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MANAGEMENT OPTIMIZATION PROBLEMS ON FUZZY GRAPHS

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ABSTRACT

In applied economics on different levels of management (of a technological process, an enterprise, region, etc.) very often there arise optimization problems that are naturally interpreted in graph terminology. Due to various reasons mainly associated with the necessity to take into account insufficient definiteness such graphs may be treated as fuzzy ones. For example, along with different metric characteristics (distance, capacity, etc.) it is possible to set up a correspondence between edges and/or vertices and such ill-measurable characteristics as reliability, stability and other values subjective by their nature. Metric characteristics can also be interpreted in terms of fuzzy mathematics (e.g. a correspondence can be set up between lower/upper capacity and the interval thus leading to the so-called interval arithmetics).

Keywords: Fuzzy graphs, Optimization problems.

INTRODUCTION

Voluminous literature is dedicated to different optimization problems on graphs in partially or even really indistinct definition. For example, a number of problems dealing with the arrangement of service centres with weights on vertices, construction of cycles in graphs with double weights of edges are considered in [1]. In [2] for the first time the notion of a fuzzy graph is introduced, some optimization problems close to similar problems for usual graphs are formulated and solved. Certain problems regarding dynamic programming with step-wise optimization of management for graphs where a correspondence is set up between edges and confidence intervals, threes are considered and transitive closures of fuzzy graphs are studied in [3].

Nevertheless, the current state of the art in the range of problems under consideration may be thought of as not quite satisfactory. The point is that in optimization problem statements the practical interpretation is of paramount importance. The formal aspect of a problem is also important but it is very bad if a problem loses meaning when passing from fuzzy to usual (Kantor's) sets. As an example we refer to the notion of the strongest [2] or the most stable [3] path in a graph. The notion is merely absurd for usual graphs because no algorithms for calculating the corresponding characteristics are needed. Generally speaking, the problem deals with the methodological aspect of relation between non-fuzzy and fuzzy mathematics, but so far there is not a unified point of view regarding this methodology. The way out is usually found either in well made subject interpretation or in the formalization that is based on classical problem statements for usual graphs and that contains solutions of the graphs as special cases. The fact that such formalization is possible and constructive is shown below. In this case we deliberately narrow the generality of construction because it is beyond the scope of our paper.

First, we introduce some definitions and notation borrowed from [1,2]. By $\mathbf{G} = (\mathbf{X}, \mathbf{V})$, as it usually is, we denote a graph ($\mathbf{X}$ - vertices, $\mathbf{V}$ - edges). For simplicity of further presentation we assume that $\mathbf{G}$ is an undirected, connected, finite and weighted graph. And to avoid consideration of cycles of a negative summarized value we restrict ourselves to graphs with non-negative matrices of weights $\mathbf{D} = [d_{ij}]$ i.e. $d_{ij} > 0$ for all vertices $x_i, x_j \in \mathbf{X}$. Graph $\mathbf{G}$ can be transformed into a fuzzy one by various means. In principle it can be achieved by one of the following three ways or by any permissible combination of them:

1. A set of vertices $\mathbf{X}$ may be associated with the function of belonging $\mu_{\mathbf{X}}: \mathbf{X} \to [0,1]$. We obtain a fuzzy set of vertices $\tilde{\mathbf{X}} = \{(x_1, \mu_{\mathbf{X}}(x_1)), \ldots, (x_n, \mu_{\mathbf{X}}(x_n))\}$;

2. A set of edges $\mathbf{V}$ (as relation of adjacency on $\mathbf{X} \times \mathbf{X}$) may be also associated with the function of belonging $\mu_{\mathbf{V}}: \mathbf{X} \times \mathbf{X} \to [0,1]$. Let us denote a fuzzy set of edges by $\tilde{\mathbf{V}} = \{(v_{ij}, \mu_{\mathbf{V}}(v_{ij})) \mid i, j \in \{1, \ldots, |\mathbf{X}|\}\}$;

3. Each number of weight matrix $\mathbf{D}$ can be made a fuzzy one by the use of the function of belonging $\mu_{\mathbf{D}}: \mathbf{D} \to [0,1]$. Let us denote the corresponding weight matrix by $\tilde{\mathbf{D}}$.

There are various interesting interpretations of fuzzy graphs $\tilde{\mathbf{G}}$, obtained by each of the above-listed ways and their combinations [2,3]. But we restrict ourselves to the study of a fuzzy graph $\tilde{\mathbf{G}} = (\mathbf{X}, \tilde{\mathbf{V}})$ with a usual weight matrix $\mathbf{D}$. The point is that graphs $\tilde{\mathbf{G}}$ with matrices $\tilde{\mathbf{D}}$ are considered in some detail in [3]. There the authors point out a pathological heuristic features of the corresponding algorithms due to partial definiteness of comparison operations on a set of fuzzy arithmetic numbers. That is why such graphs are not of much interes from the point of view of problem statement. Besides, too much effort should be made for calculating the function of belonging that corresponds to the result and it is difficult to justify that fact taking into account the above-mentioned heuristic features of comparison operation. We do not also consider graphs $\tilde{\mathbf{G}}$ with fuzzy sets of vertices $\tilde{\mathbf{X}}$ but due to other reasons. First, they have been studied in [1] in quite a different context. Second, as it will become clear from the presentation that follows, function $\mu_{\mathbf{X}}$ can be easily taken into account in statements that we propose. In an algorithmic sense its use does not most likely change anything. Notice that precisely for graphs $\tilde{\mathbf{G}} = (\mathbf{X}, \tilde{\mathbf{V}})$ (but for unweighted ones) concepts are introduced in [2,3] that we will use for further considerations.

So, assume that there is a fuzzy graph $\tilde{\mathbf{G}} = (\mathbf{X}, \tilde{\mathbf{V}}), |\mathbf{X}| = n$. To define function $\mu_{\mathbf{X}}$ we will use a matrix $\mathbf{I}_{\tilde{\mathbf{G}}} = [\mu_{\mathbf{V}}(v_{ij})]$ of size nxn similar to adjacency matrix. Due to undirected nature of graph $\tilde{\mathbf{G}}$ it would appear reasonable that matrix $\mathbf{I}_{\tilde{G}}$ is a symmetric one, i.e. $\mu_V(v_{ij}) = \mu_V(v_{ji})$ for all $i, j \in \{1, \dots n\}$. Along with $\tilde{\mathbf{G}}$ we will consider subgraphs $\tilde{\mathbf{G}}' = (\mathbf{X}', \tilde{\mathbf{V}}')$ where $\mathbf{X}' \subseteq \mathbf{X}$ while $\tilde{\mathbf{V}}' \subseteq \tilde{\mathbf{V}}$ and $\tilde{\mathbf{V}}'$ do not contain edges adjacent to vertices from $\mathbf{X} \setminus \mathbf{X}'$. Each subgraph $\tilde{\mathbf{G}}'$ can be associated with several numerical characteristics:

- weight of the subgraph $\omega(\mathbf{G}') = \sum_{v_{ij} \in \tilde{\mathbf{V}}'} d_{ij}$ (where d_{ij} - weight of edge v_{ij});

- value of the subgraph $l(\tilde{\mathbf{G}}') = |\mathbf{V}'|$;

- stability [3] of the subgraph $\mu(\mathbf{G}') = \min_{v_{ij} \in \tilde{\mathbf{V}}'} \{\mu_{\mathbf{V}}(v_{ij})\}$

Let us now fix the vertex $x_i \in \mathbf{X}$ and consider a random path $\mathbf{P}_k(x_i, x_j) = (x_{i_1}^k, \dots, x_{i_s}^k)$ where $x_{i_1}^k = x_i$, $x_{i_s}^k = x_j$, k passes through a set $\mathbf{N}(x_i, x_j) \in \mathbf{N}$ characterizing the number of paths from X_i to X_j. It is evident that a definite connected subgraph $\tilde{G}'$ corresponds to each path and that is why the numeric characteristics introduced above can also be calculated for the path $\mathbf{P}_k(x_i, x_j)$. Let us denote them by $\omega(\mathbf{P}_k(x_i, x_j))$, $l(\mathbf{P}_k(x_i, x_j))$ and $\mu(\mathbf{P}_k(x_i, x_j))$ respectively. There can be more than one path (if $\tilde{\mathbf{G}}$ is not a tree) in accordance with the assumptions with respect to for vertices $x_i, x_j \in \mathbf{X}$. And it is natural to single out peculiar paths from the whole set $\{\mathbf{P}_k(x_i, x_j) \mid k \in \mathbf{N}(x_i, x_j)\}$. For example, in [3] the authors introduce the notion of the most stable path $\mathbf{P}_{k_0}(x_i, x_j)$:

$$\mu(\mathbf{P}_{k_0}(x_i, x_j)) = \max \{\mu(\mathbf{P}_k(x_i, x_j)) \mid k \in \mathbf{N}(x_i, x_j)\}$$

A procedure for finding such paths is also proposed there. The procedure is based on construction of transitive closure of the graph, i.e. exponentiation of matrix $\mathbf{I}_{\tilde{G}}$. Evidently, the complexity of the corresponding algorithm $\mathbf{O}(n^{n-1})$ is overstated. Besides, as mentioned above the notion of the most stable path is a non-algorithmic for usual graphs $\mathbf{G}$. Below we will show how to overcome these drawbacks.

Let us consider subgraph $\tilde{\mathbf{G}}'(x_i)$ that can be obtained by collecting vertices and edges that are in the most stable paths from x_i to all other vertices x_j. It is evident that

Assertion 1. $\tilde{\mathbf{G}}'(x_i)$ - *is a spanning tree.*

The proof of this assertion can be based on the rule of contraries.

It is also easy to prove

Assertion 2. $\mu(\tilde{\mathbf{G}}'(x_i)) = \mu(\tilde{\mathbf{G}}'(x_j))$ *for all* $i, j \in \{1, \dots, n\}$.

To prove this let us assume that the assertion is not satisfied. Then there can be two possible cases.

1. There is such $x_j \in \mathbf{X}$ that $\mu(\tilde{\mathbf{G}}'(x_j)) > \mu(\tilde{\mathbf{G}}'(x_i))$ for a fixed vertex $x_i \in \mathbf{X}$. But this means that in $\tilde{\mathbf{G}}'(x_j)$ the path from x_j to other vertices (including x_i – $\mathbf{P}_{k_1}(x_j, x_i)$) has the following feature: $\mu(\mathbf{P}_{k_1}(x_j, x_i) > \mu(\mathbf{P}_k(x_i, x_j)$ where $\mathbf{P}_k(x_i, x_j)$ is the most stable path from x_i to x_j. But it is not possible for an undirected graph.

2. The case $\mu(\tilde{\mathbf{G}}'(x_j)) < \mu(\tilde{\mathbf{G}}'(x_i))$ is considered in a similar manner.

And, finally, the following assertion holds true.

Assertion 3. *Let* $\tilde{\mathbf{G}}' = (\mathbf{X}', \tilde{\mathbf{V}}')$ *and* $\mathbf{X}' \subset \mathbf{X}$. *If* v_{st} – *is an edge connecting vertices* $x_s \in \mathbf{X}'$ *and* $x_t \in \mathbf{X} \setminus \mathbf{X}'$ *for which*

$$\mu_{\mathbf{V}}(v_{st}) = \mathbf{max}\ \{\mu(v_{fh}) \big| (x_f, x_h) \in \mathbf{X}' \times \mathbf{X} \setminus \mathbf{X}'\}$$

then for all $x_i \in \mathbf{X}'$ *the edge* v_{st} *is certain to belong to* $\tilde{\mathbf{G}}'(x_i)$.

Let us apply the rule of contraries and assume that for vertex $x_i \in \mathbf{X}'$ the spanning tree $\tilde{\mathbf{G}}'(x_i)$ contains an edge v_{fh} ($x_f \in \mathbf{X}'$ and $x_h \in \mathbf{X} \setminus \mathbf{X}'$) but not v_{st}. Let us add the edge v_{st} to $\tilde{\mathbf{G}}'(x_i)$ and exclude v_{fh}. This operation is reasonable because edges v_{st} and v_{fh} are certain to enter one cycle. As a result we obtain a new spanning tree $\tilde{\mathbf{G}}''(x_i)$ for which by the construction we have:

$$\mu(\tilde{\mathbf{G}}''(x_i)) \geq \mu(\tilde{\mathbf{G}}'(x_i)).$$

It should be noted that all the above-mentioned assertions are close by ideology to similar assertions for problems dealing with the construction of a minimal spanning tree (MST). Moreover, *assertion 3* shows that for the construction of the most stable paths from x_i to all other vertices x_j it is possible to use an algorithm for constructing MST. Such a unified algorithmic basis is a foundation for ideological combination of these problems with the help of the following functional

$$(1) \qquad \frac{\mu(\tilde{G}'(x_i))}{\omega(\tilde{G}'(x_i))} \rightarrow \mathbf{max}$$

It is apparent that problem (1) includes problems of constructing the most stable paths and MST. Really, to solve the first one it is sufficient to put weights of all edges equal to one positive number taken at random. For the solution of the second problem the similar operation should be made with values $\mu_V(v_{ij})$. In so doing the structure of the subgraph $\tilde{G}'(x_i)$ does not depend on the choice of numbers. In the general case we obtain a tree with maximal stability of paths.

Let us describe now an algorithm for the solution of problems dealing with the construction of a tree of the most stable paths and MST on the basis of the functional (1).

Algorithm.

Step 0. $\mathbf{Y} = \{x_i\}$, $\mathbf{Z} = \varnothing$, $\alpha(v_{st}) = \dfrac{\mu_V(v_{st})}{\omega(v_{st})}$ in all $v_{st} \in \mathbf{V}$.

Step 1. Choose such a vertex $x_t \in \mathbf{X} \setminus \mathbf{Y}$ that $x_s \in \mathbf{Y}$ and

$$\alpha(v_{st}) = \mathbf{max}\{v_{fh} \mid x_f \in \mathbf{Y}, x_h \in \mathbf{X} \setminus \mathbf{Y}\}.$$

Step 2. $\mathbf{Y} = \mathbf{Y} \cup x_t$, $\mathbf{Z} = \mathbf{Z} \cup v_{st}$.

Step 3. If $|\mathbf{Y}| = n$ then the algorithm ends its work and $\tilde{G}'(x_i) = (\mathbf{Y}, \mathbf{Z})$ is the graph to be found. Otherwise we go back to step 1.

Unfortunately, the algorithm does not construct trees with maximal stability of paths on the basis of functional (1). The development of such an algorithm is beyond the scope of the present paper.

Example. Let us assume that the fuzzy graph $\tilde{G} = (X, \tilde{V})$ is defined by the following matrices:

$$I_{\tilde{G}} = \begin{bmatrix} 0 & .8 & 0 & .6 & .5 \\ .8 & 0 & .5 & 0 & 0 \\ 0 & .5 & 0 & .8 & .6 \\ .6 & 0 & .8 & 0 & .4 \\ .5 & 0 & .6 & .4 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 5 & 0 & 2 & 3 \\ 5 & 0 & 3 & 0 & 0 \\ 0 & 3 & 0 & 2 & 3 \\ 2 & 0 & 2 & 0 & 4 \\ 3 & 0 & 3 & 4 & 0 \end{bmatrix}$$

Using the above-mentioned algorithm we construct in the graph a tree of the most stable paths $\tilde{G}'(x_1)$ and a minimal spanning tree.

The results are graphically shown as:

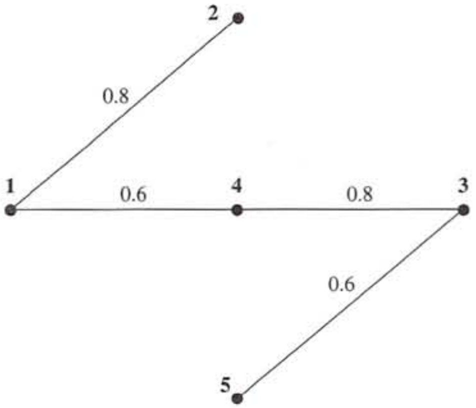

Fig.1 A tree of the most stable paths.

$\mu(\tilde{\mathbf{G}}'(x_1) = \mathbf{0.6}$

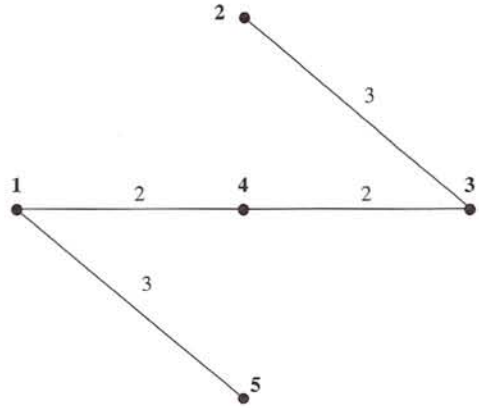

Fig.2 A minimal spanning tree.

$\omega(\tilde{\mathbf{G}}'(x_1)) = \mathbf{10}$

It is not difficult to see that the algorithm has the complexity $\mathbf{O}(n^2)$. It is hardly possible to lower the assessment even in case of constructing a tree of the most stable paths because in any step-wise procedure at each n steps it will be necessary to compare no less than $n-1$ numeric characteristics for the whole graph.

It is interesting to note that the solution of the problem dealing with the construction of a tree of the most stable paths can be obtained in terms of paths using the following functional

$$\text{(2)} \qquad \sum_{\substack{x_j \in \mathbf{X} \\ x_j \neq x_i}} \frac{\mu(\mathbf{P}_k(x_i, x_j)}{\omega(\mathbf{P}_k(x_i, x_j)) \cdot l^{-1}(\mathbf{P}_\mathbf{k}(x_i, x_j))} \rightarrow \mathbf{max}$$

Unlike the case with functional (1) weights of all edges in $\mathbf{V}$ should be put equal to **1**. But, it remains unclear whether it is possible in problem (2) to use the same algorithm or to apply methods similar to those for constructing the shortest paths in a usual graph.

And, finally, a particular case of functional (2) is also of interest

$$\text{(3)} \qquad \sum_{\substack{x_j \in \mathbf{X} \\ x_j \neq x_i}} \frac{\mu(\mathbf{P}_k(x_i x_j))}{\omega(\mathbf{P}_k(x_i, x_j))} \rightarrow \mathbf{max}$$

Evidently, if we put all values $\mu_V(v_{ij})$ equal to any positive number taken at random then problem (3) amounts to construction of the shortest paths from vertex x_i to all other $x_j \in \mathbf{X}$. On the other hand, at no values d_{ij} problem (3) is connected with a problem of constructing a tree of the most stable paths. That is why methods that form the foundation of algorithms for constructing the shortest paths should be used precisely in (3).

The present paper is not aimed at considering all possible functionals (1) - (3). We have only made an attempt to combine standard problems for usual and fuzzy graphs. And one of the functional has been considered in greater detail. Of course, there is still a problem of interpreting all functionals in terms of practical tasks because the last-named are, for all that, primary ones in optimization.

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