(B'_E)$ such that (see Chapter 20 in [14])

$$\begin{aligned} &\Big(\int_{B_{F'}} |<Tx, y'>|^{2(1-\sigma)} d\mu(y')\Big)^{\frac{1}{2(1-\sigma)}} \\ &\leq M_{(2(1-\sigma),1)}(T)\Big(\int_{B_{E'}} |<x, x'>| d\nu(x')\Big) \quad \forall x \in E. \end{aligned}$$

Then, for all $x \in E$,

$$\begin{aligned} &\Big(\int_{B_{F'}} |<Tx, y'>|^2 d\mu(y')\Big)^{\frac{1}{2}} \\ &\leq \Big(\int_{B_{F'}} |<Tx, y'>|^{2(1-\sigma)} d\mu(y')\Big)^{\frac{1}{2}} \|Tx\|^\sigma \\ &\leq M^{1-\sigma}_{(2(1-\sigma),1)}(T)\|T\|^\sigma \Big(\int_{B_{E'}} |<x, x'>| d\nu(x')\Big)^{1-\sigma} \|x\|^\sigma. \end{aligned}$$

∎

Proposition 6. *Let $1 \leq p \leq \infty$ and $0 \leq \sigma < 1$. Then*

1) $\mathcal{P}_{(p,\frac{1}{1-\sigma})} \subset \mathcal{P}_{(p,(1,\sigma))} \subset \mathcal{P}_{(p,1)}$.

2) $\mathcal{P}_{(p(1-\sigma),1)} \subset \mathcal{P}_{(p(1-\sigma),1),\sigma} \subset \mathcal{P}_{(p,(1,\sigma))}$, *where* $\mathcal{P}_{(p(1-\sigma),1),\sigma}$ *is the interpolated operator ideal related to* $\mathcal{P}_{(p(1-\sigma),1)}$ *following the interpolative procedure given in Definition* 1.

3) $\text{Space}\,(\mathcal{P}_{(p(1-\sigma),1)}) = \text{Space}\,(\mathcal{P}_{(p(1-\sigma),1),\sigma}) = \text{Space}\,(\mathcal{P}_{(p,(1,\sigma))})$.

Proof. 1) is a direct application of the inequalities $w_{\frac{1}{1-\sigma}} \leq \delta_{1,\sigma} \leq w_1$.

2) The first inclusion follows from definition 1. For the second one, if $T \in \mathcal{P}_{(p(1-\sigma),1),\sigma}$, then there is an operator S and a Banach space G such that $S \in \mathcal{P}_{(p(1-\sigma),1)}(E,G)$ and $\|Tx\| \leq \|Sx\|^{1-\sigma}\|x\|^{\sigma}$. Then if $x_1, \ldots, x_m \in E$ is a finite family of vectors,

$$l_p((Tx_i)) \leq \Big(\sum_{j=1}^{m} \|Sx_j\|x_j\|^{\frac{\sigma}{1-\sigma}}\|^{p(1-\sigma)} \Big)^{\frac{1}{p}}$$

$$\leq \pi^{1-\sigma}_{(p(1-\sigma),1)}(S) w_1^{1-\sigma}\Big((x_j\|x_j\|^{\frac{\sigma}{1-\sigma}}) \Big) = \pi^{1-\sigma}_{(p(1-\sigma),1)}(S)\delta_{1,\sigma}((x_j)).$$

3) Let $E \in \text{Space}\,(\mathcal{P}_{(p,(1,\sigma))})$. Consider $x_1, \ldots, x_m \in E$. Then

$$l_p((x_j)) \leq \pi_{(p,(1,\sigma))}(Id_E)\delta_{1,\sigma}((x_j)).$$

Since $\|x_j\|^{p(1-\sigma)} = \|x_j\|x_j\|^{-\sigma}\|^p$, we get

$$l_{p(1-\sigma)}((x_j)) = \Big(\sum_{j=1}^{m} \|x_j\|x_j\|^{-\sigma}\|^p \Big)^{\frac{1}{p(1-\sigma)}}$$

$$\leq \pi^{\frac{1}{1-\sigma}}_{(p,(1,\sigma))}(Id_E) \sup_{x' \in B_{E'}} \Big(\sum_{j=1}^{m} | < x_j\|x_j\|^{-\sigma}, x' > | \|x_j\|x_j\|^{-\sigma}\|^{\frac{\sigma}{1-\sigma}} \Big)$$

$$\leq \pi^{\frac{1}{1-\sigma}}_{(p,(1,\sigma))}(Id_E) w_1((x_j)).$$

Therefore, $E \in \text{Space}\,(\mathcal{P}_{(p(1-\sigma),1)}$. This, together with 2), implies the result. ∎

THEOREM 7. *Let* $2 \leq p < \infty$ *and* $\sigma = \frac{p-2}{2(p-1)}$. *Then*

$$\text{Space}\,(\mathcal{M}_{(2,(1,\sigma))}) \subset \text{Space}\,(\mathcal{P}_{(p,1)}).$$

Proof. Let $x_1, \ldots, x_m \in E$. Theorem 4 gives

$$m_{(2,\frac{1}{1-\sigma})}((x_j)) \leq M_{(2,(1,\sigma))}(Id_E)\delta_{1,\sigma}((x_j)).$$

Put $r = \frac{2}{1-2\sigma}$. Then $\frac{1}{2} + \frac{1}{r} = 1 - \sigma$ and

$$l_r((x_j)) \leq m_{(2,\frac{1}{1-\sigma})}((x_j)) \leq M_{(2,(1,\sigma))}(Id_E)\delta_{1,\sigma}((x_j)),$$

and then, according to 6.3),

$$E \in \text{Space}\,(\mathcal{P}_{(r,(1,\sigma))}) = \text{Space}\,(\mathcal{P}_{(r(1-\sigma),1)}) = \text{Space}\,(\mathcal{P}_{(p,1)}).$$

∎

Remark 8. Let $p \geq 2$. By application of Talagrand´s result [19], Corollary 5 and Proposition 7 we get both a necessary and a sufficient condition for a Banach space to have cotype p. Suppose that the relation between p and σ is $p = \frac{2(1-\sigma)}{1-2\sigma}$. Then

1) If E has cotype p, then for each $0 < \sigma' < \frac{1}{2}$ satisfying $\sigma < \sigma'$ there exists $C > 0$ such that for each $\mu \in W(B_{E'})$ there is $\nu \in W(B_{E'})$ satisfying

$$\Big(\int_{B_{E'}} |<x,x'>|^2 d\mu(x')\Big)^{\frac{1}{2}} \leq C\Big(\int_{B_{E'}} |<x,x'>| d\nu(x')\Big)^{1-\sigma'} \|x\|^{\sigma'}, \forall x \in E.$$

2) Let $0 \leq \sigma < \frac{1}{2}$. If there is $C > 0$ such that for each $\mu \in W(B_{E'})$ there exists $\nu \in W(B_{E'})$ satisfying

$$\Big(\int_{B_{E'}} |<x,x'>|^2 d\mu(x')\Big)^{\frac{1}{2}} \leq C\Big(\int_{B_{E'}} |<x,x'>| d\nu(x')\Big)^{1-\sigma} \|x\|^{\sigma},\ \forall x \in E,$$

then E has cotype p.

3. Applications. Absolutely summing operators between spaces of cotype q ans spaces of cotype p.

Proposition 9. *Let $0 \leq \sigma < \frac{1}{2}$ and let E, F and G be Banach spaces. Then $\mathcal{P}_{(\frac{2}{1-2\sigma},2)}\mathcal{M}_{(2,(1,\sigma))} \subset \mathcal{P}_{(\frac{1}{1-2\sigma},(1,\sigma))}$ and*

$$\pi_{(\frac{1}{1-2\sigma},(1,\sigma))}(ST) \leq \pi_{(\frac{2}{1-2\sigma},2)}(S)M_{(2,(1,\sigma))}(T),$$

for each $T \in \mathcal{M}_{(2,(1,\sigma))}(E,F)$ and each $S \in \mathcal{P}_{(\frac{2}{1-2\sigma},2)}(F,G)$.

Proof. Consider a finite family of vectors $x_1, \ldots, x_m \in E$ and $\epsilon > 0$. There are $\tau_1, \ldots, \tau_m$ and $y_1, \ldots, y_m \in F$ such that $Tx_j = \tau_j y_j$ and

$$l_r((\tau_j))w_2((y_j)) \leq (1+\epsilon)m_{(2,\frac{1}{1-\sigma})}((Tx_j)),$$

where $\frac{1}{2} + \frac{1}{r} = 1 - \sigma$. Then

$$\begin{aligned} l_{\frac{1}{1-2\sigma}}((STx_j)) &\leq l_{\frac{2}{1-2\sigma}}((\tau_j))l_{\frac{2}{1-2\sigma}}((Sy_j)) \\ &\leq \pi_{(\frac{2}{1-2\sigma},2)}(S)l_{\frac{2}{1-2\sigma}}((\tau_j))w_2((y_j)) \\ &\leq \pi_{(\frac{2}{1-2\sigma},2)}(S)(1+\epsilon)m_{(2,\frac{1}{1-\sigma})}((Tx_j)) \\ &\leq \pi_{(\frac{2}{1-2\sigma},2)}(S)M_{(2,(1,\sigma))}(T)(1+\epsilon)\delta_{1,\sigma}((x_j)). \end{aligned}$$

∎

The following corollary is closely related to some classical results about operators between L_p spaces of integrable functions. The reader can find more information about this problem in [9] and [1].

COROLLARY 10. *Let $q > 2, \sigma = \frac{1}{2} - \frac{1}{q}$. Then $\mathcal{L}(L_1, L_q) = \mathcal{P}_{(\frac{q}{2},(1,\sigma))}(L_1, L_q)$.*

Proof. Corollary 11.12 in [20] gives $\mathcal{L}(L_1, L_q) = \mathcal{P}_{(q,2)}(L_1, L_q)$. Since $q = \frac{2}{1-2\sigma}$, L_1 has cotype 2, and cotype 2 spaces are in Space $(\mathcal{M}_{(2,1)}) \subset$ Space $(\mathcal{M}_{(2,(1,\sigma))})$ (see Chapter 32.2 in [6]), the result follows just by applying proposition 9. ∎

PROPOSITION 11. *Let $q \geq 2$ and let E, F and G be Banach spaces. If $1 \leq s < \infty$, then $\mathcal{C}_q\mathcal{P}_s \subset \mathcal{P}_{(q,2)}$. Moreover, if $S : F \to G$ is an operator of cotype q, then $\pi_{(q,2)}(ST) \leq C_q(S)b_s\pi_s(T)$ for each $T \in \mathcal{P}_s(E, F)$, where b_s is the second constant from the Khintchine inequalities.*

The proof of this proposition can be obtained following the lines of the proof of the proposition 26.4.(3) in [6]. It is a direct consequence of the definition of cotype q and the Grothendieck-Pietsch domination theorem for s-summing operators.

THEOREM 12. *Let E, F and G be Banach spaces.*

1) *Let $S : F \to G$ be an operator of cotype 2 and $p \geq 2$. Let E be of cotype $p \geq 2$. Let $0 < \sigma < \frac{1}{2}$ such that $p < \frac{2(1-\sigma)}{1-2\sigma}$. Then for each $1 \leq s < \infty$ and every $T \in \mathcal{P}_s(E, F)$, ST is $(1,\sigma)$-absolutely continuous. Moreover,*

$$\pi_{1,\sigma}(ST) \leq M_{(2,(1,\sigma))}(E)C_2(S)b_s\pi_s(T).$$

2) *If $S : F \to G$ is an operator of cotype $q \geq 2$, the Banach space E has cotype $p \geq 2$, $T \in \mathcal{P}_s(E,F)$ and for $\sigma = \frac{1}{2} - \frac{1}{q}$ the inequality $p < q(1-\sigma)$ holds, then ST is a $(\frac{q}{2},(1,\sigma))$- absolutely summing operator. Moreover, for each $1 \leq s < \infty$*

$$\pi_{(\frac{q}{2},(1,\sigma))}(ST) \leq M_{(2,(1,\sigma))}(E) C_q(S) b_s \pi_s(T).$$

3) *If F has cotype $q \geq 2$, E has cotype $p \geq 2$, and for $\sigma = \frac{1}{2} - \frac{1}{q}$ the inequality $p < q(1-\sigma)$ holds, then each operator $T \in \mathcal{L}(E,F)$ factoring through a $C(K)$-space is $(\frac{q}{2},(1,\sigma))$-absolutely summing.*

Proof. 1) On one hand, since E has cotype p with $p < \frac{2(1-\sigma)}{1-2\sigma}$, then $E \in \text{Space}\,(\mathcal{M}_{(2,(1,\sigma))})$ on account of Remark 8.1. On the other hand, the proposition 26.4.3 in [6] gives

$$\mathbb{U}_{1 \leq s} \mathcal{P}_s(E,F) \subset \mathcal{P}_2(E,F)$$

and $\pi_2(ST) \leq C_2(S) b_s \pi_s(T)$. Then

$$\mathbb{U}_{1 \leq s} \mathcal{P}_s(E,F) \subset \mathcal{P}_{1,\sigma}(E,F)$$

since $E \in \text{Space}\,(\mathcal{M}_{(2,(1,\sigma))})$, and $\pi_{1,\sigma}(ST) \leq M_{(2,(1,\sigma))}(E) C_2(S) b_s \pi_s(T)$.

2) Put $\sigma := \frac{1}{2} - \frac{1}{q}$. If E has cotype p, if $p < q(1-\sigma) = \frac{2(1-\sigma)}{1-2\sigma}$, then

$$E \in \text{Space}\,(\mathcal{M}_{(2,(1,\sigma))})$$

according to Remark 8. For each $1 \leq s < \infty$, $(\mathcal{C}_q \mathcal{P}_s)(E,F) \subset \mathcal{P}_{(q,2)}(E,F)$ and $\pi_{(q,2)}(ST) \leq C_q(S) b_s \pi_s(T)$ according to Proposition 11. Since $q = \frac{q}{1-2\sigma}$, an application of Proposition 9 gives

$$\pi_{(\frac{1}{1-2\sigma},(1,\sigma))}(ST) \leq \pi_{(\frac{2}{1-2\sigma},2)}(ST) M_{(2,(1,\sigma))}(E) \leq C_q(S) b_s \pi_s(T) M_{(2,(1,\sigma))}(E).$$

3) It is enough to consider that every $T \in \mathcal{L}(E,F)$ factoring through a $C(K)$-space is $(q,2)$-summing if F has cotype q (see theorem 21.4 in [20]). Then the same argument given for the proof of 2) gives the result. ∎

Nowadays we know the cotype of a broad class of Banach spaces. For the case of the real interpolation spaces the reader can find the results in the paper of Xu [22]. In the general context of the Calderón-Lozanovskii spaces, the paper of Cerdá and Mastylo gives nice characterizations of the spaces of finite cotype [2]. Creekmore got the results for the case of the Lorentz function spaces in 1981 [3] although a direct argument related to the properties of concavity and convexity of Banach lattices due to Defant may be used to show them [5]. This means that the former theorem can be applied in a straightforward fashion for couples of these spaces in order to give a $(1,\sigma)$-domination theorem that is satisfied for all the s-summing operators between them (for every $s \geq 1$). Another interesting application can be found for spaces of real functions that are integrable with respect to a vector measure since *Curbera* got several results about the type and cotype of these spaces (see [4]). We give the following example.

We know that all 2-summing operators from a space $L_1(\mu)$ in a Banach space F are 1-summing (see Corollary 2 of 23.10 in [6]). Moreover if F has cotype 2 we get $\mathcal{P}_p(L_1(\mu),F) \subset \mathcal{P}_1(L_1(\mu),F)$ for each $p > 1$ (see 26.4(3) in [6]). We can give a general version of this fact if μ is a vector measure. Let ν be a countably additive vector measure with values in the Banach space E. We consider the space $L_1(\nu)$ of real functions that are integrable with respect to the vector measure ν in the sense of Lewis (see [4]).

Corollary 13. *Let E be a Banach space of cotype $p \geq 2$ and ν a countable additive E-valued vector measure. Let F be a Banach space of cotype 2, $0 < \sigma < \frac{1}{2}$ and $p < \frac{2(1-\sigma)}{1-2\sigma}$. Then for each $1 \leq s < \infty$,*

$$\mathcal{P}_s(L_1(\nu),F) \subset \mathcal{P}_{1,\sigma}(L_1(\nu),F).$$

The proof is a direct consequence of Theorem 1 in [4] and theorem 12.1.

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