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# Quadratic Systems Equivalent by Domains to a Linear One: Global Phase Portraits

M. Ndiaye and H. Giacomini

*Laboratoire de Mathématique et Physique Théorique*
*Faculté des Sciences et Techniques, Université de Tours* 37200-*Tours, France*

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## Introduction

Quadratic polynomial vector fields have been intensively studied over the last thirty years. The problem of determining all the possible phase portraits of such systems is important in many branches of science. Quadratic systems have been employed in mathematical ecology to model the populations of a predator-prey system, in chemistry to model the concentrations of two chemical reactions, in astrophysics, fluid mechanics, etc. [23].

Moreover, quadratic systems present also an interest in pure mathematics. The problem of determining the maximum number of limit cycles and their distribution in phase space has aroused a lot of interest in the last years. In spite of the considerable effort that has been devoted to the problem, it remains unsolved.

Considerable progress has been made in the study of quadratic systems that do not present limit cycles. All possible topologically different phase portraits have been determined for several families of systems.

Homogeneous quadratic systems have been studied by Lyagina [16], Markus [17], Vulpe and Sibirskii [26], Korol [14], Newton [19], Date [11] and Vdovina [25].

Systems possessing a star nodal point have been studied by Berlinskii [7]. Quadratic systems that have two invariant straight lines have been considered by Reyn [22].

Chordal systems have been analyzed by Gassul, Li-Ren and Llibre [13] and Gassul and Llibre [12]. Hamiltonian and gradient systems have been recently studied by Artés and Llibre [5], [6].

All possible phase portraits of quadratic systems with a center have been obtained by Lunkevich and Sibirskii [15] and by Vulpe [27].

The topological classification of structurally stable quadratic systems without limit cycles was discussed for the first time by Tavares Dos Santos [24]. later on, Cai Sui-Lin pointed out in [8] that the classification of [24] was incomplet and he added eight more examples. The complete topological classification of this type of system has been recently obtained in [4].

Cairó and Llibre [9] determined all the possible phase portraits of a semi-homogeneous family of quadratic systems.

None of these systems possess limit cycles.

In this paper we study a new family of quadratic systems that do not present limit cycles.

We prove that this family has a general first integral and that it is equivalent by domain to a linear system. From these two results we have been able to find all the topologically different phase portraits and we have proved that there are exactly 47. (Unless we have critical points which are not isolated).

The paper is organized as follows: In section 1 we give some results concerning the general system (1). In section 2 we consider the quadratic case and analyze the nature of finite critical points. Finally in section 3, we analyze the nature of critical points at infinity.

## 1. General results.

We consider the following system:

$$\begin{cases} \dot{x} = P(x,y), \\ \\ \dot{y} = Q(x,y), \end{cases} \tag{1}$$

where

$$P(x,y) = c_1 f_1(x,y)\frac{\partial f_2(x,y)}{\partial y} - c_2 f_2(x,y)\frac{\partial f_1(x,y)}{\partial y},$$
$$Q(x,y) = c_2 f_2(x,y)\frac{\partial f_1(x,y)}{\partial x} - c_1 f_1(x,y)\frac{\partial f_2(x,y)}{\partial x},$$

and $c_1 \neq 0$, $c_2 \neq 0$ and $f_1(x,y),\ f_2(x,y) \in C^1$.

The system (1) admits $f_1(x,y) = 0$ and $f_2(x,y) = 0$ as invariant curves as we can see from the relations:

$$\frac{\partial f_i(x,y)}{\partial x}P(x,y) + \frac{\partial f_i(x,y)}{\partial y}Q(x,y) = c_i f_i(x,y) J(x,y) \quad i = 1,2$$

where

$$J(x,y) = \frac{\partial f_1(x,y)}{\partial x}\frac{\partial f_2(x,y)}{\partial y} - \frac{\partial f_1(x,y)}{\partial y}\frac{\partial f_2(x,y)}{\partial x}.$$

LEMMA 1.1. *The system* (1) *has the general integral:*

$$I(x,y) = |f_2(x,y)|^{c_1}|f_1(x,y)|^{-c_2}.$$

*Proof.* By a direct calculation, it is easy to verify that $\dot{I}(x,y) = 0$ ∎

In consequence, the system (1) has no limit cycles.

LEMMA 1.2. *If* $J(x,y) \neq 0$ *and the application* $H$ *defined by* $H(x,y) = (f_1(x,y), f_2(x,y))$ *is injective, then the system* (1) *is equivalent to a linear one in the two domains:* $D_1 = \{(x,y) \in \mathbb{R}^2 / J(x,y) > 0\}$ *and* $D_2 = \{(x,y) \in \mathbb{R}^2 / J(x,y) < 0\}$. *Furthermore, when critical points exist in the domain* $J(x,y) \neq 0$, *then they are either saddles when* $c_1c_2 < 0$ *or nodes when* $c_1c_2 > 0$.

*Proof.* we make the change of variables $X = f_1(x,y)$, $Y = f_2(x,y)$, and obtain the following system:

$$\dot{X} = c_1 J(x,y) X, \quad \dot{Y} = c_2 J(x,y) Y. \tag{2}$$

The Jacobian matrix of $H(x,y)$ is $J(x,y)$ which is nonzero, furthermore $H$ is injective; therefore, $H$ is a global homeomorphism in $D_1$ and $D_2$ which transforms the system (1) to the system (2). Furthermore, since the critical point $(0,0)$ of the system (2) is either a saddle when $c_1c_2 < 0$ or a node when $c_1c_2 > 0$, the corresponding critical points of the system (1) when $J(x,y) \neq 0$, given by $f_1(x,y) = 0$ and $f_2(x,y) = 0$, are saddles if $c_1c_2 < 0$ and nodes if $c_1c_2 > 0$, whenever they exist. Besides, in $D_1$ and $D_2$, system (2) is topologically equivalent to the linear system $\dot{X} = c_1X$, $\dot{Y} = c_2Y$. ∎

*Remark.* Concerning the stability of the critical points of (1) in the domain $J(x,y) \neq 0$, it is easy to show that when critical points exist in $D_1$ and in $D_2$, and when the critical points in $D_1$ are stable nodes, then those in $D_2$ are unstable nodes, and vice-versa. To show this, it is sufficient to consider system (2); the eigenvalues associated to $(0,0)$ are $c_1J(x_0,y_0)$ and $c_2J(x_0,y_0)$, where $(x_0,y_0)$ are the coordinates of the corresponding critical points of (1) in the domain $J(x,y) \neq 0$. These two quantities change signs when we pass from $D_1$ to $D_2$.

## 2. Behaviour of the system in the neighborhood of finite critical points

In this section, we consider the case where (1) is a quadratic system, with

$$f_1(x,y) = a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2,$$

$$f_2(x,y) = b_0 + b_1x + b_2y.$$

By a change of coordinates, we can take without loss of generality $b_0 = b_1 = 0$ and $b_2 = 1$.

We obtain a quadratic system of the form:

$$\begin{aligned} P(x,y) &= a_0c_1 + a_1c_1x + a_3c_1x^2 + a_2(c_1-c_2)y + a_4(c_1-c_2)xy \\ &\qquad + a_5(c_1-2c_2)y^2, \\ Q(x,y) &= c_2y(a_1 + 2a_3x + a_4y), \end{aligned} \tag{3}$$

and our goal is to prove the following theorem:

Theorem. *The phase portraits of the system* (3) *are homeomorphic to one of the figures* 1–47 *and* 22 *(except for the orientation).*

Let us remark that $H = (f_1, f_2)$ is injective. This can be shown directly by using the Rolle theorem. Therefore $H$ is a global homeomorphism in $D_1$ and $D_2$.

Let us also remark that in this section, $D_1$ and $D_2$ design two half-planes separated by the straight line $J(x,y) = 0$.

Before studying the nature of critical points, we start by localizing their positions. When $J(x,y) \neq 0$, the critical points are given by the solutions of $f_1(x,y) = 0$ and $f_2(x,y) = 0$, which lead us to a second degree equation $h(x) = 0$ whose discriminant is $\Delta = a_1^2 - 4a_0a_3$, with $a_3 \neq 0$.

When $\Delta > 0$ it is easy to verify that the two critical points $M_1(x_\Delta^1, y_\Delta^1)$ and $M_2(x_\Delta^2, y_\Delta^2)$ are symmetric with respect to a point belonging to $J(x,y) = 0$.

The other critical points, whenever they exist, are situated on the line $J(x,y) = a_1 + 2a_3x + a_4y = 0$. After expressing $x$ in terms of $y$ and replacing in $P(x,y) = 0$, we obtain a second degree equation $h_1(y) = 0$ whose discriminant is

$$\Delta_1 = \frac{(-2a_2a_3 + a_1a_4)^2(c_1-c_2)^2}{4a_3^2} - \frac{(c_1-2c_2)c_1\Delta\,\Delta_2}{4a_3^2},$$

where $\Delta_2 = a_4^2 - 4a_3a_5$ and $a_3\Delta_2(c_1 - 2c_2) \neq 0$.

Therefore, we obtain the following conclusion:

On the line $J(x,y)=0$, when $\Delta_1>0$, we have two critical points; when $\Delta_1=0$, we have only one critical point; and when $\Delta_1<0$, we have no critical points.

One can remark that when $\Delta=0$, the equation $h(x)=0$ has a double solution but it is easy to verify that this double solution belongs to $J(x,y)=0$.

Now, since the nature of critical points in the domain $J(x,y)\neq 0$ has been already studied, we can focus our study only on the nature of critical points on $J(x,y)=0$.

### 2.1. Nature of critical points if $\Delta>0$.

PROPOSITION 2.1. *We suppose that* $\Delta_1>0$. *If* $c_1<0$, $c_2<0$ *and* $(c_1-2c_2)\Delta_2>0$ *or* $c_1>0$, $c_2>0$ *and* $(c_1-2c_2)\Delta_2<0$, *then there are four critical points for the system (3). Two of them are saddles and are situated on the straight line* $J(x,y)=0$ *and the two others are nodes and they are situated in the domain* $J(x,y)\neq 0$. *The corresponding global phase portraits are homeomorphic to figures* 1 *and* 2.

*Proof.* If $\Delta_1>0$ and $a_3(c_1-2c_2)\Delta_2\neq 0$ then we have two critical points $M_3(x^1_{\Delta_1},y^1_{\Delta_1})$ and $M_4(x^2_{\Delta_1},y^2_{\Delta_1})$ on the line $J(x,y)=0$.

Let us denote by $\lambda^1_{M_i}$ and $\lambda^2_{M_i}$ $(i=3,4)$ their eigenvalues calculated in terms of $y$; we obtain:

$\lambda^1_{M_i}=\sqrt{c_2 f(y)}$ and $\lambda^2_{M_i}=-\sqrt{c_2 f(y)}$ $(i=3,4)$, where

$$f(y)=y((c_1-c_2)(2a_2a_3-a_1a_4)-(c_1-2c_2)\Delta_2 y).$$

Hence $M_3$ is either a saddle or a linear center and similarly for $M_4$. Let us remark that $f(y)=2a_3yP'(y)$ where

$$P(y)=P\left(x=-\frac{(a_1+a_4y)}{2a_3},y\right)\quad\text{and}\quad P'(y)=\frac{dP(y)}{dy}.$$

The function $f(y)$ has two zeros:

$$y_1=0\quad\text{and}\quad y_2=\frac{(-2a_2a_3+a_1a_4)(-c_1+c_2)}{(c_1-2c_2)\Delta_2}.$$

We also have

$$P(y_1)=-\frac{\Delta c_1}{4a_3},\qquad P(y_2)=\frac{a_3\Delta_1}{(c_1-2c_2)\Delta_2}.$$

We first suppose that $a_3 > 0$ , $c_1 > 0$, $c_2 > 0$ and $(c_1 - 2c_2)\Delta_2 < 0$; then $P(y_1) < 0$ and $P(y_2) < 0$ and we have the following distribution of signs of the quantities $P'(y), P(y), y$ and $f(y)$ as shown in tables 1 and 2:

| $y$ | $y_{\Delta_1}^1$ | $y_1$ | $y_2$ | $y_{\Delta_1}^2$ | |
|---|---|---|---|---|---|
| $P'(y)$ | - | - | - | + | + |
| $P(y)$ | + | - | - | - | + |
| $y$ | - | - | + | + | + |
| $f(y)$ | + | + | - | + | + |

Table 1

| $y$ | $y_{\Delta_1}^1$ | $y_2$ | $y_1$ | $y_{\Delta_1}^2$ | |
|---|---|---|---|---|---|
| $P'(y)$ | - | - | + | + | + |
| $P(y)$ | + | - | - | - | + |
| $y$ | - | - | - | + | + |
| $f(y)$ | + | + | - | + | + |

Table 2

In this case, $y_1$ is either in the interval $]y_{\Delta_1}^1, y_2[$, as represented in table 1 or in the interval $]y_2, y_{\Delta_1}^2[$, as represented in table 2.

We see that at $y_{\Delta_1}^1$ and $y_{\Delta_1}^2$, $f(y) > 0$. Since $c_2 > 0$, we see that $M_3$ and $M_4$ are saddle points.

The same procedure can be applied when $a_3 < 0$, and we will find the same conclusion; this is also the case when $c_1 < 0, c_2 < 0$ and $(c_1 - 2c_2)\Delta_2 > 0$.

Since $\Delta > 0$, we have two critical points in $J(x, y) \neq 0$ which are saddles when $c_1c_2 < 0$ or nodes when $c_1c_2 > 0$ according to Lemma 1.2, and then the proposition 2.1 follows. ∎

Proposition 2.2. *We suppose that* $\Delta_1 > 0$. *If* $c_1 > 0$ *and* $(c_1 - 2c_2)\Delta_2 > 0$, *or* $c_1 < 0$ *and* $(c_1 - 2c_2)\Delta_2 < 0$, *then* (1) *has one saddle and one center on* $J(x, y) = 0$ *and either two nodes when* $c_1c_2 > 0$ *or two saddles when* $c_1c_2 < 0$ *in the domain* $J(x, y) \neq 0$. *For these cases we have the phase portraits corresponding to the figures* 2 *and* 3.

*Proof.* We apply the same procedure as for the previous proposition. ∎

Proposition 2.3. *We suppose that* $\Delta_1 > 0$. *If* $c_1c_2 < 0$ *and* $\Delta_2 < 0$ *then the system* (3) *has two centers on* $J(x, y) = 0$ *and two saddles in the domain* $J(x, y) \neq 0$. *The corresponding phase portrait is homeomorphic to figure* 5.

*Proof.* We follow the method employed in the proof of proposition 2.1 ∎

PROPOSITION 2.4. *We suppose now that* $\Delta_1 < 0$, *then the system* (3) *has only two nodes in* $J(x,y) \neq 0$ *if* $c_1c_2 > 0$ *and two saddles if* $c_1c_2 < 0$. *The phase portraits are topologically equivalent to the figures* 6, 7 *and* 47.

*Proof.* We apply the same procedure as in the proposition 2.1 ∎

PROPOSITION 2.5. *We suppose that* $\Delta_1 = 0$, *then the system has three critical points; one is situated on* $J(x,y) = 0$; *the neighborhood of this point is a union of hyperbolic sectors or a cusp; the two other critical points in the domain* $J(x,y) \neq 0$ *are saddles in the case* $c_1c_2 < 0$ *and nodes if* $c_1c_2 > 0$. *These cases correspond to the figures* 8, 9 *and* 10.

*Proof.* If we suppose that $\Delta_1 = 0$ then we have

$$\Delta = \frac{(2a_2a_3 - a_1a_4)^2(-c_1 + c_2)^2}{c_1(c_1 - 2c_2)\Delta_2}.$$

Since $\Delta > 0$ then $c_1(c_1 - 2c_2)\Delta_2 > 0$.

The critical point on $J(x,y) = 0$ is given by $M_5(\alpha_1, \alpha_2)$, where

$$\alpha_1 = \frac{2a_2a_3a_4c_1 - 4a_1a_3a_5c_1 - 2a_2a_3a_4c_2 - a_1a_4^2c_2 + 8a_1a_3a_5c_1}{2a_3(c_1 - 2c_2)\Delta_2},$$

$$\alpha_2 = -\frac{(2a_2a_3 - a_1a_4)(c_1 - c_2)}{(c_1 - 2c_2)\Delta_2}.$$

It is easy to verify that $M_5$ is a critical point of type *N.E.* (the determinant and the trace of the jacobian matrix are both zero). We translate the origin at $M_5$ and perform the transformation $x = -\frac{a_4}{2a_3}u + v, \quad y = u, \quad t' = \alpha_3 t$, where

$$\alpha_3 = \frac{2a_3(-2a_2a_3 + a_1a_4)c_2(-c_1 + c_2)}{(c_1 - 2c_2)\Delta_2}.$$

We obtain finally the following system:

$$P_1(u,v) = v + \frac{2a_3c_2}{\alpha_3}uv,$$

$$Q_1(u,v) = -\frac{(c_1 - 2c_2)\Delta_2}{4a_3\alpha_3}u^2 + \frac{a_3c_1}{\alpha_3}v^2.$$

Therefore, we have a union of hyperbolic sectors or a cusp at the origin according to theorem *N.E.* (see appendix and [1]). ∎

2.2. Nature of critical points if $\Delta < 0$. The proof of the following propositions are similar to the last one. We can summarize the situation as follows:

Proposition 2.6. *We suppose that* $\Delta_1 > 0$. *If* $c_1c_2 < 0$ *and* $\Delta_2 > 0$, *then system* (1) *has only two saddles and they are situated on* $J(x, y) = 0$. *This case corresponds to the phase portrait* 11.

Proposition 2.7. $\Delta_1 > 0$. *If* $c_1 > 0$ *and* $(c_1 - 2c_2)\Delta_2 < 0$, *or* $c_1 < 0$ *and* $(c_1 - 2c_2)\Delta_2 > 0$, *then, the system has only one saddle and one center and they are situated on the line* $J(x, y) = 0$. *The phase portraits are topologically equivalent to the figures* 12 *and* 13.

Proposition 2.8. $\Delta_1 > 0$. *If* $c_1 > 0$, $c_2 > 0$ *and* $(c_1 - 2c_2)\Delta_2 > 0$, *or* $c_1 < 0$, $c_2 < 0$ *and* $(c_1 - 2c_2)\Delta_2 < 0$, *then the system has only two critical points which are centers and they are situated on* $J(x, y) = 0$. *We have the figures* 14 *and* 15.

Proposition 2.9. $\Delta_1 < 0$. *We have no finite critical point in this case and we obtain regular curves in the vicinity of any point on the plane, (see for instance Nemytskii and V.V. Stepanov* [18]*). The phase portraits correspond to* 16 *and* 17.

Proposition 2.10. $\Delta_1 = 0$. *The system has only one critical point and it is situated on* $J(x, y) = 0$. *The neighborhood of this critical point is a union of hyperbolic sectors or a cusp. We obtain figures* 18 *and* 19.

2.3. Nature of critical points if $\Delta = 0$.

Proposition 2.11. *We suppose that* $\Delta = 0$ *and* $a_3(c_1 - 2c_2)\Delta_2 \neq 0$, *then* $\Delta_1 \geq 0$ *and we have two cases.*

*In the case where* $\Delta_1 > 0$, *we have only two critical points and they are situated on* $J(x, y) = 0$. *The first is a non-elementary critical point. It is a saddle when* $c_1c_2 < 0$ *and when* $c_1c_2 > 0$, *its neighborhood is a union of a hyperbolic sector and an elliptic sector . The second point is an elementary critical point. It is a saddle when* $c_2(c_1 - 2c_2)\Delta_2 < 0$ *and a center when* $c_2(c_1 - 2c_2)\Delta_2 > 0$. *We obtain the phase portraits* 20, 21, 22 *and* 23.

*In the case* $\Delta_1 = 0$, *we have a critical point of type* $Z$ *(the matrix of the linear part is zero). So, we distinguish two cases: the case where* $-2a_2a_3 + a_1a_4 = 0$ *and the case where* $c_1 - c_2 = 0$.

a) *Case where* $-2a_2a_3 + a_1a_4 = 0$. *For this case we obtain the following results:*

*When* $\Delta_2 < 0$, $c_1(c_1 - 2c_2) < 0$, *the phase portrait of the system is homeomorphic to the figure* 24.

*When* $\Delta_2 < 0$, $c_1(c_1 - 2c_2) > 0$, *the phase portrait of the system is topologically equivalent to the figure* 25.

*When* $\Delta_2 > 0, c_1c_2 < 0$, *the phase portrait of the system is topologically equivalent to the figure* 26.

*When* $\Delta_2 > 0$, $c_1(c_1 - 2c_2) < 0$, *the phase portrait of the system is homeomorphic to the figure* 27.

*When* $\Delta_2 > 0$, $c_2(c_1 - 2c_2) > 0$, *the phase portrait of the system is homeomorphic to the figure* 28.

b) *Case where* $c_1 = c_2$. *When* $\Delta_2 < 0$, *the phase portrait of the system is homeomorphic to the figure* 24, *and when* $\Delta_2 > 0$, *it is topologically equivalent to the figure* 27.

*Proof.* When $\Delta_1 > 0$, the proof is direct using theorem *N.E.*

When $\Delta_1 = 0$, we obtain an homogeneous quadratic system and we use theorem A1 and A2 in the appendix (see for instance [28]). ∎

2.4. Particular cases. We summarize the different cases in the following proposition:

Proposition 2.12. *We suppose now that* $a_3(c_1 - 2c_2)\Delta_2 = 0$, *then we must consider the following cases:*

a) $a_3 = 0$, $a_4(c_1 - 2c_2) \neq 0$: *We have two critical points; one is situated in the domain* $J(x, y) \neq 0$ *and it is a saddle or a node and the other is on* $J(x, y) = 0$ *and it is a saddle if* $a_1 \neq 0$. *The phase portraits for these conditions correspond to the figures* 29, 30 *and* 31. *When* $a_1 = 0$ *and* $a_0 \neq 0$, *we have no finite critical points. This case corresponds to the figures* 32 *and* 33. *When* $a_1 = 0$ *and* $a_0 = 0$ *then the line* $y = 0$ *is composed of critical points.*

b) $\Delta_2 = 0$, $a_3(c_1 - 2c_2) \neq 0$: *If we suppose that* $(2a_2a_3 - a_1a_4)(c_1 - c_2) \neq 0$, *then we have two critical points* ($\Delta > 0$) *in* $J(x, y) \neq 0$ *which are both either nodes or saddles and a third one on* $J(x, y) = 0$ *which is a saddle point when* $c_1c_2\Delta > 0$ *or a center when* $c_1c_2\Delta < 0$. *When* $\Delta = 0$ *we have only one critical point on* $J(x, y) = 0$ *which is a saddle when* $c_1c_2 < 0$ *and we have a union of a hyperbolic sector and an elliptic sector when* $c_1c_2 > 0$. *The corresponding* *phase portraits are represented in figures* 34, 35, 36, 37, 38 *and* 39. *We suppose now that* $2a_2a_3 - a_1a_4 = 0$. *If* $\Delta < 0$ *we have no critical points; if* $\Delta > 0$ *we have only two critical points in* $J(x,y) \neq 0$ *which are nodes or saddles according to lemma* 1.2. *When* $\Delta = 0$ *the line* $J(x,y) = 0$ *is composed of critical points. The corresponding phase portraits are represented in figures* 40, 41, 42, 43 *and* 44. *The case* $c_1 = c_2$ *is a particular case of the above results.*

c) $a_3 = 0$, $a_4 = 0$: *When* $a_1 \neq 0$, *we have only one critical point which is either a node when* $c_1c_2 > 0$ *or a saddle when* $c_1c_2 < 0$ *in the domain* $J(x,y) \neq 0$. *This corresponds to the figures* 45 *and* 46. *When* $a_1 = 0$ *and* $a_0 \neq 0$, *we have no critical points. When* $a_1 = 0$ *and* $a_0 = 0$ *then the line* $y = 0$ *is composed of critical points.*

*Proof.* The proof is straightforward. ∎

## 3. Behaviour of the system $(S)$ in the neighborhood of critical points at infinity

For the study of the critical points at infinity, we introduce the Poincaré compactification of the plane vector field via the central projection. We use the variables $(u,z)$ defined by: $z = \frac{1}{x}$, $u = \frac{y}{x}$ $(x \neq 0)$ and $(w,z)$ defined by: $z = \frac{1}{y}$, $w = \frac{x}{y}$ $(y \neq 0)$. These transformations define the charts that are needed in the analysis of the critical points at infinity.

After rescaling the time by setting $dt' = zdt$, we obtain, respectively, for the two charts the following systems:

$$\begin{aligned} P_2 =& u(-(c_1 - 2c_2)a_3 - (c_1 - 2c_2)a_4u - (c_1 - 2c_2)a_5u^2 - (c_1 - c_2)a_1z \\ & - (c_1 - c_2)a_2uz - a_0c_1z^2), \\ Q_2 =& z(-a_3c_1 - (c_1 - c_2)a_4u - (c_1 - 2c_2)a_5u^2 - a_1c_1z - (c_1 - c_2)a_2uz \\ & - a_0c_1z^2), \end{aligned}$$

$$\begin{aligned} P_3 =& (c_1 - 2c_2)a_5 + (c_1 - 2c_2)a_4w + (c_1 - 2c_2)a_3w^2 + (c_1 - c_2)a_2z \\ & + (c_1 - c_2)a_1wz + a_0c_1z^2, \\ Q_3 =& - c_2z(a_4 + 2a_3w + a_1z). \end{aligned}$$

Proposition 3.1. *We suppose that* $a_3(c_1 - 2c_2)\Delta_2 \neq 0$. *When* $\Delta_2 < 0$ *the origin* $O$ *is the only critical point at infinity. It is a node when* $c_1(c_1 - 2c_2) > 0$

*and a saddle when* $c_1(c_1 - 2c_2) < 0$. *When* $a_4^2 - 4a_3a_4 > 0$, *there are three critical points at infinity:* $O$, $M_6$, $M_7$. *We have the following results:*

*If* $c_1c_2 < 0$ *then* $M_6$, $M_7$ *and* $O$ *are nodes.*

*If* $c_2(c_1 - 2c_2) > 0$ *then* $M_6$, $M_7$ *are saddles and* $O$ *is a node.*

*If* $c_1(c_1 - 2c_2) < 0$ *then* $M_6$, $M_7$ *are nodes and* $O$ *is a saddle.*

*Proof.* The proof is direct. ∎

3.1. PARTICULAR CASES.

PROPOSITION 3.2. *We suppose that* $a_3\Delta_2 = 0$; *then we have the following cases:*

a) $\Delta_2 = 0$ *and* $a_3(c_1 - 2c_2) \neq 0$.

a1.- *If* $\left(a_2 - \frac{a_1a_4}{2a_3}\right)(c_1 - c_2) \neq 0$ *then we have two critical points:* $O$ *which is either a saddle or a node and the second is* $M_6$. *If* $c_1(c_1 - 2c_2) < 0$ *then it's neighborhood is either a union of a hyperbolic sector and an elliptic sector otherwise* $M_6$ *is a topological saddle when* $c_2(c_1 - 2c_2) > 0$. *The corresponding phase portraits are represented in figures* 34, 35, 36, 37, 38 *and* 39.

a2.- *If* $a_2 - \dfrac{a_1a_4}{2a_3} = 0$ *then we have the following results:*

*If* $\Delta < 0$ *we have two cases:*

$c_1(c_1 - 2c_2) < 0$; *the local behaviour in the neighborhood of* $M_8$ *is given by scheme* 1 *(see page* 113*) and then we have the phase portrait* 40.

$c_1(c_1 - 2c_2) > 0$; *we have the scheme* 2 *and therefore the phase portrait* 41.

*If* $\Delta > 0$ *then we have three different cases:*

$c_1c_2 < 0$; *we have the scheme* 5 *and then the phase portrait* 42.

$c_1(c_1 - 2c_2) < 0$; *we have scheme* 4, *and therefore the phase portrait* 43.

$c_2(c_1 - 2c_2) > 0$; *we obtain scheme* 3 *and the corresponding phase portrait is represented in figure* 44.

a3.- *If* $c_1 = c_2$ *then we have scheme* 1 *for* $\Delta < 0$, *and scheme* 5 *for* $\Delta > 0$.

b) $a_3 = 0$, $a_4(c_1 - 2c_2) \neq 0$.

*First, we suppose that* $a_1 \neq 0$. *In this case we have two critical points: The origin* $O$ *and another point* $M_7$. *The point* $M_7$ *is a saddle when*

*$c_2(c_1-2c_2)>0$ and a node when $c_2(c_1-2c_2)<0$, while the origin $O$ is degenerate. For this latter critical point we have the following results:*

*When $c_1c_2<0$ or $c_2(c_1-2c_2)>0$ the behaviour of the system in the vicinity of $O$ is homeomorphic to scheme 5. This corresponds to the figures 29 and 30.*

*When $c_1(c_1-2c_2)<0$, we have scheme 4 and therefore the phase portrait 31.*

*Second, we suppose that $a_1=0$ and $a_0\neq 0$ we must consider two cases:*

*$c_2(c_1-2c_2)>0$; the local phase portrait is given by scheme 6 and the global phase portrait is represented in the figure 32.*

*$c_2(c_1-2c_2)<0$; we have scheme 7 and then the phase portrait 33.*

*Third, when $a_1=0$ and $a_0=0$, the origin $O$ is not an isolated critical point.*

c) $a_3=0$, $a_4=0$. *In this case $O$ is the only critical point at infinity and it is degenerate. If we suppose that $a_1\neq 0$ and $a_5\neq 0$, then the local phase portrait of the system in the vicinity of $O$ is homeomorphic to scheme 6 when $c_1c_2<0$ and to scheme 7 when $c_1c_2>0$. This corresponds respectively to figures 45 and 46. If $a_1=0$ or $a_5=0$ then the critical point $O$ is not isolated.*

*Proof.* For the proof of a), we use theorem N.E.

For b), when $a_1\neq 0$ it is easy to determine the nature of critical point $M_7$. The point $O$ is degenerate with Jacobian Matrix be identically zero. By using Forster Theorem (see for instance [18]), the proposition follows.

We suppose now that $a_1=0$ and $a_0\neq 0$. In this case we use the technic of "Blow-up" (see for instance [2]).

Since $z(-(c_1-2c_2)a_4u^2-u(-(c_1-c_2)a_4uz)=a_4c_2u^2z$, we make the change of variables:

$u=u_1,\;\; z=u_1+z_1$ and we find the following system:

$$\begin{aligned}P_4=&-(c_1-2c_2)a_4u_1^2-(a_0c_1+a_2c_1+a_5c_1-a_2c_2-2a_5c_2)u_1^3\\&-(2a_0c_1+a_2c_1-a_2c_2)u_1^2z_1-a_0c_1u_1z_1^2,\end{aligned}$$

$$\begin{aligned}Q_4=&-a_4c_2u_1^2-(c_1-c_2)a_4u_1z_1-(a_0c_1+a_2c_1+a_5c_1-a_2c_2-2a_5c_2)u_1^2z_1\\&-(2a_0c_1+a_2c_1-a_2c_2)u_1z_1^2-a_0c_1z_1^3.\end{aligned}$$

We now make a horizontal "Blow-up" by introducing the transformation: $u_1 = z_1 u_2,\ \ z_1 = z_2$. We obtain, after rescaling the time ($dt' = z_2 dt$), the following system:

$$P_5(u_2, z_2) = a_4 c_2 u_2^2 (1 + u_2),$$

$$\begin{aligned} Q_5(u_2, z_2) =& z_2(-(c_1 - c_2) a_4 u_2 - a_4 c_2 u_2^2 - a_0 c_1 z_2 - (2a_0 c_1 + a_2 c_1 - a_2 c_2) u_2 z_2 \\ &- (a_0 c_1 + a_2 c_1 + a_5 c_1 - a_2 c_2 - 2a_5 c_2) u_2^2 z_2. \end{aligned}$$

For this system we have two critical points: $O$ which is degenerate and $M_8(-1, 0)$ whose associated eigenvalues are $\lambda^1_{M_8} = (c_1 - 2c_2) a_4$ and $\lambda^2_{M_8} = c_2 a_4$. Then $M_8$ is a saddle when $c_2(c_1 - 2c_2) < 0$ or a node when $c_2(c_1 - 2c_2) > 0$.
Since $z_2(a_4 c_2 u_2^2) - u_2(z_2(c_1 - c_2) a_4 u_2 - a_0 c_1 z_2) = u_2 z_2 (a_4 u_2 + a_0 z_2) c_1$, we make another change of variables: $u_2 = u_3,\ \ z_2 = u_3 + z_3$ and supposing that $a_0 + a_4 \neq 0$, we obtain the system:

$$P_6(u_3, z_3) = a_4 c_2 u_3^2 (1 + u_3),$$

$$\begin{aligned} Q_6(u_3, z_3) =& -(a_0 c_1 + a_4 c_1) u_3^2 - (2a_0 c_1 + a_2 c_1 - a_2 c_2 + 2a_4 c_2) u_3^3 \\ &- (a_0 c_1 + a_2 c_1 + a_5 c_1 - a_2 c_2 - 2a_5 c_2) u_3^4 \\ &- (2a_0 c_1 + a_4 c_1 - a_4 c_2) u_3 z_3 \\ &- (4a_0 c_1 + 2a_2 c_1 - 2a_2 c_2 + a_4 c_2) u_3^2 z_3 - (2a_0 c_1 + 2a_2 c_1 \\ &+ 2a_5 c_1 - 2a_2 c_2 - 4a_5 c_2) u_3^3 z_3 - a_0 c_1 z_3^2 \\ &- (2a_0 c_1 + a_2 c_1 - a_2 c_2) u_3 z_3^2 - (a_0 c_1 + a_2 c_1 + a_5 c_1 \\ &- a_2 c_2 - 2a_5 c_2) u_3^2 z_3^2. \end{aligned}$$

Let us introduce another "Blow-up" defined by $u_3 = u_4 z_3$, $z_3 = z_4$. After rescaling the time by setting $dt' = z_4 dt$ we obtain the following system:

$$\begin{aligned} P_7(u_4, z_4) =& u_4(1 + u_4)(a_0 c_1 + a_0 c_1 u_4 + a_4 c_1 u_4 + 2a_0 c_1 u_4 z_4 \\ &+ a_2 c_1 u_4 z_4 - a_2 c_2 u_4 z_4 + 2a_0 c_1 u_4^2 z_4 + a_2 c_1 u_4^2 z_4 - a_2 c_2 u_4^2 z_4 \\ &+ 2a_4 c_2 u_4^2 z_4 + a_0 c_1 u_4^2 z_4^2 + a_2 c_1 u_4^2 z_4^2 + a_5 c_1 u_4^2 z_4^2 \\ &- a_2 c_2 u_4^2 z_4^2 - 2a_5 c_2 u_4^2 z_4^2 + a_0 c_1 u_4^3 z_4^2 + a_2 c_1 u_4^3 z_4^2 \\ &+ a_5 c_1 u_4^3 z_4^2 - a_2 c_2 u_4^3 z_4^2 - 2a_5 c_2 u_4^3 z_4^2), \end{aligned}$$

$$\begin{aligned}Q_7(u_4,z_4) =& z_4(-a_0c_1 - 2a_0c_1u_4 - a_4c_1u_4 + a_4c_2u_4 - a_0c_1u_4^2\\ & - a_4c_1u_4^2 - 2a_0c_1u_4z_4 - a_2c_1u_4z_4 + a_2c_2u_4z_4\\ & - 4a_0c_1u_4^2z_4 - 2a_2c_1u_4^2z_4 + 2a_2c_2u_4^2z_4 - a_4c_2u_4^2z_4\\ & - 2a_0c_1u_4^3z_4 - a_2c_1u_4^3z_4 + a_2c_2u_4^3z_4 - 2a_4c_2u_4^3z_4\\ & - a_0c_1u_4^2z_4^2 - a_2c_1u_4^2z_4^2 - a_5c_1u_4^2z_4^2 + a_2c_2u_4^2z_4^2\\ & + 2a_5c_2u_4^2z_4^2 - 2a_0c_1u_4^3z_4^2 - 2a_2c_1u_4^3z_4^2 - 2a_5c_1u_4^3z_4^2\\ & + 2a_2c_2u_4^3z_4^2 + 4a_5c_2u_4^3z_4^2 - a_0c_1u_4^4z_4^2 - a_2c_1u_4^4z_4^2\\ & - a_5c_1u_4^4z_4^2 + a_2c_2u_4^4z_4^2 + 2a_5c_2u_4^4z_4^2).\end{aligned}$$

We obtain three hyperbolic critical points for this system: $O$, $M_9(-1,0)$ and $M_{10}(-\frac{a_0}{a_0+a_4},0)$. Their eigenvalues are, respectively:

$$(a_0c_1,-a_0c_1),\ \ (a_4c_1,-a_4c_2),\ \ \left(-\frac{a_0a_4}{a_0+a_4}c_1,\ -\frac{a_0a_4}{a_0+a_4}c_2\right).$$

Hence, $O$ is a saddle point. When $c_1c_2>0$ $M_9$ is a saddle and $M_{10}$ is a node. When $c_1c_2<0$ $M_9$ is a node while $M_{10}$ is a saddle.

Now, "going down" through the transformation, we can conclude that if $c_2(c_1-2c_2)>0$ the local phase portrait is homeomorphic to the scheme 6 and if $c_2(c_1-2c_2)<0$ we obtain scheme 7. The phase portraits are respectively 32 and 33. We obtain the same results when $a_4+a_0=0$; in this case, it is sufficient to employ another change of variables.

c) $a_3=0,\ a_4=0$

If $a_5\neq 0$ and $a_1\neq 0$ we can again use the technic of "Blow-up" as above, but vertically, and we find that the local behaviour in the neighborhood of the origin is homeomorphic to the scheme 8 when $c_1c_2<0$ and to scheme 9 when $c_1c_2>0$. We have respectively the phase portraits 45 and 46. ∎

*Remark.* Let us make some remarks concerning proposition 2.4. When $\Delta>0$, $\Delta_1<0$ and $c_1(c_1-2c_2)<0$; from the expression of $\Delta_1$ we conclude that $\Delta_2<0$. Then the origin $O$ is the only critical point at infinity (with the exception of its antipodal point). Both points are saddles. Another way of finding this result is to employ the Poincaré-Hopf theorem which says that the sum of the indices of all critical points is independent of the vector field and is equal to the Euler-Poincaré characteristic, which is 2 for the sphere. Then the sum of the indices is 2. Now, if we suppose that $\Delta_2>0$, we will have three infinite critical points, two finite critical points and their antipodal.

The sum of their indices is equal to 6, in contradiction with the Poincaré-Hopf theorem. In conclusion, we must have $\Delta_2 < 0$.

We can make the same analysis when $\Delta > 0$, $\Delta_1 < 0$ and $c_1(c_1 - 2c_2) > 0$. For this case, we must have $\Delta_2 > 0$.

Let us also remark that the most of phase portraits that we have drawn are already known. For example, for the cases where we have a center [27], those originated from homogeneous quadratic systems [28] (figures 24,25,26,27,28), systems with one finite critical point [10] (figures 18,19,24,25,28,38,45,46) and chordal quadratic systems [13] (figures 16,17,32,33,41).

ACKNOWLEDGEMENTS

We would like to thank J. Llibre for many useful suggestions.

## 4. Appendix

THEOREM N.E. ([1]) *Let $(0,0)$ be an isolated critical point of the system*

$$\dot{x} = y + X(x,y)$$

$$\dot{y} = Y(x,y)$$

*where $X$ and $Y$ are analytical in a neighborhood of the origin and they have expansions that begin with second degree terms in $x$ and $y$. Let $y = F(x) = k_2x^2 + k_3x^3 + ...$ be a solution of the equation $y + X(x,y) = 0$ in the neighborhood of $(0,0)$, and assume that we have the series expansions*

$$f(x) = Y(x, F(x)) = ax^{\alpha}(1 + ...) \quad \text{and}$$
$$\Phi(x) = \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)(x, F(x)) = bx^{\beta}(1 + ...),$$

*where $a \neq 0$, $\alpha \geq 2$ and $\beta \geq 1$.*

(a) *If $\alpha$ is even, and*

(a.1) *$\alpha > 2\beta + 1$, then the origin is a saddle-node (index $0$), see scheme 10;*

(a.2) *either $\alpha < 2\beta + 1$ or $\Phi(x) \equiv 0$, then the origin is a critical point whose neighborhood is a union of hyperbolic sectors (index $0$), see scheme 11.*

(b) *If $\alpha$ is odd and $a > 0$, then the origin is a saddle (index $-1$), see scheme 12.*

(c) *If $\alpha$ is odd and $a < 0$, and*

(c.1) *either $\alpha > 2\beta + 1$ and $\beta$ even; or $\alpha = 2\beta + 1$, $\beta$ even and $b^2 + 4a(\beta+1) \geq 0$, the the origin is a node (index 1), the node is stable if $b < 0$, or unstable if $b > 0$;*

(c.2) *either $\alpha > 2\beta+1$ and $\beta$ odd; or $\alpha = 2\beta+1$, $\beta$ odd and $b^2+4a(\beta+1) \geq 0$, then in the neighborhood of the origin, there are a union of a hyperbolic sector and an elliptic sector (index +1), see scheme 14.*

(c.3) *either $\alpha = 2\beta+1$ or $b^2+4a(\beta+1) < 0$, or $\alpha < 2\beta+1$ (or $\Phi \equiv 0$), then the origin is either a focus, or a center (index +1).*

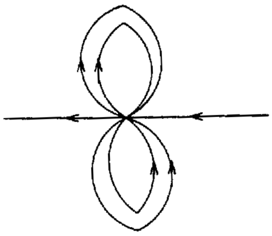
Theorem A1. ([18]) *Consider the following system:*

$$\dot{x} = a_{11}x^2 + a_{12}xy + a_{22}y^2$$
$$\dot{y} = b_{12}xy + b_{22}y^2$$

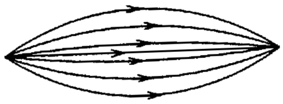
*We set $\Delta = (b_{22} - a_{12})^2 + 4a_{22}(b_{12} - a_{11})$. For the above system, suppose that the right sides of the two equations do not have a common factor, and suppose $\Delta < 0$. Then, when $a_{11}(b_{12} - a_{11}) > 0$ the global phase portrait is as shown in figure 24, and for $a_{11}(b_{12} - a_{11}) < 0$ the global phase portrait is as shown in figure 25.*

Theorem A2. ([18]) *For the above system, whose right sides do not have a common factor, suppose $a_{22} = 0$ and $A_1 = (b_{12} - a_{11})(a_{12} - b_{22}) > 0$. Then when $A_2 = a_{11}(b_{12} - a_{11})$, $A_3 = b_{22}(a_{12} - b_{22})$, and $A_4 = a_{11}b_{22} - a_{12}b_{12}$ are all negative, the phase portrait is as shown in figure 26; when two of the quantities $A_2, A_3$ and $A_4$ are negative and the other is positive, the global phase portrait is as shown in figure 27; when two of them are positive and the other is negative, the global phase portrait is as shown in figure 28.*

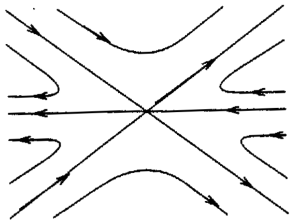
## 5. Schemes



scheme 1



scheme 2



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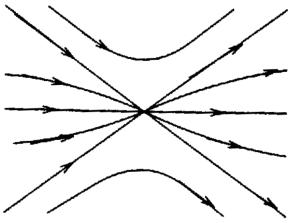

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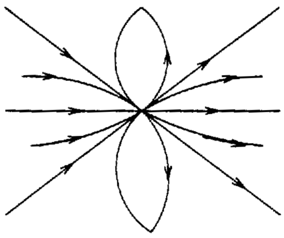

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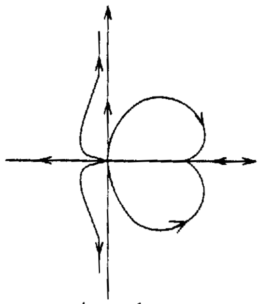

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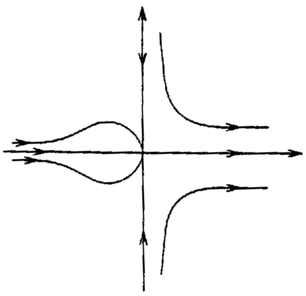

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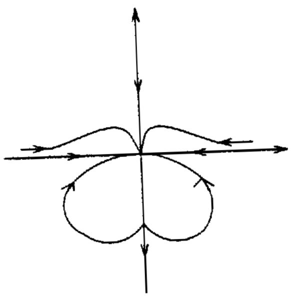

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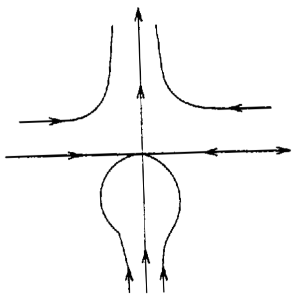
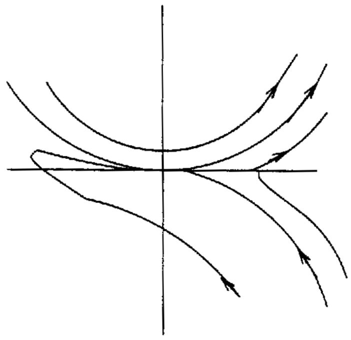
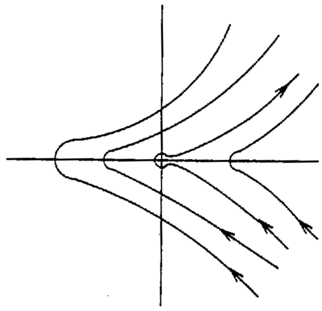


scheme 9



scheme 10



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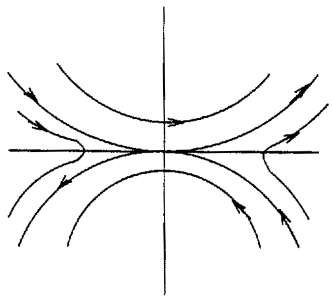

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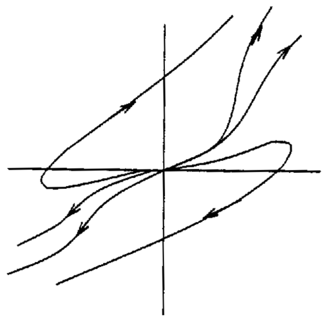

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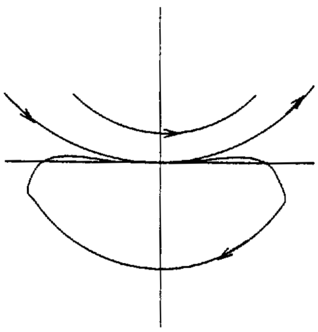
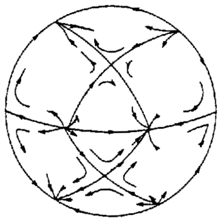


scheme 14

## 6. Figures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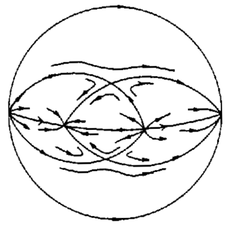
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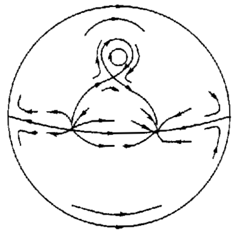
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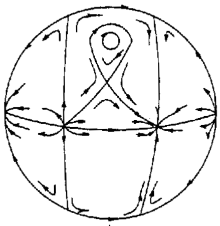
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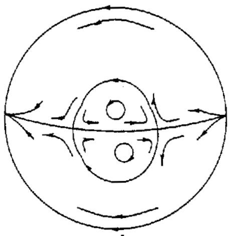
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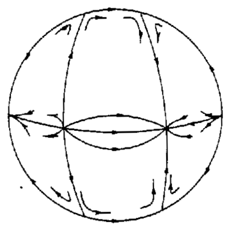
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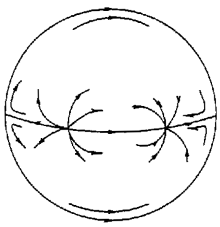
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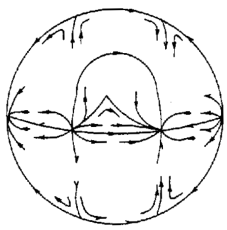
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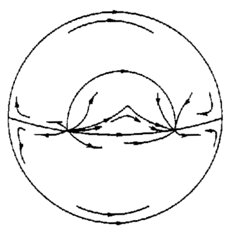
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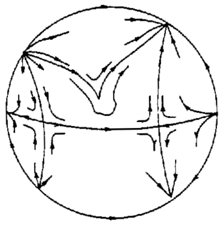
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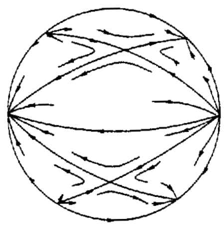
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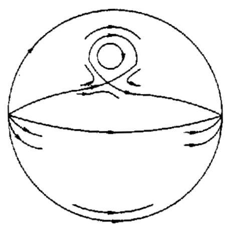
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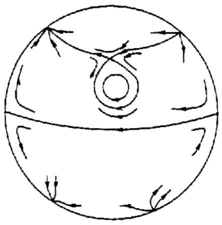
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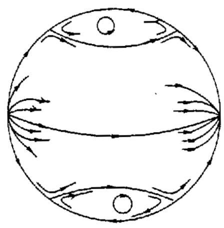
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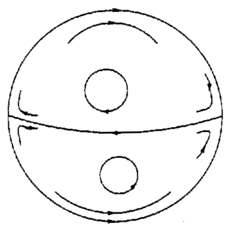
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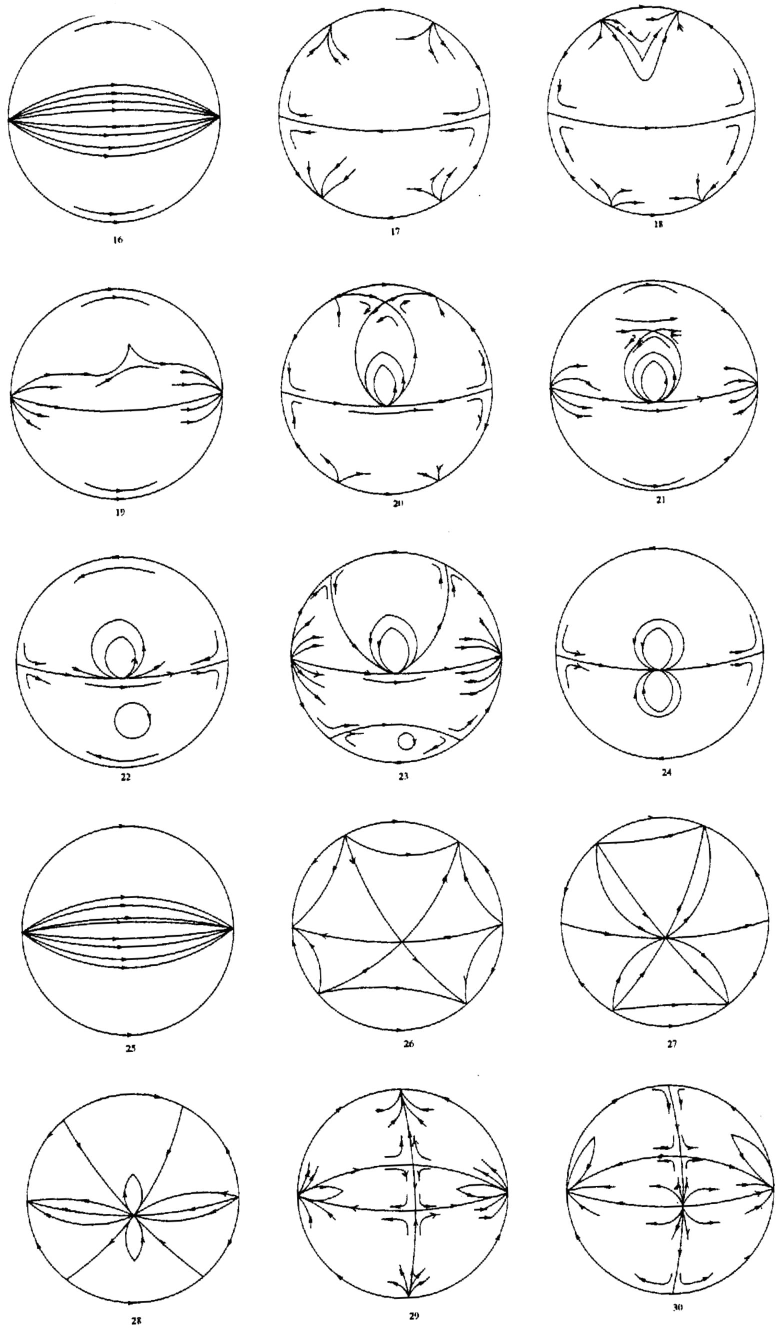
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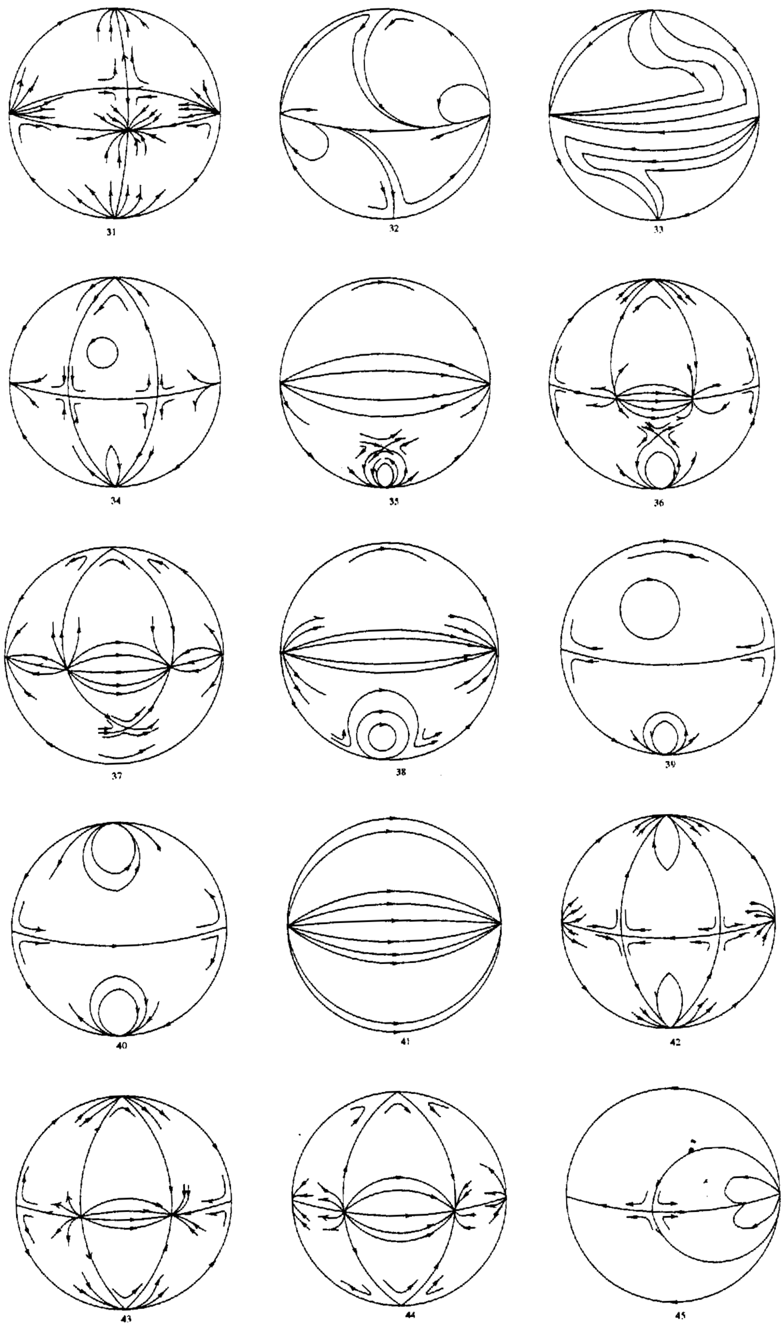
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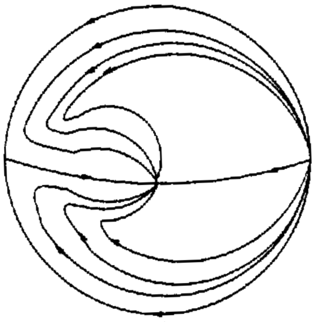
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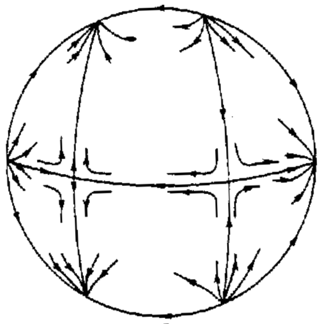
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