This is a reprint of
Lecturas Matemáticas
Volumen **25** *(2004), páginas 5–13*

The convergence of Mann iteration with errors is equivalent to the convergence of Ishikawa iteration with errors

Stefan M. Şoltuz
Cluj-Napoca, Romania

Lecturas Matemáticas
Volumen **25** *(2004), páginas 5–13*

The convergence of Mann iteration with errors is equivalent to the convergence of Ishikawa iteration with errors

Ştefan M. Şoltuz
Cluj-Napoca, Romania

Abstract. We prove that the convergence of Mann iteration with errors and the convergence of Ishikawa iteration with errors are equivalent when dealing with Lipschitzian strongly pseudocontractive maps.

Key words and phrases. Mann iteration with errors, Ishikawa iteration with errors, strongly pseudocontraction, strongly accretive and accretive.

1991 Mathematics Subject Classification. Primary (2000): 47H10.

Resumen. Se demuestra que en un espacio de Banach real la convergencia de la iteración de Mann con errores es equivalente la convergencia de la iteración de Ishikawa con errores cuando se trabaja con aplicaciones lipschitzianas fuertemente seudocontractivas, fuertemente acretivas y acretivas.

1. Preliminaries

Let X be a real Banach space and B be a nonempty, convex subset of X. Let $T : B \to X$ be a map and $u_1, x_1 \in B$. Let us denote by $F(T)$

tne fixed points of T. We consider the iteration (see [10]):

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n T u_n + e_n. \tag{M}$$

The sequence $(\alpha_n)_n \subset (0,1)$, is such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. A prototype is $\alpha_n = 1/n$, $n \geq 1$. The errors $(e_n) \subset X$ satisfy $\sum_{n=1}^{\infty} \|e_n\| < \infty$. This iteration is known as *Mann iteration with errors.* We consider the following iteration (see [10]):

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n + p_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n + q_n, \quad n = 1, 2, \ldots . \end{aligned} \tag{I}$$

The sequences (α_n), $(\beta_n) \subset (0,1)$, are such that $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Furthermore, the sequence (α_n) is the same (M) and (I). For $\beta_n = 0$, $n \in N$ we get Mann iteration with errors. The errors (p_n) and $(q_n) \subset X$ satisfy $\sum_{n=1}^{\infty} \|p_n\| < \infty$, $\sum_{n=1}^{\infty} \|q_n\| < \infty$. When $p_n = q_n = 0$, $n \in N$, we deal with Ishikawa iteration (see [7]).

Mann and Ishikawa iterations with errors (M) and (I) were criticized in [17]. The main argument was that the errors should be bounded sequences without being necessary convergent to zero. New iteration types were introduced in [17]. According to [5], these new iterations are redundant (i.e. lead us to (M) and (I)), when the operator T has a bounded range.

In [12] an example can be found in which iteration (I) but not (M) converges. The map T is a Lipschitz pseudocontraction, without being strongly pseudocontractive. In this note we prove that the convergence of Mann iteration is actually equivalent to the convergence of Ishikawa iteration when dealing with Lipschitz strong pseudocontractions.

The following definition is in [9].

Definition 1. *Let X be a real Banach space and let B be a nonempty subset of X. A map $T : B \to B$ is called strongly pseudocontractive if there exists $k \in (0,1)$ such that, denoting with I the identity map of X, we have*

$$\|x - y\| \leq \|x - y + r[(I - T - kI)x - (I - T - kI)y]\|, \qquad (1)$$

for all $x, y \in B$, and all $r > 0$.

The following lemma is in [10].

Lemma 2. *Let (a_n) be a nonnegative sequence satisfying*

$$a_{n+1} \leq (1 - \lambda_n)a_n + \sigma_n + c_n, \qquad (2)$$

where $\lambda_n \in (0,1)$, $\sigma_n = \varepsilon_n \lambda_n$, and $\varepsilon_n \geq 0$, for all $n \in N$, $\lim_{n\to\infty} \varepsilon_n = 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

2. The convergence of Mann iteration with errors is equivalent to the convergence of Ishikawa iteration with errors

We are now able to establish our the main result:

Theorem 3. *Let X be a Banach space, B be a nonempty, bounded, convex and closed subset of X, and $T : B \to X$ be a Lipschitzian, with $L > 1$ strongly pseudocontractive map with $T(B)$ bounded and $F(T) \neq \emptyset$. If $u_1 = x_1 \in B$, then the following assertions are equivalent:*

(i) *Mann iteration with errors (M) converges to $x^* \in F(T)$.*
(ii) *Ishikawa iteration with errors (I) converges to $x^* \in F(T)$.*

Proof. The proof is similar to the proof of Theorem 4 from [15]. From (M) it follows that

$$\begin{aligned}
u_n &= u_{n+1} + \alpha_n u_n - \alpha_n T u_n - e_n \\
&= (1+\alpha_n)u_{n+1} + \alpha_n(I - T - kI)u_{n+1} \\
&- (2-k)\alpha_n x_{un+1} + \alpha_n u_n + \alpha_n(Tu_{n+1} - Tu_n) - e_n \\
&= (1+\alpha_n)u_{n+1} + \alpha_n(I - T - kI)u_{n+1} \\
&- (2-k)\alpha_n[(1-\alpha_n)u_n + \alpha_n T u_n] + \alpha_n u_n + \alpha_n(Tu_{n+1} - Tu_n) - e_n \\
&= (1+\alpha_n)u_{n+1} + \alpha_n(I - T - kI)u_{n+1} \\
&- (1-k)\alpha_n u_n + (2-k)\alpha_n^2(u_n - Tu_n) + \alpha_n(Tu_{n+1} - Tu_n) - e_n.
\end{aligned}$$

Therefore

$$\begin{aligned}
u_n &= (1+\alpha_n)u_{n+1} + \alpha_n(I - T - kI)u_{n+1} \\
&\quad - (1-k)\alpha_n u_n + (2-k)\alpha_n^2(u_n - Tu_n) + \alpha_n(Tu_{n+1} - Tu_n) - e_n.
\end{aligned} \tag{3}$$

Similarly, from (I) we obtain

$$\begin{aligned}
x_n &= x_{n+1} + \alpha_n x_n - \alpha_n T y_n - p_n \\
&= (1+\alpha_n)x_{n+1} + \alpha_n(I - T - kI)x_{n+1} \\
&\quad - (2-k)\alpha_n x_{n+1} + \alpha_n x_n + \alpha_n(Tx_{n+1} - Ty_n) - p_n \\
&= (1+\alpha_n)x_{n+1} + \alpha_n(I - T - kI)x_{n+1} \\
&\quad - (2-k)\alpha_n[(1-\alpha_n)x_n + \alpha_n T y_n] + \alpha_n x_n + \alpha_n(Tx_{n+1} - Ty_n) - p_n \\
&= (1+\alpha_n)x_{n+1} + \alpha_n(I - T - kI)x_{n+1} \\
&\quad - (1-k)\alpha_n x_n + (2-k)\alpha_n^2(x_n - Ty_n) + \alpha_n(Tx_{n+1} - Ty_n) - p_n\ ,
\end{aligned}$$

and therefore

$$\begin{aligned}
x_n &= (1+\alpha_n)x_{n+1} + \alpha_n(I - T - kI)x_{n+1} \\
&\quad - (1-k)\alpha_n x_n + (2-k)\alpha_n^2(x_n - Ty_n) + \alpha_n(Tx_{n+1} - Ty_n) - p_n.
\end{aligned} \tag{4}$$

From (3) and (4) we finally get

$$\begin{aligned}
x_n - u_n &= (1+\alpha_n)(x_{n+1} - u_{n+1}) \\
&\quad + \alpha_n\left((I - T - kI)x_{n+1} - (I - T - kI)u_{n+1}\right) \\
&\quad - (1-k)\alpha_n(x_n - u_n) + (2-k)\alpha_n^2(x_n - u_n - Ty_n + Tu_n) \\
&\quad + \alpha_n(Tx_{n+1} - Tu_{n+1} - Ty_n + Tu_n) + e_n - p_n.
\end{aligned} \tag{5}$$

Now,

$$\begin{aligned}
&\|(1+\alpha_n)(x_{n+1}-u_{n+1})+\alpha_n\left((I-T-kI)x_{n+1}-(I-T-kI)u_{n+1}\right)\| \\
&\quad = (1+\alpha_n)\bigg\|(x_{n+1}-u_{n+1})+ \\
&\qquad\qquad \frac{\alpha_n}{1+\alpha_n}\left((I-T-kI)x_{n+1}-(I-T-kI)u_{n+1}\right)\bigg\| ,
\end{aligned}$$

and using (1) with $x = x_{n+1}$ and $y = u_{n+1}$ produces

$$\begin{aligned}
&\|(1+\alpha_n)(x_{n+1}-u_{n+1})+\alpha_n\left((I-T-kI)x_{n+1}-(I-T-kI)u_{n+1}\right)\| \\
&\qquad \geq (1+\alpha_n)\|x_{n+1}-u_{n+1}\| \qquad (6)
\end{aligned}$$

Taking norms in (5) and then using (6) yields

$$\begin{aligned}
\|x_n-u_n\| \geq\ & \|(1+\alpha_n)(x_{n+1}-u_{n+1}) \\
& +\alpha_n\left((I-T-kI)x_{n+1}-(I-T-kI)u_{n+1}\right)\| \\
& \quad -(1-k)\alpha_n\|x_n-u_n\|-(2-k)\alpha_n^2\|x_n-u_n-Ty_n+Tu_n\| \\
& -\alpha_n\|Tx_{n+1}-Tu_{n+1}-Ty_n+Tu_n\|-\|e_n-p_n\| \\
\geq\ & (1+\alpha_n)\|x_{n+1}-u_{n+1}\|-(1-k)\alpha_n\|x_n-u_n\| \\
& \quad -(2-k)\alpha_n^2\|x_n-u_n-Ty_n+Tu_n\| \\
& \qquad -\alpha_n\|Tx_{n+1}-Tu_{n+1}-Ty_n+Tu_n\|-\|e_n-p_n\| \\
\geq\ & (1+\alpha_n)\|x_{n+1}-u_{n+1}\|-(1-k)\alpha_n\|x_n-u_n\|-(2-k)\alpha_n^2\cdot 4D \\
& \quad -\alpha_n\|Tx_{n+1}-Tu_{n+1}-Ty_n+Tu_n\|-\|e_n-p_n\| ,
\end{aligned}$$

where $D = \operatorname{diam}(B, T(B)) < \infty$ (recall that the sets B and $T(B)$ are bounded). Thus the following inequality holds

$$\begin{aligned}
(1+\alpha_n)\|x_{n+1}-u_{n+1}\| \leq\ & (1+(1-k)\alpha_n)\|x_n-u_n\|+4D(2-k)\alpha_n^2+ \\
& +\alpha_n\|Tx_{n+1}-Tu_{n+1}-Ty_n+Tu_n\|+\|e_n-p_n\| .
\end{aligned}$$

Finally, noting that $(1+\alpha_n)^{-1} \leq 1-\alpha_n+\alpha_n^2$ and $(1+\alpha_n)^{-1} \leq 1$ leads to

$$\begin{aligned}
\|x_{n+1}-u_{n+1}\| &\leq (1+(1-k)\alpha_n)(1+\alpha_n)^{-1}\|x_n-u_n\| \\
&\quad + 4D(2-k)\alpha_n^2(1+\alpha_n)^{-1} \\
&\quad + \alpha_n(1+\alpha_n)^{-1}\|Tx_{n+1}-Tu_{n+1}-Ty_n+Tu_n\| \\
&\quad + (1+\alpha_n)^{-1}\|e_n-p_n\| \\
&\leq (1+(1-k)\alpha_n)(1-\alpha_n+\alpha_n^2)\|x_n-u_n\| + 4D(2-k)\alpha_n^2 \\
&\quad + \alpha_n\|Tx_{n+1}-Tu_{n+1}-Ty_n+Tu_n\| + \|e_n-p_n\| \\
&\leq (1-k\alpha_n)\|x_n-u_n\| + 4D(2-k)\alpha_n^2 \\
&\quad + \alpha_n\|Tx_{n+1}-Tu_{n+1}-Ty_n+Tu_n\| + \|e_n-p_n\|.
\end{aligned}$$

For short,

$$\begin{aligned}
\|x_{n+1}-u_{n+1}\| &\leq (1-k\alpha_n)\|x_n-u_n\| + 4D(2-k)\alpha_n^2 + \\
&\quad + \alpha_n\|Tx_{n+1}-Tu_{n+1}-Ty_n+Tu_n\| + \|e_n-p_n\|. \qquad (7)
\end{aligned}$$

We now estimate $\|Tx_{n+1}-Tu_{n+1}-Ty_n+Tu_n\|$. Using (M) and (I), we get for some Lipschitzian constant L that

$$\begin{aligned}
&\|Tx_{n+1}-Tu_{n+1}-Ty_n+Tu_n\| \\
&\leq \|Tx_{n+1}-Ty_n\| + \|Tu_{n+1}-Tu_n\| \\
&\leq L\|x_{n+1}-y_n\| + L\|u_{n+1}-u_n\| \\
&\leq L\|(1-\alpha_n)(x_n-y_n)+\alpha_n(Ty_n-y_n)+p_n\| + L\alpha_n\|Tu_n-u_n\| \\
&\leq L(1-\alpha_n)\|x_n-y_n\| + L\alpha_n\|Ty_n-y_n\| + L\|p_n\| + L\alpha_n\|Tu_n-u_n\| \\
&\leq L\|\beta_n(Tx_n-x_n)+q_n\| + L\alpha_n\|Ty_n-y_n\| + L\|p_n\| + L\alpha_n\|Tu_n-u_n\| \\
&\leq L\beta_n\|Tx_n-x_n\| + L\|q_n\| + L\alpha_n\|Ty_n-y_n\| + L\|p_n\| + L\alpha_n\|Tu_n-u_n\| \\
&\leq 2DL\beta_n + L\|q_n\| + 2DL\alpha_n + L\|p_n\| + 2DL\alpha_n \\
&\leq L[2D(\beta_n+2\alpha_n) + \|q_n\| + \|p_n\|].
\end{aligned}$$

Using (7) yields

$$\begin{aligned}
\|x_{n+1}-u_{n+1}\| &\leq (1-k\alpha_n)\|x_n-u_n\| + 4D(2-k)\alpha_n^2 \\
&\quad + \alpha_n L[2D(\beta_n+2\alpha_n) + \|q_n\| + \|p_n\|] + \|e_n-p_n\| \ ,
\end{aligned}$$

or the same,

$$\begin{aligned}\|x_{n+1}-u_{n+1}\| \leq & (1-k\alpha_n)\|x_n-u_n\| \\ & +\alpha_n\left\{4D(2-k)\alpha_n+L[2D(\beta_n+2\alpha_n)+\|q_n\|+\|p_n\|]\right\} \\ & +\|e_n-p_n\| \ .\end{aligned}$$

Let

$$\begin{aligned} a_n &:= \|x_n-u_n\|, \\ \lambda_n &:= k\alpha_n, \\ \sigma_n &:= \alpha_n\left\{4D(2-k)\alpha_n+L[2D(\beta_n+2\alpha_n)+\|q_n\|+\|p_n\|]\right\}, \\ c_n &:= \|e_n-p_n\| . \end{aligned}$$

The conditions $\sum_{n=1}^{\infty}\|p_n\|<\infty$ and $\sum_{n=1}^{\infty}\|q_n\|<\infty$ imply that $\lim_{n\to\infty}\|p_n\|=0$ and $\lim_{n\to\infty}\|q_n\|=0$, so that $\sigma_n=o(\lambda_n)$. Also

$$\sum_{n=1}^{\infty}c_n=\sum_{n=1}^{\infty}\|e_n-p_n\|\leq\sum_{n=1}^{\infty}\|e_n\|+\sum_{n=1}^{\infty}\|p_n\|<\infty.$$

So, from relation (2) in Lemma 2 we get $\lim_{n\to\infty}a_n=0$. Hence, t

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \tag{9}$$

Now assume that $\lim_{n\to\infty}u_n=x^*$. Then

$$\|x_n-x^*\|\leq\|x_n-u_n\|+\|u_n-x^*\| \ , \to 0,(n\to\infty) \ .$$

Thus, we get $\lim_{n\to\infty}x_n=x^*$. For the converse we suppose that $\lim_{n\to\infty}x_n=x^*$. Relation (9) and the following inequality

$$\|u_n-x^*\|\leq\|x_n-u_n\|+\|x_n-x^*\|\to 0,(n\to\infty),$$

lead us to $\lim_{n\to\infty}u_n=x^*$. ☑

A map S is (strongly) accretive if and only if $I-S$) is (strongly) pseudocontractive. Using the same argumentation asi in [15] one can see that the above equivalence holds when we deal with a Lipschitz, strongly accretive and accretive map S. Cases in which we take the operators $Tx=f+(x-Sx)$ and $Tx=f-Sx$ in (M) and (I). A fixed point for the above T are solutions for $Sx = f$ and $x + Sx = f$, with f given. The assumption that $T(B)$ is bounded will change into $(I - S)(B)$ bounded for the strongly accretive case and $S(B)$ bounded for the accretive case. All our results will hold in the set-valued case, if the operators admit a selection.

References

[1] S. S. Chang, Y. J. Cho, B. S. Lee, J.S. Jung, S. M. Kang, *Iterative Approximations of Fixed Points and Solutions for Strongly Accretive and Strongly Pseudo-contractive Mappings in Banach Spaces*, J. Math. Anal. Appl. **224** (1998), 149–165.

[2] S. S. Chang, J. Y. Park, Y. J. Cho, *Iterative Approximations of Fixed Points for Asymtotically Nonexpansive Mappings in Banach Spaces*, Bull. Korean Math. Soc. **37** (2000), 109–119.

[3] C. E. Chidume, *Iterative Approximation of Fixed Points of Lipschitzian Strictly Pseudo-Contractive Mappings*, Proc. Amer. Math. Soc. **99** (1987), 283–287.

[4] C. E. Chidume, *Approximation of Fixed Points of Strongly Pseudocontractive Mappings*, Proc. Amer. Math. Soc. **120** (1994), 546–551.

[5] C. E. Chidume, C. Moore, *Steepest Descent Method for Equilibrium Points of Nonlinear Systems with Accretive Operators*, J. Math. Anal. Appl. **245** (2000), 142–160.

[6] K. Deimling, *Zeroes of Accretive Operators,* Manuscripta Math. **13** (1974), 365–374.

[7] S. Ishikawa, *Fixed Points by a New Iteration Method*, Proc. Amer. Math. Soc. **44** (1974), 147–150.

[8] T. H. Kim, H. K. Xu, *Some Hilbert Space Characterizations and Banach Space Inequalities*, Mathematical Inequalities and Applications **1** (1998), 113-121.

[9] Liwei Liu, *Approximation of Fixed Points of a Strictly Pseudocontractive Mapping*, Proc. Amer. Math. Soc. **125** (1997), 1363–1366.

[10] L.-S. Liu, I*shikawa and Mann Iterative Process with Errors for Nonlinear Strongly Accretive Mappings in Banach Spaces*, J. Math. Anal. Appl. **194** (1995), 114–125.

[11] W. R. Mann, *Mean Value in Iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[12] S. A. Mutangadura, C. E. Chidume, *An Example on the Mann Iteration Method for Lipschitz Pseudocontractions*, Proc. Amer. Math. Soc. **129** (2000), 2359–2363.

[13] J. A. Park, *Mann-iteration for Strictly Pseudocontractive Maps*, J. Korean Math. Soc. **31** (1994), 333–337.

[14] J. Y. PARK, J. UG JEONG, *Iteration Processes of Asymptotically Pseudocontractive Mappings in Banach Spaces*, Bull. Korean Math. Soc. **38** (2001):3, 611–622.

[15] B. E. RHOADES & ŞTEFAN M. ŞOLTUZ, *On the Equivalence of Mann and Ishikawa Iteration Methods*, International Journal of Mathematics and Mathematical Science **33**: 7 (2003), 451–459.

[16] X. WENG, *Fixed Point Iteration for Local Strictly Pseudocontractive Mapping*, Proc. Amer. Math. Soc. **113** (1991), 727–731.

[17] Y. XU, *Ishikawa and Mann Iterative Process with Errors for Nonlinear Strongly Accretive Operator Equations*, J. Math. Anal. Appl. **224** (1998), 91–101.

[18] HAIYUN ZHOU, YUTING JIA, *Approximation of Fixed Points of Strongly Pseudocontractive Maps without Lipschitz Assumption*, Proc. Amer. Math. Soc. **125** (1997), 1705–1709.

(Recibido en julio de 2001; versión revisada en mayo de 2003)

ŞTEFAN M. ŞOLTUZ, STR. AVRAM IANCU 13
3400 CLUJ-NAPOCA, ROMANIA
KURT SCHUMACHER 48, AP. 38, 67663 KAISERSLAUTERN, GERMANY
e-mail: soltuz@itwm.fhg.de soltuzul@yahoo.com