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MIXED TACTICAL ASSET ALLOCATION

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MIXED TACTICAL ASSET ALLOCATION

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Resumen
En un típico marco de Asignación Táctica de Activos un administrador de cartera es compensado por su exceso de retorno, una vez que se da un objetivo de Tracking Error. Las críticas a este enfoque apuntan a su falta de control del riesgo total del portafolio. Los métodos actuales recomiendan aquello que llamamos una asignación mixta, que considera el retorno y riesgo tanto relativo como absoluto.
Este trabajo provee el marco analítico de una Asignación Táctica de Activos Mixta. Nos basamos en la premisa que una vez que el inversionista fija el Tracking Error aún debe resolver un segundo trade off, aquel entre el exceso de retorno y el riesgo total del portafolio. Este artículo deriva un teorema de separación para la asignación táctica, donde se muestra que el portafolio resultante es una combinación lineal entre un portafolio “alfa” – el que provee exceso de retorno – y un portafolio “beta” – el que provee cobertura para el riesgo total –. El autor muestra como la expresión formal para la asignación táctica contiene a todos los estudios anteriores. Más aún, éste también incluye el caso más sencillo de la asignación táctica de Black Litterman.

Abstract
In a typical tactical asset allocation set up a manager receives compensation for his excess of return given a tracking error target. Critics of this framework cite its lack of control over the total portfolio risk. Current approaches recommend what we call a mixed allocation, derived from concerns about relative and absolute return and risk.
This work provides an analytical framework for mixed tactical asset allocation, based on the premise that after the investor sets a tracking error target, a fundamental trade off remains unsolved: the one between excess of return and total risk. The article derives a separation theorem for tactical allocation, wherein the portfolio is a linear combination of an alpha portfolio providing excess returns and a beta portfolio providing overall risk hedge. The author shows how the formal expression summarizes all previous works. Moreover, it also includes the simplest Black-Litterman allocation.

I am grateful to Marcelo Reyes, Sergio Godoy, Alvaro Rojas, Cristian Salinas and Esteban Jadresic from the Central Bank of Chile, and Robert Patch and Rumi Masih from Goldman Sachs, and an anonimous referee for their useful comments. Any shortcomings are mine alone.
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1 INTRODUCTION

Nowadays the investment industry uses Tactical Asset Allocation (TAA) widely. Under this approach the investor selects a *strategic* or benchmark (BMK) portfolio, usually an index (S&P 500 for US stocks, or the Lehman Bond index) or a long-run equilibrium portfolio. Next he engages an active manager who takes *tactical* positions relative to the BMK based on short-run forecasts.

Investors assessment of managers is often based on the latter's return performance relative to BMK, in the belief that the investment accrues value if they systematically beat the BMK. However, a long time can elapse before manager performance can be measured with statistical significance. Investors therefore impose a limit on the volatility of the manager's excess of return or Tracking Error (TE).

Under a tactical setup the manager is induced to maximize his excess of return, known as alpha, subject to a TE constraint. Although commonly used in practice, this approach appears to be contrary to first principles. The sponsor gives his managers incentives to worry about relative rather than absolute performance. From an analytical point of view, Roll [1992] has shown that the alpha optimization solution is not a global mean-variance efficient portfolio. From a practical point of view, Jorion [2002] finds that a sample of enhanced index funds, formally similar to an alpha optimal portfolio, has systematically greater risk than the BMK. Jorion [2003] shows that diversification across managers does not eliminate the problem. The fact that the bet is independent of the BMK explains this lack of overall efficiency. Simply put, the overweight position that an investor takes on a rising stock is the same regardless of the original stock endowment.

To provide a solution to this problem, Roll [1992] examines an additional constraint on the BMK-active portfolio correlation, also known as tactical beta. Jorion [2003] moves the constraint to the total portfolio risk. Chow [1995] proposes a total utility function with an extra TE dissatisfaction term. All of the above approaches include relative and absolute variables in the optimization framework, so we name them all *Mixed TAA*. Today, practitioners set mix targets over TE and beta.

This article provides an analytical framework for mixed TAA. We hold that once the TE is fixed, the fundamental trade off between total risk and alpha remains unsolved. The arguments presented herein arrives at a general solution that incorporates all previous works. Here the tactical allocation is the sum of two portfolios: an alpha portfolio designed to obtain excess of return and a beta portfolio hedging total risk. This distinction is in line with Anson's division [2004] between alpha and beta drivers.

Once derived, the tactical portfolio provide the basis for relating mixed TAA to mixed estimation setups. We show that the simplest application of the Black Litterman method [1992], proposed by Sharpe [1981], it is an example of a mixed TAA. This formal association assures the BL investor of the global efficiency of his investment. The BL method provides a natural bound for the TE without imposing further constraints, not even those of TE or beta.

We stress that our work has none math complexity.

2 TOTAL AND RELATIVE TAA

Basically, in TAA a manager uses a new information set to take positions from an investor BMK[1]. We will note the BMK as $\mathbf{q}_B$ and the bet derived from the new information set as $\mathbf{x}$. Thus, the final portfolio $\mathbf{q}_P$ contains strategic as well as tactical perspective

$$\mathbf{q}_P = \mathbf{q}_B + \mathbf{x}$$

The manager obtains his information from a forecast of future market movements. We summarize this set of information in an estimated vector of returns $\mathbf{R}$ and an estimated covariance matrix $\mathbf{V}$. Formally, the TAA problem is to derive $\mathbf{x}$ as a function of $\mathbf{q}_B$, $\mathbf{R}$ and $\mathbf{V}$ given some efficiency criteria.

Hereafter we make the following assumptions. There are N risky securities traded in the first period and paid off in the second. Total investor wealth is normalized to one. We avoid any delegation problem by assuming that the manager has no private information, leaving the compensation issue for the final comments. Hence, the investor and the manager can be seen as a single operator. However, we will remark on each separately because of their unique tasks in the allocation process.

2.1 Absolute efficient TAA

We derive the global efficient portfolio using the new information set. Lee [2000] call this approach *total return/total risk* perspective. The problem is as follows

$$\text{Max } U(\mathbf{q}_P) = \mathbf{q}_p'\mathbf{R} - \frac{1}{2\tau}\mathbf{q}_P'\mathbf{V}\mathbf{q}_P \tag{1}$$
$$\text{s.t. } \mathbf{q}_P'\mathbf{I} = 1 \text{ (or } \mathbf{x}'\mathbf{I} = 0)$$

where $\boldsymbol{\tau}$ is the investor risk tolerance, assumed constant. $\mathbf{I}$ is an N - vector of ones, so the constraint reflects that all wealth is invested. No short sales constraint is imposed in order to obtain an analytical solution.

The following portfolios are introduced:

$\mathbf{q}_o$ = the global minimum variance portfolio.

$\mathbf{q}_1$ = the portfolio located at the intersection of the efficient frontier and a ray passing through the origin and $\mathbf{q}_o$.

$\mathbf{x}_1 = \mathbf{q}_1 - \mathbf{q}_o$

$\mathbf{x}_B = \mathbf{q}_B - \mathbf{q}_o$

The first two portfolios $\mathbf{q}_0$ and $\mathbf{q}_1$ are the basis to construct any MV efficient portfolio. The second pair $\mathbf{x}_1$ and $\mathbf{x}_B$ are the basis for TAA, with their associated risks noted as σ_{X1} and σ_{XB} respectively, while ρ is the correlation among them. The tactical bet that solves (1) is

$$\mathbf{x}_{MV} = \varepsilon^{-1}(\mathbf{x}_1 - \varepsilon \mathbf{x}_B) \quad (2)$$

where $\varepsilon^{-1} = \tau b$. Appendix A1 reminds definitions for $\mathbf{q}_0$, $\mathbf{q}_1$ and b as well as (2) derivation. Exhibit 1 depicts the efficient frontier for $\mathbf{R}$ and $\mathbf{V}$, the BMK portfolio $\mathbf{q}_B$, the bet $\mathbf{x}$ and the final portfolio $\mathbf{q}_P$, in the *total return/total risk* space

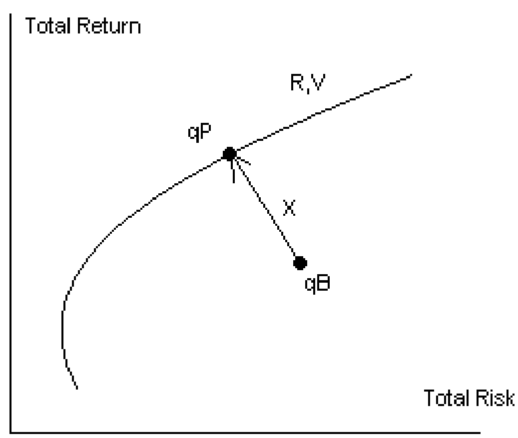

Exhibit 1

The solution (2) assigns a null bet if the BMK is optimal for $\mathbf{R}$ and $\mathbf{V}$ by itself[2]. This can represent a manager with no additional information. The problem with this total efficient TAA is that the final portfolio does not depend on the BMK, as noted in the state of (1). The final portfolio can depart significantly from the BMK, contrary to the very notion of TAA, as seen in Exhibit 1.

2.2 Relative Efficient TAA

The alpha volatility or Tracking Error $T = \sqrt{x'Vx}$ is defined as a measure of the final portfolio's deviation from the BMK[3]. In practice, investors set a TE target for the manager and claim for alpha inside this limit. Lee [2000] calls this approach *active return/active risk* perspective. The problem is stated as

$$\begin{aligned} &\text{Max } \alpha(\mathbf{x}) = \mathbf{x}'\mathbf{R} \\ &\text{s.t. } \mathbf{q}_P'\mathbf{I} = 1 \text{ (or } \mathbf{x}'\mathbf{I} = 0) \\ &\sqrt{\mathbf{x}'\mathbf{V}\mathbf{x}} = T_0 \end{aligned} \tag{3}$$

The solution $\mathbf{x}_\alpha$ is derived in Roll [1992]

$$\mathbf{x}_\alpha = \frac{T_0}{\sigma_{X1}} \mathbf{x}_1 \tag{4}$$

Setting T_0 the investor controls the size of the bet. Such bet is displayed in Exhibit 2, where.we use the result of Jorion [2003] who shows that a fixed TE depicts an ellipse in the total return total risk space

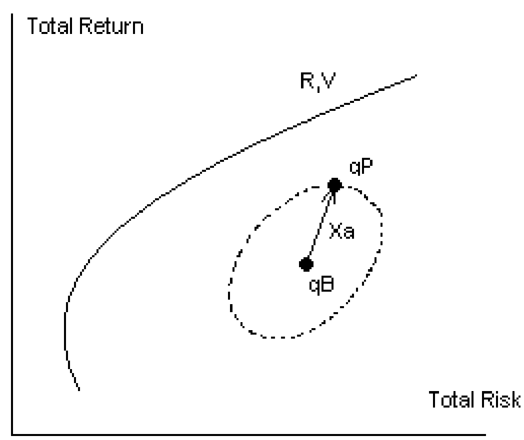

Exhibit 2

Roll notes that the above tactical allocation does not take into account $\mathbf{q}_B$ but only the expected returns. This independence causes inefficiency in overall terms. Comparing (2) with (4) we observe that $\mathbf{x}_\alpha$ not only bounds the bet but changes its direction.

3 MIXED TAA

We hold that a desirable TAA must seek return, while taking into account total and incremental risk. Solution (2) satisfies only the first condition, (4) only the second. We call this desirable allocation a *mixed TAA* as it involves total and relative variables in its optimization framework.

Given a TE target, the mixed TAA trades off primarily between alpha and global risk. This decision is captured in a global utility function[4]. The problem is stated as

$$\text{Max } U(\mathbf{q}_P) = \mathbf{q}_P'\mathbf{R} - \frac{1}{2\tau}\mathbf{q}_P'\mathbf{V}\mathbf{q}_P \tag{5}$$
$$\text{s.t. } \mathbf{q}_P'\mathbf{I} = 1 \text{ (or } \mathbf{x}'\mathbf{I} = 0)$$
$$\sqrt{\mathbf{x}'\mathbf{V}\mathbf{x}} = T_0$$

The Appendix A.2. derives the solution $\mathbf{x}_M$

$$\mathbf{x}_M = \frac{T_0}{\sigma_{X\varepsilon}}(\mathbf{x}_1 - \varepsilon\mathbf{x}_B) \tag{6}$$

where $\sigma^2_{X\varepsilon} = \sigma^2_{X1} + 2\varepsilon\rho\sigma_{X1}\sigma_{XB} + \varepsilon^2\sigma^2_{XB}$. The tactical portfolio (6) has the same direction as the total efficient allocation (2) but bounded by the TE condition.

The problem statement (5) and its solution (6) are well known and simple; we make no claim for originality or complexity. However, the mixed allocation provides a powerful insight about TAA, which is embodied in the following two propositions

Proposition1: The mixed tactical allocation (6) can also be written as the linear combination of two portfolios

$$\mathbf{x}_M = \frac{\sigma_{X1}}{\sigma_{X\varepsilon}}(\mathbf{x}_\alpha + \varepsilon'\mathbf{x}_\beta) \tag{7}$$

where

$\mathbf{x}_\alpha$ = Alpha Portfolio, maximizing alpha for a given TE
$\mathbf{x}_\beta$ = Beta Portfolio, maximizing beta for a given TE
and $\varepsilon' = (\sigma_{XB}/\sigma_{X1})\varepsilon$.

Appendix A.3 derives (7) and provides the definition of $\mathbf{x}_\beta$ (expression (4) defines $\mathbf{x}_\alpha$). Under this TAA separation theorem the bet is a linear combination of a portfolio that looks for returns and a second portfolio hedging global risk. The relative weight of both is the parameter ε, fixed by the investor based on his particular willingness to pay for reducing overall risk. $\varepsilon = 0$, in which solution (4) is recovered, represents the limit case in which the investor has a complete aversion to paying for total risk.Exhibit 3 depicts the mixed allocation.

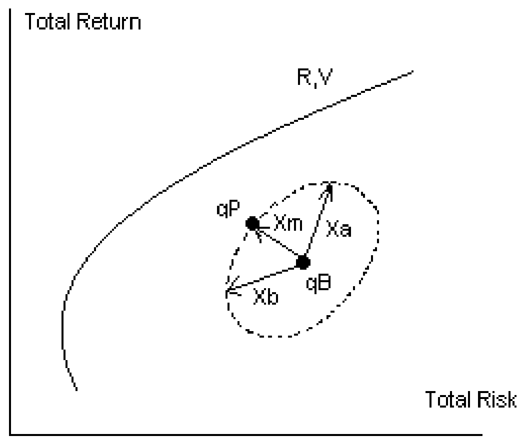

Exhibit 3

Some years ago the industry summarized TAA as the ratio between alpha and TE. However the Information Ratio does not capture the overall tactical decision. Indeed, the industry has shifted substantially towards recognizing alpha and beta movements as fundamental components of TAA. In an insightful paper, Anson [2004] explains the allocation process as a choice between alpha and beta assets, with alpha being more tactical looking-for-return assets, and beta more strategical hedging-risk assets. The portfolio (7) is the analytical expression for the alpha-beta trade off.

Proposition 2: The alpha maximization with additional constraints in beta or total risk can be obtained from (6) by selecting a proper value of ε.

A beta constraint is equivalent to a total risk constraint, as the total of the portfolio is the sum of the fixed BMK risk, the fixed TE, and the beta term. For analytical purposes we will set in γ the correlation between $\mathbf{x}$ and $\mathbf{x}_B$. This means setting beta in $\beta = 1 + \gamma(T_0\sigma_{B'}\sigma_B^{-2})$, or fixing the total risk in $\mathbf{q}_P'\mathbf{V}\mathbf{q}_P = \sigma_B^2 + T_0^2 + 2\gamma(T_0\sigma_{B'})$. Appendix A.4 derives the following proposition.

Solving the problem

$$\begin{aligned} \text{Max } \alpha(\mathbf{x}) &= \mathbf{x}'\mathbf{R} \\ \text{s.t. } \mathbf{q}_p'\mathbf{I} &= 1 \text{ (o } \mathbf{x}'\mathbf{I} = 0) \\ \mathbf{x}'\mathbf{V}\mathbf{x} &= T_0^2 \\ \frac{\mathbf{x}'\mathbf{V}\mathbf{x}_B}{\sigma_{B'}T_0} &= \gamma \end{aligned} \tag{8}$$

is equivalent to solving problem (5) with ε given for

$$\varepsilon = \frac{\sigma_{x_1}}{\sigma_{B'}}\left(\rho + \gamma\sqrt{\frac{1-\rho^2}{1-\gamma^2}}\right) \tag{9}$$

The proposition establishes that the investor's global risk aversion drives the final portfolio to any total risk or beta value he selects. As an example, Roll suggests that an investor could look for a beta of one or a bet uncorrelated with the BMK. In this case $\gamma = 0$ and $\varepsilon = \rho(\sigma_{x_1}/\sigma_{B'})$, with an explicit bet $\mathbf{x}_M^{(\beta=0)} = (1-\rho^2)^{-1/2}(\mathbf{x}_\alpha + \rho\mathbf{x}_\beta)$[5].

Proposition 2 allows the investor to achieve mixed TAA by controlling the beta or total portfolio risk. In practical applications this is better than handling a noisy utility function. The different mixed allocation approaches are summarized in (6), which, we will show, also contains the simplest case of the widely used Black Litterman allocation model.

4 MIXED ESTIMATION PRODUCES MIXED TAA

A second approach in TAA is to mix the BMK and the forecast information to derive a new return vector and a new covariance matrix. Next, this new mixed information set produces the tactical allocation. The method can be separated into three steps.

The first step is to associate a return and a variance with the BMK. We assume that the BMK is MV efficient for a return $\mathbf{R}^B$ and a variance $\mathbf{V}^B$, given a risk aversion parameter τ^B. For a long-run strategic portfolio $\mathbf{R}^B$ and $\mathbf{V}^B$ are obtained from historical realizations over a given period, and the risk parameter is the investor's. In this scheme the BMK is derived a posteriori as a mean variance efficient portfolio associated with the long-run equilibrium. However, BMK is usually not the solution to an optimization problem, so we have to focus the problem in the opposite way. We do this calculating the variance $\mathbf{V}^B$ again from historical realizations, but selecting returns $\mathbf{R}^B$ to obtain an efficient portfolio in the BMK. We state that $\mathbf{R}^B$ is compatible with $\mathbf{q}_B$. When the BMK is a market index, the compatible returns are those of the equilibrium, as suggest by CAPM theory, and risk aversion is the ratio between market risk premium and market variance. However, we emphasize that BMK weights are not generally market capitalization weights[6]. Best y Grauer [1985] provide analytical methods to derive compatible returns.

The second step involves using the strategic information $\mathbf{R}^B$ and $\mathbf{V}^B$ and the tactical information $\mathbf{R}$ and $\mathbf{V}$ to obtain a mixed information set $\mathbf{R}^M$ and $\mathbf{V}^M$. The statistical framework is Theil's mixed estimation [1962]. We assume that there are n_1 BMK data normally distributed with mean R^B and V^B, and n_2 forecast data normally distributed with mean R and V. The mixed estimation provides an estimation of the returns and variances using both information sets. The ratio n_1/n_2 appears in the final estimation, underlining the importance of each data set. This is a key issue: the weights of both sources of information can be set a priori, hence the tactical movements can be controlled in a scope other than TE.

In the third and final step, the mixed estimations become inputs of an optimization problem, where the solution is the TAA.

Sharpe [1981] proposes a simple framework which includes only a mixed estimation for the returns. This is a simple case of the general use of mixed estimation for TAA provided by Black and Litterman [1992].

4.1 "Simple" Black Litterman or Sharpe tactical allocation

Sharpe [1981] states that TAA is the mean variance efficient allocation once the BMK compatible returns are modified by forecast returns. He defines the mixed return $\mathbf{R}^M$ as the linear combination between these two sources of information

$$\mathbf{R}^M = (1-w)\mathbf{R}^B + w\mathbf{R} \tag{10}$$

where the parameter $0 < w < 1$ controls the distance between the mixed and BMK returns, hence generating a measure of confidence in the prediction. w is the ratio between both sets of information in the Theil's context.

Sharpe assumes that the BMK and forecast variances are the same, $\mathbf{V}^B = \mathbf{V}$. All new information available to the manager is contained in the forecast return vector. Although restrictive, this is not unusual in practice. Commonly the manager has a worse variance forecast than a return forecast, so he uses historical variance as the forecast. The TAA is a mean variance efficient portfolio for the mixed returns $\mathbf{R}^M$. Optimization assumes the same risk parameter $\tau = \tau^B$ in order to generate the BMK if $\mathbf{R}^B = \mathbf{R}$. The problem is as follows

$$\begin{aligned} \text{Max } U(\mathbf{q}_P) &= \mathbf{q}_p'\mathbf{R}^M - \frac{1}{2\tau}\mathbf{q}_P'\mathbf{V}\mathbf{q}_P \\ \text{s.a. } \mathbf{q}_P'\mathbf{I} &= 1 \ (\text{ o}\mathbf{x}'\mathbf{I} = 0) \end{aligned} \tag{11}$$

We emphasize that no TE constraint is imposed. Appendix A.5 derives the solution $\mathbf{x}_S$

$$\mathbf{x}_S = w\varepsilon^{-1}(\mathbf{x}_1 - \varepsilon\mathbf{x}_B) \tag{12}$$

The tactical allocation (12) is a mixed allocation of the form (6). Indeed, both bets are the same if w is chosen to satisfy $w = T_0\varepsilon/\sigma_{X\varepsilon}$. The investor controls the bet size in the parameter w, thus controlling the TE level indirectly.

BL provides a complete framework for TAA using mixed estimation as the base point. The method has several advantages: it allows for generating forecasts for only some assets, or for controlling the confidence level of every forecast; it also permits the inclusion of a new variance matrix[7]. This flexibility in applying the manager's views accounts for the wide use of BL nowadays.

The estimation of returns given in (10) is the simplest example of BL estimation[8]. We give a naive interpretation for w as the Theil ratio between tactical and strategic information's sets. This is different from the usual explanation given by BL practitioners, for whom w represents the uncertainty in the CAPM assumption. Our perspective emphasizes that BL allocation has an implicit TE constraint. In real applications practitioners suggest setting a TE and a beta target in the optimization process (see Bevan and Wilkenman [1998]). We stress that the method indirectly constrains both sources of risk by fixing w and ε. This last statement does not eliminate the need of set targets on investment practice.

5 SUMMARY AND PRACTICAL IMPLICATIONS

This article provides a simple analytical framework for mixed TAA. We express such allocation as a typical separation theorem that containts the choice between a large alpha or small beta, once TE is set up. As a primary application we establish a link with mixed estimation approaches, and relate alpha maximization with the BL method, surely the most commonly used TAA frameworks.

Finally we depict two practical issues. First, we noticed that the Information Ratio is the usual tactical allocation performance measure. However, the alpha/TE relationship alone does not capture all the aspects of the manager's performance. The alpha reported can be obtained at the expense of a high global exposure. We hold that as the TAA involves two fundamental decisions, two ratios are needed for measuring performance. Hence, an "alpha-beta" ratio (or simply the usual Sharpe ratio) must be added to the Information Ratio[9]. Empirical discussion of this issue would be valuable.

The second point is the compensation issue. Our work assumes that investors and managers are the same person in order to obtain the optimal TAA. In practice however, these two actors not only have different sources of information, they also do not share the same goals for the investment process. The delegation scheme can not be separated from the tactical investment. "BMK adjusted compensation" is widely used nowadays, but induces the manager to look only for relative variables obtaining allocation (4). In contrast, pay off for total return creates incentives to take a global position with no regard for the BMK, allocation (2)[10]. Mixed allocation requires a mixed compensation scheme, with pay offs for total and relative returns. The amount of each will depend on the investor's choice of alpha versus beta portfolios, which compounds the bet. Although further research is needed, we propose moving from the BMK related compensation to more global schemes, in order to make managers more aware of the overall portfolio risk.

APPENDIX

A.1. THE MEAN VARIANCE PROBLEM

The solution $\mathbf{q}$ of the mean variance problem (1) satisfies the following LaGrangian equation

$$\mathbf{R} - \frac{1}{\tau}\mathbf{V}\mathbf{q} + \mu\mathbf{I} = 0$$

where μ is an undetermined LaGrange multiplier. Solving for $\mathbf{q}$

$$\mathbf{q} = \tau b\left(\frac{\mathbf{V}^{-1}\mathbf{R}}{b}\right) - \tau\mu c\left(\frac{\mathbf{V}^{-1}\mathbf{I}}{c}\right)$$

where we introduce b and c for normalization purposes, that is $b = \mathbf{I}'\mathbf{V}^{-1}\mathbf{R}$ and $c = I'V^{-1}I$. Thus $\mathbf{q}_o = (\mathbf{V}^{-1}\mathbf{I})/c$ and $\mathbf{q}_1 = (\mathbf{V}^{-1}\mathbf{R})/b$, are two porfolios satisfying $\mathbf{q}_0'\mathbf{I} = \mathbf{q}_1'\mathbf{I} = 1$. Portfolio $\mathbf{q}_0$ is the global minimum variance portfolio while $\mathbf{q}_1$ is located at the intersection between the efficient frontier and the ray passing through the origin and $\mathbf{q}_0$. μ is eliminated from the constraint $\mathbf{q}\prime\mathbf{I} = 1$. The solution is

$$\mathbf{q} = \tau b(\mathbf{q}_1 - \mathbf{q}_0) + \mathbf{q}_0 \tag{A1}$$

The optimal bet is obtained by subtracting BMK to (A1)

$$\mathbf{x} = \tau b(\mathbf{q}_1 - \mathbf{q}_0) - (\mathbf{q}_B - \mathbf{q}_0)$$

Denoting $\mathbf{x}_1 = \mathbf{q}_1 - \mathbf{q}_o$, $\mathbf{x}_B = \mathbf{q}_B - \mathbf{q}_o$ and $\varepsilon^{-1} = \tau b$ we obtain the solution (2). The portfolios $\mathbf{x}_1$ and $\mathbf{x}_B$ are the basis for tactical allocation as $\mathbf{q}_0$ and $\mathbf{q}_1$ for strategic allocation.

A.2. MIXED TAA

The mixed allocation problem (5) has the following LaGrangian

$$L = \mathbf{x}'\mathbf{R} - \frac{1}{\tau}\mathbf{x}'\mathbf{V}\mathbf{q}_B - \frac{1}{2\tau}\mathbf{x}'\mathbf{V}\mathbf{x} + U(\mathbf{q}_B) - \left(\frac{b}{2\mu_1} - \frac{1}{2\tau}\right)(\mathbf{x}'\mathbf{V}\mathbf{x} - T_0^2) - \frac{\mu_2 b}{c}(\mathbf{x}'\mathbf{I})$$

where $(b/2\mu_1 - 1/2\tau)$ y $\mu_2 b/c$ are LaGrange multipliers. The FOC is

$$\begin{aligned} x &= \frac{\mu_1}{b}(\mathbf{V}^{-1}\mathbf{R} - \mu_2 b\frac{\mathbf{V}^{-1}\mathbf{I}}{c} - \frac{1}{\tau}\mathbf{q}_B) \\ &= \mu_1(\mathbf{q}_1 - \mu_2\mathbf{q}_0 - \varepsilon\mathbf{q}_B) \end{aligned}$$

Constraint $\mathbf{x}'\mathbf{I} = 0$ implies $(1 - \mu_2 - \varepsilon) = 0$. μ_1 is eliminated from the TE constraint. The solution is

$$\mathbf{x} = \frac{T_0}{((\mathbf{x}_1 + \varepsilon\mathbf{x}_B)'V(\mathbf{x}_1 + \varepsilon\mathbf{x}_B))^{1/2}}(\mathbf{x}_1 - \varepsilon\mathbf{x}_B)$$

A.3. ALPHA AND BETA PORTFOLIO

The portfolio that maximizes alpha for a fixed TE level of T_0 is given by (4)

$$\mathbf{x}_\alpha = \frac{T_0}{\sigma_{X1}}\mathbf{x}_1$$

The portfolio that minimizes beta for a fixed TE level of T_0 is the solution of the following LaGrangian

$$L = \mathbf{x'Vq}_B - \frac{1}{2\mu_1}(\mathbf{x'Vx} - T_0^2) - \frac{\mu_2}{c}(\mathbf{x'I})$$

where $1/2\mu_1$ y μ_2/c are LaGrange multipliers. The FOC is

$$\mathbf{x} = \mu_1(\mathbf{q}_B - \mu_2\mathbf{q}_0)$$

The two constraint implies

$$\mathbf{x} = \pm\frac{T_0}{\sigma_{XB}}\mathbf{x}_B$$

The negative solution is the minimum beta ($q_B V x_B = \sigma_B^2 - \sigma_0^2 > 0$), thus

$$\mathbf{x}_\beta = -\frac{T_0}{\sigma_{XB}}\mathbf{x}_B$$

The equality of (6) and (7) follows directly from the former definitions of $\mathbf{x}_\alpha$ and $\mathbf{x}_\beta$.

A.4. EQUIVALENCE BETWEEN MIXED TAA

The problem (8) has the following LaGrangian

$$L = \mathbf{x'R} - \frac{b}{2\mu_1}(\mathbf{x'Vx} - T_0^2) + b\mu(\mathbf{x'Vx}_B - \gamma\sigma_{B'}T_0) - \frac{\mu_2 b}{c}(\mathbf{x'I})$$

where $b/2\mu_1$, $b\mu$ and $\mu_2 b/c$ are LaGrange multipliers. The FOC is

$$x = \mu_1(\mathbf{q}_1 - \mu\mathbf{q}_0 - \mu_3\mathbf{q}_B)$$

Constraint $\mathbf{x'I} = 0$ implies that $(1 - \mu - \mu_3) = 0$. μ_1 is eliminated from the TE constraint. Thus

$$x = \frac{T_0}{((\mathbf{x}_1 + \mu\mathbf{x}_B)'V(\mathbf{x}_1 + \mu\mathbf{x}_B))^{1/2}}(\mathbf{x}_1 - \mu\mathbf{x}_B)$$

μ is eliminated from the beta constraint that it is written as

$$\frac{(\mathbf{x}_1 - \mu\mathbf{x}_B)V\mathbf{x}_B}{((\mathbf{x}_1 + \mu\mathbf{x}_B)'V(\mathbf{x}_1 + \mu\mathbf{x}_B))^{1/2}\sigma_{B'}} = \gamma$$

This gives a second order equation for μ. The solution is

$$\mu = \frac{\sigma_{x_1}}{\sigma_{B'}} \left(\rho + \gamma \sqrt{\frac{1-\rho^2}{1-\gamma^2}} \right) \tag{A2}$$

This expression is positive for every $\gamma < 1$ and $\rho < 1$. The solution $\mathbf{x}$ if the same of (6) with $\varepsilon = \mu$ in (A2).

A.5. SIMPLE BL AS A MIXED ALLOCATION

The values relative to the BMK are noted with the superscript B. The BMK is

$$\mathbf{q}_B = \tau b^B (\mathbf{q}_1^B - \mathbf{q}_0) + \mathbf{q}_0 \tag{A3}$$

The solution of the MV problem is given in (A1)

$$\mathbf{x} = \tau b^M (\mathbf{q}_1^M - \mathbf{q}_0) - (\mathbf{q}_B - \mathbf{q}_0)$$

where $b^M = \mathbf{I}'\mathbf{V}^{-1}\mathbf{R}^M$ and $\mathbf{q}_1^M = (\mathbf{V}^{-1}\mathbf{R}^M)/b^M$, where R^M is given by (10). We expand at first order in w

$$\begin{aligned} b^M &= b^B + w(b - b^B) \\ \mathbf{q}_1^M &= \mathbf{q}_1^B + w\frac{b}{b^B}(\mathbf{q}_1 - \mathbf{q}_1^B) \end{aligned}$$

Thus $\mathbf{x}$ at first order in w is

$$\mathbf{x} = \tau b^B (\mathbf{q}_1^B - \mathbf{q}_0) - (\mathbf{q}_B - \mathbf{q}_0) + w\tau(b - b^B)(\mathbf{q}_1^B - \mathbf{q}_0) + w\tau b(\mathbf{q}_1 - \mathbf{q}_1^B)$$

The order zero in w is null given (A3). The solution is

$$\begin{aligned} \mathbf{x} &= w(\tau b(\mathbf{q}_1 - \mathbf{q}_0) - (\mathbf{q}_B - \mathbf{q}_0) \\ &= w\varepsilon^{-1}(\mathbf{x}_1 - \varepsilon \mathbf{x}_B) \end{aligned}$$

It is easy to prove that the relation is true for all orders in w.

Notes

[1] A complete review of definitions is presented in Lee [2000].

[2] In this case $\mathbf{q}_B = \mathbf{q}_0 + \varepsilon^{-1}(\mathbf{q}_1 - \mathbf{q}_0)$ and $\mathbf{x}_{MV} = \mathbf{0}$.

[3] Roll [1992] uses the tem volatility of TE for what we call TE.

[4] Jorion [2003] stated the same problem but refuses a deeper analysis since the problem measure of a utility function. We will cover the implementation issues in proposition 2.

[5] A second example, proposed by Jorion, is the more restrictive case in which total risk is equal to BMK risk, and thus beta $\beta < 1$, $\gamma = -(T_0/2\sigma_{B'})$, and the equations (6) and (9) can be evaluated again to obtain the explicit solution.

[6] As an example many central banks perform active investment of their reserves. The strategic portfolio is obtained from macroeconomic criteria such as fiscal debt or the export composition.

[7] The importance of using better information to build the variance matrix must not be understated. Litterman and Winkelmann [1998] show that the final allocation is quite sensitive to variance estimation.

[8] BL assumes that the forecast returns $\boldsymbol{\mu}$ can be expressed as $\mathbf{P}\boldsymbol{\mu} = \mathbf{Q} + \boldsymbol{\epsilon}$, where $\mathbf{P}$ is a KxN matrix, $\mathbf{Q}$ a K vector, and $\boldsymbol{\epsilon}$ is normally distributed with zero mean and covariance $\boldsymbol{\Omega}$. The mixed estimation is

$$\mathbf{R}^M = (\mathbf{V}^{-1} + w\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{P})^{-1}(\mathbf{V}^{-1}\mathbf{R}^B + w\mathbf{P}'\boldsymbol{\Omega}^{-1}\mathbf{Q})$$

We assume that $\mu = R + \boldsymbol{\epsilon}$ ($\mathbf{P} = \mathbf{I}$ and $\mathbf{Q} = \mathbf{R}$) and $\boldsymbol{\Omega} = \mathbf{V}$, so we obtain (10).

[9] Anson [2004] stated that performance measure must include both Information and Sharpe ratio.

[10] This result is probed in Admati and Pfleiderer [1997], where a mixed compensation scheme is presented.

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