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RESUMEN
En este artículo analizamos las propiedades de convergencia del algoritmo de bandas móviles propuesto por Maliar y Maliar (2003) para inicializar el método de Parametrización de Expectativas. Realizamos un experimento de Monte Carlo para estudiar el comportamiento del citado algoritmo frente a otras alternativas existentes basadas en los principios de la homotopía. El marco del experimento son dos modelos estándar de crecimiento económico. Nuestros resultados muestran que: (i) la velocidad de convergencia del algoritmo es modesta cuando se compara con otras alternativas disponibles; (ii) si el algoritmo no se inicializa apropiadamente puede diverger incluso en condiciones relativamente simples. Estos resultados sugieren la necesidad de refinar el método de Maliar y Maliar (2003) para mejorar sus propiedades de convergencia hacia una solución.

Palabras clave: modelos no lineales; métodos numéricos de solución; algoritmo de Parametrización de Expectativas; Crecimiento económico

ABSTRACT
In this paper we analyze the convergence properties of the moving bounds algorithm to initialize the Parameterized Expectations Algorithm suggested by Maliar and Maliar (2003) [*Journal of Business and Economic Statistics* 1, pp. 88-92]. We carry out a Monte Carlo experiment to check its performance against some initialization alternatives based on homotopy principles. We do so within the framework of two standard neoclassical growth models. We show that: (i) speed of convergence is poor as compared to alternatives; (ii) starting from a not very accurate initial guess might prevent convergence in relatively simple models. The results suggest the need to fine tune Maliar and Maliar's method to improve its convergence properties.

Keywords: Nonlinear models; Numerical solution methods; Parameterized Expectations algorithm; Optimal growth
JEL classification: C63; E17

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1 Motivation

The Parameterized Expectations Algorithm (PEA) is a widely applied method for solving nonlinear stochastic dynamic models with rational expectations (see for example den Haan and Marcet (1990), Marcet and Marshall (1994), Marcet and Lorenzoni (1999) or Christiano and Fisher (2000), and the references quoted therein). The PEA scheme involves the approximation of the conditional expectation functions in the Euler equations with certain parametric functions, and the use of a numerical optimization method to determine the parameterization of these functions. The PEA tends to be a convenient algorithm, especially when there are a large number of state variables and stochastic shocks in key conditional expectations terms.

The main drawback of the PEA in practical applications is related to the obtention of appropriate initial conditions that would guarantee convergence to a fixed point solution. Indeed, the PEA is not a contraction mapping technique and thus does not guarantee a solution will be found. If the initial decision rule happens to be far from the final solution, the algorithm is likely to diverge.

The usual approach to systematically find a good initial condition for PEA is based on the principles of homotopy (see, for example, den Haan and Marcet, 1990). The basic idea behind homotopy is to slowly move from a simple case, where the solution in known or easy to compute, to the desired case where the solution is difficult to solve for and typically unknown. As long as the intermediate versions of the model are continuous with respect to the parameter/s that drives the model from the known to the desired solution, one would always be solving models with appropriate initial conditions. In this way, only local stability of the algorithm that solves for the fixed point is needed. For an introduction to homotopy theory see Garcia and Zangwill (1981); Eaves and Schmedders (1999) provide an introduction to using homotopies in economics. Nevertheless, depending on the complexity of the model at hand, this type of initialization methods may be of difficult and/or cumbersome application. Fast and accurate homotopy alternatives to initialize the PEA on the basis of log-linear approximations are given by Christiano and Fisher (2000), and Pérez (2004).

As an alternative to homotopy methods, Maliar and Maliar (2003) proposed an appeal-

ing modification of PEA based on restricting the simulated series within certain bounds. Their intuitive idea was to rule out the possibility of (ex)implosive behavior by artificially restricting the simulated series within certain bounds. As the solution is refined along the iterations, the bounds are gradually removed. The authors claim that: "Introducing the moving bounds resolves the problem of finding a good initial guess in the sense that the modified PEA is able to converge even if the initial guess is not very accurate" (p. 88). In particular, Maliar and Maliar solve the basic neoclassical growth model and provide two simulations to illustrate that the modified PEA can find the stochastic solution starting from the nonstochastic steady state. Nevertheless, the authors do not provide any systematic or theoretical assessment of the convergence properties of their algorithm.

The aim of this note is to study the convergence properties of the modification of PEA proposed by Maliar and Maliar (2003). Upon the basis of a Monte Carlo experiment, we analyze the robustness of the statement that starting from a poor initial guess would lead to the fixed point solution under the moving bounds approach. We also check in a systematic way its speed of convergence. We have chosen two models to frame the discussion: the simple neoclassical growth model, and the Cooley and Hansen (1989) model, that adds to the previous model a non convexity, indivisible labor, and introduces money via a cash-in-advance constraint in consumption. In our experiment we solve both models by PEA under Maliar and Maliar's initialization scheme, and also under two alternative schemes based on the principles of homotopy.

In the remainder of the paper we describe the PEA method in Marcet and Marshall (1994) and the modification by Maliar and Maliar (2003), provide the framework for evaluating the latter proposal, and provide illustrative results on its convergence properties.

2 PEA and the moving bounds

Consider an economy which is described by a vector of n variables, z_t, and a vector of s exogenously given shocks, u_t. Let the process $\{z_t, u_t\}$ be represented by a system

$$g(E_t[\phi(z_{t+1}, z_t)], z_t, z_{t-1}, u_t) = 0, \quad \text{for all} \;\; t \tag{1}$$

where $g : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^q$ and $\phi : \mathbb{R}^{2n} \to \mathbb{R}^m$; the vector z_t includes all

endogenous and exogenous variables that are inside the expectation, and u_t follows a first-order Markov process. It is assumed that z_t is uniquely determined by (1) if the rest of arguments are given. The functions $g(\cdot)$ and $\phi(\cdot)$ are known functions once the structural parameters of the economy are fixed. Alternatively, let the solution be expressed as a law of motion h such that the vector z_t generated by

$$z_t = h(z_{t-1}, u_t) \tag{2}$$

fulfills (1), given that all past information relevant to forecast $\phi(z_{t+1}, z_t)$ can be summarized in a finite-dimension function of $\{z_{t-1}, u_t\}$.

Obtaining a solution to (1) using PEA consists of finding a parametric function $\psi(\beta; z_{t-1}, u_t)$, such that for a positive integer ν, $\beta \in \mathbb{D}^\nu$, where $\mathbb{D}^\nu \subset \{\beta \in \mathbb{R}^\infty : i\text{th element of } q \text{ is zero if } i > \nu\}$, the process $\{z_t(\beta)\}$ satisfies for all t the set of equations

$$g(\psi(\beta; z_{t-1}, u_t), z_t(\beta), z_{t-1}(\beta), u_t) = 0 \tag{3}$$

and the order of ν is such that when solving $G(\beta) = \arg\min_{\beta \in \mathbb{D}^\nu} E_t[\phi(z_{t+1}(\beta), z_t(\beta)) - \psi(\beta; z_{t-1}(\beta), u_t)]^2$, then $\beta = G(\beta)$. Given these conditions, the stochastic process $\{z_t(\beta)\}$ is the PEA approximated solution. Under certain regularity conditions over the functions defining the equilibrium (1), the function $g(\cdot)$ is invertible in its second argument, and equation (3) can be written as (see Marcet and Marshall, 1994)

$$z_t(\beta) = h_\beta(z_{t-1}(\beta), u_t) \tag{4}$$

for stationary and ergodic processes. Marcet and Marshall (1994) show that under those regularity conditions, fulfilled by standard business cycle models, it is always possible to find an approximated function $h_\beta(\cdot)$ arbitrarily close to the true law of motion of the system $h(\cdot)$. Under the true law of motion $h(z_{t-1}, u_t)$, the true process $\{z_t, u_t\}_{t=-\infty}^{\infty}$ verifying (1) is stationary. For the approximation to be acceptable, it is necessary that, given initial conditions $\{z_0, u_0\}$ and an initial vector β, the resulting process $\{z_t(\beta)\}_{t=1}^T$ verifying (3) has to be stationary.

In order to achieve stationarity in $\{z_t(\beta)\}_{t=1}^T$ starting for an arbitrary initial value for β, Maliar and Maliar (2003) bounds artificially the solution to (3) to induce the stationarity

of possibly (ex)implosive simulated series by not allowing such series to go beyond a fixed range $\underline{z} < z_t(\beta) < \overline{z}$. The range becomes irrelevant as the number of iterations increases. The PEA algorithm as modified by Maliar and Maliar (2003) can be written as follows:

- Step 1. Fix upper and lower bounds, $\underline{z}$ and $\overline{z}$ for the process $\{z_t(\beta)\}$. For an initial iteration $i = 0$ fix $\beta = \beta(0)$. Fix initial conditions u_0 and z_0; draw and fix a random series $\{u_t\}_{t=1}^T$ from a given definition. Replace the conditional expectation in (1) with a function $\psi(\beta; z_{t-1}, u_t)$ and compute $z_t(\beta)$ from (4).

- Step 2. For a given $\beta \in \mathbb{D}^\nu$ recursively calculate $\{z_t(\beta)\}_{t=1}^T$ according to

$$\begin{aligned} z_t(\beta) &= \underline{z}, && \text{if } z_t(\beta) < \underline{z} \\ z_t(\beta) &= \overline{z}, && \text{if } z_t(\beta) \geq \overline{z} \\ z_t(\beta) &= h_\beta(z_{t-1}(\beta), u_t), && \text{if } \underline{z} < z_t(\beta) < \overline{z} \end{aligned}$$

- Step 3. Find a $G(\beta)$ that satisfies $G(\beta) = \arg\min_{\xi \in \mathbb{D}^\nu} E_t[\phi(z_{t+1}(\beta), z_t(\beta)) - \psi(\xi; z_{t-1}(\beta), u_t)]^2$. In order to perform this step, one can run a nonlinear least squares regression with the sample $\{z_t(\beta), u_t\}$, taking $\phi(z_{t+1}(\beta), z_t(\beta))$ as a dependent variable, $\psi(\cdot)$ as an explanatory function, and ξ as a parameter vector to be estimated.

- Compute the vector $\beta(i+1)$ for the next iteration,

$$\beta(i+1) = (1-\lambda)\beta(i) + \lambda G(\beta(i)), \quad \lambda \in (0,1)$$

- Step 5. Compute $\underline{z}(i+1)$ and $\overline{z}(i+1)$ for the next iteration,

$$\begin{aligned} \underline{z}(i+1) &= \underline{z}(i) - \underline{\Delta}(i) \\ \overline{z}(i+1) &= \overline{z}(i) + \overline{\Delta}(i) \end{aligned}$$

  where $\underline{\Delta}(i)$ and $\overline{\Delta}(i)$ are the corresponding steps.

Iterate on Steps $2-5$ until $\|\beta(i+1) - \beta(i)\|$ is below a certain tolerance value, and $\underline{z} < z_t(\beta(i+1)) < \overline{z}$, $\forall t$.

3 The models

As stated before, we have selected two models to frame the discussion: the one-sector stochastic growth model and the Cooley and Hansen (1989) model. Consider firstly the one-sector stochastic growth model,

$$\max_{\{c_t,k_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \delta^t \frac{c_t^{1-\gamma}-1}{1-\gamma}, \qquad s.t. \quad c_t + k_t = (1-d)k_{t-1} + \theta_t k_{t-1}^{\alpha},$$

where $\log\theta_t = \rho\log\theta_{t-1} + \epsilon_t$ with $\epsilon_t \backsim N(0,\sigma^2)$, the initial condition (k_{-1},θ_0) is given. c_t is consumption at time t, k_{t-1} the beginning of period t capital stock, $0<\delta<1$ is the subjective discount factor, $0<\alpha<1$ the capital share in production, $0<d<1$ the depreciation rate, and $0<\rho<1$. But for the case with logarithmic utility, $\gamma=1$, and full depreciation of capital, $d=1$, a closed-form solution to this model is not known. Following den Haan and Marcet(1990), we approximate the conditional expectation by

$$E_t[c_{t+1}^{-\gamma}(1-d+\alpha\theta_{t+1}k_{t-1}^{\alpha})] \approxeq \exp(\beta_0+\beta_1\log\theta_t+\beta_2\log k_{t-1})$$

where $\beta=(\beta_0,\beta_1,\beta_2)$ is a vector of coefficients to be found. To simulate the model, parameter values are fixed as: $\alpha=0.33$, $\delta=0.95$, $\gamma=1$, $d=0.02$, $\rho=0.95$, $\sigma=0.01$, $k_{-1}=k_{ss}$ and $\theta_0=1$. The moving-bound parameters are defined by Maliar and Maliar (2003) as:

$$\begin{aligned} \underline{k}(i) &= k_{ss}\exp(-ai), \\ \overline{k}(i) &= k_{ss}(2-\exp(-ai)) \end{aligned} \tag{5}$$

where $a>0$ determines the path at which bounds are moved, i is the number of iterations performed, and the variables with the subscript ss are the steady-state values. Under this choice, in the first iteration ($i=0$), both bounds coincide with the steady-state solution. On the subsequent iterations, the lower and upper bounds gradually move, approaching 0 and $2k_{ss}$, respectively.

The model given by Cooley and Hansen (1989) is a bit more complex in that it includes a non-convexity, indivisible labor. Money is introduced via a cash-in-advance constraint in consumption. The competitive equilibrium is non-Pareto-optimal, and the second welfare theorem does not apply. The representative firm solves a standard profit maximization problem, while households seek to maximize their time preferences subject to their holdings

of money balances and a set of standard budget constraints. There are two sources of uncertainty in this economy: the autoregressive shock to technology, θ_t, and an autoregressive logged money growth rate, $\log(g_t) = (1-\rho_g)\log(g_{ss}) + \rho_g \log(g_{t-1}) + \epsilon_{g_t}$. In equilibrium, there is one first order condition involving expectations that cannot be solved for analytically. den Haan and Marcet(1994)'s preferred specification for the approximating function $\psi(\cdot)$ to the expectation term is a third-order polynomial such that,

$$\begin{aligned} E_t[\mu_{t+1}(\alpha\theta_{t+1}k_t^{\alpha-1}N_{t+1}^{1-\alpha} + 1 - d)] \approxeq \quad & \exp\big(\ \beta_0 + \beta_1 \log k_{t-1} + \beta_2 \log\theta_t + \beta_3 \log g_t \\ & + \beta_4(\log k_{t-1})^2 + \beta_5 \log k_{t-1}\log\theta_t + \beta_6(\log\theta_t)^2 \\ & + \beta_7(\log\theta_t)^3\ \big) \end{aligned} \tag{6}$$

where μ_t is the Lagrange multiplier attached to household's budget constraint, and N_t denote hours worked. Following den Haan and Marcet, we will adopt as baseline parameterization: $\delta = 0.99$, $\alpha = 0.36$, $A_N = 2.86$, $\rho = 0.95$, $\rho_g = 0.48$, $\sigma = 0.00721$, $\sigma_{\epsilon_g} = 0.009$, $g_{ss} = 1.15$, and $d = 0.025$. The moving-bound parameters are defined as in (5).

4 Alternative initialization methods

The first alternative we pose in order to evaluate the convergence properties of the Moving Bounds method is based on a log-linearization of the necessary equations characterizing the equilibrium of the system. Let us denote a log-linear approximation to (1) as,

$$\hat{g}(E_t[\phi(\hat{z}_{t+1}, \hat{z}_t)], \hat{z}_t, \hat{z}_{t-1}, u_t) = 0, \quad \text{for all } \ t \tag{7}$$

where $\hat{z}$, denotes the log-linearized counterpart of z, and $\hat{g}(\cdot)$ is a log-linear function approximating $g(\cdot)$. The stable solution to (7) can be obtained in a standard way by solving for the desired recursive equilibrium law of motion

$$\hat{z}_t(\beta) = \hat{h}(\hat{z}_{t-1}(\beta), u_t) \tag{8}$$

Solving for the stable manifold of the system forces the transversality conditions to hold. This, in turn, is a necessary and sufficient condition for the stationarity of the solution. Basing the obtention of the initial conditions for the non-linear model in a log-linear version about the deterministic steady state of the very model, implicitly makes use of the ideas

of homotopy. The log-linear approximation is a local counterpart of the model about the steady state, and at least locally should be close to the nonlinear model. As described in Pérez (2004), one could exploit the parallelism between the set of PEA-approximated first order conditions in (3) and the log-linear first order conditions in (7), or between the PEA-law of motion in (4) and the log-linear law of motion in (8). The latter alternative is close to that suggested by Christiano and Fisher (2000).

We follow the first approach, for it is more generally applicable and presents better convergence properties (see Pérez, 2004). The method consists of evaluating (3) for $\hat{z}_t$, and then estimating the approximated $g_\beta(\cdot)$ and implied $h_\beta(\cdot)$ by means of nonlinear regressions. Given regressions are run between stationary variables, the resulting estimated β parameters have a stationary distribution. On other grounds, if T is long enough, the potential multicollinearity problems that might arise are kept to a minimum. Consider the Cooley and Hansen model. From a log-linear approximation one can define the variable $\hat{\psi}_t \equiv \hat{\mu}_{t+1}(\alpha\theta_{t+1}\hat{k}_t^{\alpha-1}\hat{N}_{t+1}^{1-\alpha} + 1 - d)$, and then, upon the basis of the specification of the functional form of the PEA approximating function in (6) run the nonlinear regression

$$\begin{aligned}\hat{\psi}_t = \quad & \exp(\ \varpi_0 + \rho_1 \log \hat{k}_{t-1} + \varpi_2 \log \theta_t + \varpi_3 \log g_t \\ & + \varpi_4 (\log \hat{k}_{t-1})^2 + \varpi_5 \log \hat{k}_{t-1} \log \theta_t + \varpi_6 (\log \theta_t)^2 + \varpi_7 (\log \theta_t)^3 + \varepsilon_t\ )\end{aligned}$$

The estimation of the ϖ parameters from the preceding regression would give a good starting point for the β coefficients needed to initialize PEA, as standard results in regression analysis guarantee a stationary distribution for the estimated vector of coefficients ϖ.

The second alternative we select starts the homotopy from initial conditions for β taken from solutions available from published papers for slightly different calibrations of the two selected models. In this sense, we get the PEA solution for the neoclassical model given by Marcet and Lorenzoni (1999) for the case with $d = 0.00$ as the initial point for the baseline case discussed above. For the Cooley and Hansen model we take the fixed point solution given by den Haan and Marcet (1994) for the case with $g_{ss} = 1.015$, to compute the solution for the baseline case with $g_{ss} = 1.15$.

5 Results

We solve the two models described above with PEA, using as alternative initialization methods Maliar and Maliar (2003) and the two homotopy methods described in the previous Section. We solve for 250 independent draws of the exogenous processes of size $T = 5000$ in each case (keeping the same shock for all alternatives in each simulation). The convergence criterion used is that the L^2 distance between vectors β obtained in two subsequent iterations is less than 10^{-5}.

For the simple neoclassical model we set $\lambda = 1.0$ and $a = 0.007$, following Maliar and Maliar's advice. For the Cooley and Hansen model, and in order to achieve convergence we set $\lambda = 0.3$ (with higher values of λ the solution algorithm diverged for the selected baseline parameterization and Maliar and Maliar's basic implementation) and $a = 0.0035$, which corresponds approximately to having $\underline{k} = 0.5z_{ss}$ and $\overline{k} = 1.5z_{ss}$ after 200 iterations. It is worth noticing that the two homotopy alternatives always found a fixed point solution with $\lambda = 1$, so that setting $\lambda = 0.3$ for the sake of comparison with Maliar and Maliar implies reducing substantially their speed of convergence.

The computational results of the experiment are shown in Table 1, where we present some summary statistics for the 250 simulations: average computational time to the fixed point solution; time to convergence of the simulation showing the maximum and minimum time; standard deviation of the times to convergence across simulations; percentage of simulations in which PEA failed to achieve a fixed point solution. `MATLAB` codes for solving the two selected models with the selected initialization alternatives are available from the authors upon request.

The main results are as follows. In terms of total computational time, the PEA solution based on the initialization under scrutiny took between two and two and a half times more in the simple neoclassical model than the alternatives, and almost three and a half times in the Cooley and Hansen case. Regarding the stability of the solutions, with both models the maximum and minimum time per simulation was around two to four times that of the typical simulation with the homotopy alternatives. The same picture emerges when the standard deviation of the time to convergence of the 250 simulations is considered. It is

Table 1: Computational results. Summary statistics for 250 simulations, each of size T=5000. Figures are expressed in relative terms (ratio to the method scoring minimum time) unless otherwise stated.

	Maliar and Maliar	Homotopy alternatives	
		Estimated Log-linear	Based on d / g_{ss}
One-sector stochastic growth model			
Average computational time	2.649	1.000	1.378
Maximum time simulation	2.835	1.000	1.407
Minimum time simulation	2.526	1.000	1.372
Standard deviation	4.412	1.000	1.415
Convergence failure (%)	1.2%	0.0%	0.0%
Cooley and Hansen (1989) model			
Average computational time	3.449	1.000	1.409
Maximum time simulation	3.918	1.000	1.306
Minimum time simulation	3.907	1.000	1.448
Standard deviation	3.907	1.000	1.245
Convergence failure (%)	83.6%	0.0%	0.0%

worth mentioning that for all summary statistics the estimated log-linear method presented the best numerical performance.

For almost all shocks (but 1.2% of them) Maliar and Maliar's algorithm was able to make PEA converge to the rational expectations equilibrium in the case of the one-sector neoclassical model. Nevertheless, in the case of the Cooley and Hansen model only 15% of the simulations converged. Indeed, this is in the case when we use a slight modification of Maliar and Maliar's baseline scheme where the moving bounds update was corrected to be contingent on the number of censored points in the previous iteration; with the baseline implementation the number of converged simulations remained around 5% of the total (remember: with $\lambda = 0.3$ and $a = 0.0035$).

6 Conclusions

In this paper we analyze the convergence properties of the moving bounds algorithm to initialize the Parameterized Expectations Algorithm suggested by Maliar and Maliar (2003). We carry out a Monte Carlo experiment to check its performance against some initialization alternatives based on homotopy principles. We do so by solving two standard neoclassical growth models. We show that: (i) speed of convergence is poor as compared to alternatives; (ii) starting from a not very accurate initial guess might prevent convergence in relatively simple models.

These results signal the need to fine tune Maliar and Maliar's basic alternative. For example, by updating the moving bounds according to the number of censored points in each iteration, there is an improvement in the number of converged simulations. At the same time, this modification allowed the detection of a divergent solution significantly earlier than with the standard procedure (2.5 to 1). Another fine tuning alternative that our results suggest would be the combination of a log-linear initialization scheme (that has good local properties and is quite automatic to implement) and Maliar and Maliar's approach, by starting the latter from the log-linear alternative, instead of the deterministic steady state; this alternative could be suitable for initializing PEA for solving highly nonlinear models.

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