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A SEMICONTINUOUS CONTINUUM

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To Professor Jairo A. Charris in memoriam

Abstract. The definition of *metric bundle*, over a topological space T, requires the upper semicontinuity of the resulting function obtained when an arbitrary pair α, β of local sections is followed by the distance function, that is, the upper semicontinuity of $t \longmapsto d(\alpha(t), \beta(t))$. The assigment of such a function to each pair of sections can be considered as a generalized metric between sections. This leads to the construction of the *Bundle of Upper Semicontinuous Functions* over the space T, suitable to play the role of a *real numbers object* in the Category of Metric Bundles over T and containing, as a section, the distance between any pair of arbitrary sections of a given metric bundle over T. As desired, one of the main features of this bundle is the completeness of its fibers. In this sense, this bundle could be viewed as some sort of semicontinuous continuum.

Resumen. Se construye un campo de espacios métricos para representar por secciones las funciones semicontinuas superiormente. Las fibras del campo construido resultan ser completas y conexas.

Key words and phrases. Metric bundle, upper semicontinuous function, complete metric space, representation by sections, continuum.

Palabras claves. Campo métrico, función semicontinua superiormente, espacio métrico completo, representación por secciones, continuo.

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1. Introduction

The category of bundles of metric spaces over a topological space T is a generalization of the category of metric spaces. The goal of this paper is to construct

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an object, in the category of metric bundles over T, that could be thought as the counterpart of the *real numbers object*. The existence of metric bundles and uniform bundles is discussed in [7] and [10], the data are provided by a topological space T and a family of sections. In the category of metric bundles over a fixed base space T, there is a generalized distance between global sections whose values consist of upper semicontinuous functions. The values of this generalized distance are used as data to construct a metric bundle, called the *Bundle of Upper Semicontinuous Functions* whose fibers turn out to be complete and connected metric spaces. In this sense, this bundle could be viewed as some sort of semicontinuous continuum.

2. Preliminaries

Definition 1. Let $p : G \longrightarrow T$ be a surjective function. A *selection* for p is a function $\alpha : Q \longrightarrow G$, with $Q \subset T$, such that $p \circ \alpha$ is the identity map of Q. If $Q = T$, α is a *global selection*.

If T is a topological space and Q is an open subset of T, α is a *local selection*. When both G and T are topological spaces, a continuous selection is called a *section* for p. A set Σ of sections is called *full* if for each $u \in G$, there exists $\alpha \in \Sigma$, such that $\alpha(p(u)) = u$.

Definition 2. A function $d : G \times G \longrightarrow [0, +\infty]$ such that, for all $u,\ v,\ w\ \in G$,

(1) $p(u) \neq p(v)$, if and only if, $d(u, v) = +\infty$,
(2) $d(u, v) = 0$, if and only if, $u = v$,
(3) $d(u, v) = d(v, u)$, and
(4) $d(u, v) \leqslant d(u, w) + d(w, v)$,

is called a *metric* for p.

For each $t \in T$, $G_t := p^{-1}(t)$ is a metric space, when endowed with the restriction of d to $G_t \times G_t$.

If $\alpha : Q \longrightarrow G$ is a selection for p, $\mathcal{T}_\epsilon(\alpha) = \{u \in G : p(u) \in Dom\,\alpha$ and $d(u, \alpha(p(u))) < \epsilon\}$ is called the *ϵ-tube* around α. See Figure 1.

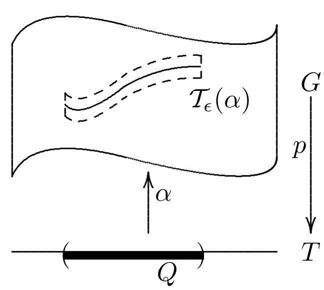

Figure 1

Definition 3. Let G and T be topological spaces, let $p : G \longrightarrow T$ be a continuous surjective function and let d be a metric for p, such that for every $u \in G$ and every $\epsilon > 0$, there is a local selection α such that $u \in \mathcal{T}_\epsilon(\alpha)$. Then (G, p, T) is called a *bundle of metric spaces*, provided that the collection of sets $\mathcal{T}_\epsilon(\alpha)$, where $\epsilon > 0$ and α runs throughout the local selections for p, is a base for the topology of G.

The space T is called the *base space*, $G_t = p^{-1}(t)$ is called the *fiber* above the point t, for each $t \in T$, and G is called the *fiber space*.

Definition 4. Given two bundles of metric spaces (G, p, T) and (H, q, T), over the same base space T, a morphism from the first one to second, is a continuous map $\Lambda : G \longrightarrow H$, such that for each $u, v \in G$, $q(\Lambda(u)) = p(u)$ and $d_H(\Lambda(u), \Lambda(v)) \leqslant d_G(u, v)$, if $p(u) = p(v)$, where d_G and d_H are the distance functions of the fiber spaces G and H respectively.

The following is a particular case of a theorem of existence of uniform bundles [7]. We sketch its proof since the cited reference is not easily accesible.

Theorem 1 (Existence Theorem of Metric Bundles)**.** Let T be a topological space, $p : G \longrightarrow T$ be a surjective function, d be a metric for p and Σ be a family of local selections for p. Assume that

(a) For every $u \in G$ and every $\epsilon > 0$, there exists $\alpha \in \Sigma$ such that $u \in \mathcal{T}_\epsilon(\alpha)$.
(b) For every $(\alpha, \beta) \in \Sigma \times \Sigma$, the function $\Phi : Dom\,\alpha \cap Dom\,\beta \longrightarrow \overline{\mathbb{R}}$, defined by $\Phi(t) = d(\alpha(t), \beta(t))$, is upper semicontinuous.

Then G can be equipped with a topology $\mathfrak{T}$ such that

(1) The family $\mathcal{B}$ of subsets of G of the form $\mathcal{T}_\epsilon(\alpha_Q)$, where $\epsilon > 0$, Q runs throughout the collection of open subsets of $Dom\,\alpha$, α throughout Σ and α_Q denotes the restriction of α to Q, is a base for $\mathfrak{T}$.
(2) Every $\alpha \in \Sigma$ is a section.
(3) (G, p, T) is a bundle of metric spaces.

Proof. We first show that the collection of all sets $\mathcal{T}_\epsilon(\alpha_Q)$, with the specifications given in conclusion (1), is a base for a topology $\mathfrak{T}$ in G.
Given two such tubes $\mathcal{T}_\epsilon(\alpha_Q)$ and $\mathcal{T}_\delta(\beta_P)$ and $u \in \mathcal{T}_\epsilon(\alpha_Q) \cap \mathcal{T}_\delta(\beta_P)$, let $\rho = \min\{\frac{1}{4}(\epsilon - d(u, \alpha(p(u)))), \frac{1}{4}(\delta - d(u, \beta(p(u))))\}$ and let $\xi \in \Sigma$ such that $u \in$

$\mathcal{T}_\rho(\xi)=\{v\in E: d(v,\xi(p(v)))<\rho\}$, then $p(u)\in\{s\in T: d(\xi(s),\alpha(s))<\epsilon'\}$, where $\epsilon'=\frac{1}{2}(d(u,\alpha(p(u)))+\epsilon)$, in fact, since $u\in\mathcal{T}_\epsilon(\alpha_Q)$ it follows that $d(u,\alpha(p(u)))<\epsilon$ and thus $d(u,\alpha(p(u)))<\frac{3}{4}d(u,\alpha(p(u)))+\frac{1}{4}\epsilon$. On the other hand, the relation $u\in\mathcal{T}_\rho(\xi)$ implies $d(u,\xi(p(u)))<\frac{1}{4}(\epsilon-d(u,\alpha(p(u))))$ and therefore $d(\xi(p(u)),\alpha(p(u)))<\epsilon'$. Similarly, $p(u)\in\{s\in T: d(\xi(s),\beta(s))<\delta'\}$ where $\delta'=\frac{1}{2}(d(u,\beta(p(u)))+\delta)$. By the semicontinuity hypothesis the sets $\{s\in T: d(\xi(s),\alpha(s))<\epsilon'\}$ and $\{s\in T: d(\xi(s),\beta(s))<\delta'\}$ are open, it follows that $S=P\cap Q\cap\{s\in T: d(\xi(s),\alpha(s))<\epsilon'\}\cap\{s\in T: d(\xi(s),\beta(s))<\delta'\}$ is a neighborhood of $p(u)$ in the space T and $\mathcal{T}_\rho(\xi_S)\subset\mathcal{T}_\epsilon(\alpha)$, indeed, the relation $v\in\mathcal{T}_\rho(\xi_S)$ implies $d(v,\xi(p(v)))<\rho<\frac{1}{2}(\epsilon-d(u,\alpha(p(u))))$, but $p(v)\in S$, then $d(\xi(p(v)),\alpha(p(v)))<\frac{1}{2}(d(u,\alpha(p(u)))+\epsilon)$, thus $d(v,\alpha(p(v)))<\epsilon$ and therefore $v\in\mathcal{T}_\epsilon(\alpha_Q)$. The inclusion $\mathcal{T}_\rho(\xi_S)\subset\mathcal{T}_\delta(\beta_P)$ is obtained in the same manner.

2) Let $\alpha\in\Sigma$ and $t\in Dom\,\alpha$. A fundamental neighborhood of $\alpha(t)$ in E is of the form $\mathcal{T}_\epsilon(\beta_Q)$, where $\beta\in\Sigma$, $Q\subset Dom\,\alpha$ is open in T, $\epsilon>0$ and $\alpha(t)\in\mathcal{T}_\epsilon(\beta_Q)$. By hypothesis b), the set $\alpha^{-1}(\mathcal{T}_\epsilon(\beta_Q))=\{s\in Q: d(\alpha(s),\beta(s))<\epsilon\}$ is open in T, therefore α is a section.

3) The tubes around arbitrary local sections are open, in fact, let $u\in G$ and let σ be a local section for p (not necessarily in Σ) such that $u\in\mathcal{T}_\epsilon(\sigma)$. To prove that (G,p,T) is a uniform bundle, we must exhibit $\eta>0$ and $\alpha\in\Sigma$ such that $u\in\mathcal{T}_\eta(\alpha)$ and $\mathcal{T}_\eta(\alpha_P)\subset\mathcal{T}_\epsilon(\sigma)$ for some neighborhood P of $p(u)$ in T.

Let $\eta=\frac{1}{4}(\epsilon-d(u,\sigma(p(u))))$ and let $\alpha\in\Sigma$ be such that $u\in\mathcal{T}_\eta(\alpha)$. Since $u\in\mathcal{T}_\epsilon(\sigma)$ we have $d(u,\sigma(p(u)))<\epsilon$ and thus $d(u,\sigma(p(u)))<\frac{3}{4}d(u,\sigma(p(u)))+\frac{1}{4}\epsilon$. On the other hand, the relation $u\in\mathcal{T}_\eta(\alpha)$ implies $d(u,\alpha(p(u)))<\eta=\frac{1}{4}(\epsilon-d_i(u,\sigma(p(u))))$, therefore $d(\sigma(p(u)),\alpha(p(u)))<\frac{1}{2}d(u,\sigma(p(u)))+\frac{1}{2}\epsilon$, then $p(u)\in\sigma^{-1}(\mathcal{T}_{\epsilon'}(\alpha))$, where $\epsilon'=\frac{1}{2}(d(u,\sigma(p(u)))+\epsilon)$. Since σ is continuous, $\sigma^{-1}(\mathcal{T}_{\epsilon'}(\alpha))$ is an open neighborhood P of $p(u)$, then $v\in\mathcal{T}_\eta(\alpha_P)$ implies $p(v)\in P$ and hence $d(\alpha(p(v)),\sigma(p(v)))<\frac{1}{2}(d(u,\sigma(p(u)))+\epsilon)$, we also have

$$d(v,\alpha(p(v)))<\eta<\frac{1}{2}(\epsilon-d(u,\sigma(p(u)))),$$

thus $d(v,\sigma(p(v)))<\epsilon$, that is $v\in\mathcal{T}_\epsilon(\alpha)$. □

The following construction does not resort to calculation of inductive limits in the category of bundles of metric spaces, alternative that although legitime, is somewhat cumbersome for the purpose of this paper.

Definition 5. Let T be a topological space and X be the set of all bounded upper semicontinuous real valued functions defined in T, endowed with the sup metric. This metric will be denoted by d. Let $t \in T$, $x, y \in X$ and denote by $\mathcal{V}(t)$ the collection of all open neighborhoods of the point t. We say that *x is related to y at the point t* or xR_ty, if and only if, the upper envelope $\overline{|x-y|}$ of the function $|x-y|$ is equal to zero at t, that is, xR_ty, if and only if, $\limsup_{s\to t}|x(s)-y(s)| = \inf_{V\in\mathcal{V}(t)}\sup_{s\in V}|x(s)-y(s)| = 0$, in symbols, $\overline{|x-y|}(t) = 0$.

With the above conventions, xR_ty, if and only if, for every $\epsilon > 0$, there exists $V \in \mathcal{V}(t)$, such that $|x(s) - y(s)| < \epsilon$, for every $s \in V$. The relation R_t is reformulated in the form made precise in the next proposition.

Proposition 1. Let $t \in T$ and $x, y \in X$. The following are equivalent:

(i) xR_ty.
(ii) The function $x - y$ is continuous at the point t and $x(t) = y(t)$.

Proof. Assume that xR_ty, then the upper envelope $\overline{|x-y|}$ of $x-y$ at the point t is 0, that is, $\overline{|x-y|}(t) = 0$. If $\epsilon > 0$, there exists an open neighborhood V of t such that $\sup_{s\in V}|x(s)-y(s)| < \epsilon$, thus $|x(s)-y(s)| < \epsilon$, for each $s \in V$, in particular, $|x(t)-y(t)| < \epsilon$, hence $x(t) = y(t)$. Furthermore, $|(x-y)(s) - (x-y)(t)| = |(x-y)(s)| < \epsilon$, for every $s \in V$, then $x - y$ is continuous at t. Conversely, suppose that $x-y$ is continuous at t and $x(t) = y(t)$, then given $\epsilon > 0$, there exists an open neighborhood V of t such that $|x(s)-y(s)| < \epsilon$, for each $s \in V$, then xR_ty. □

Proposition 2. The relation R_t, just defined, is an equivalence relation in the space X.

Proof. The relation R_t is clearly reflexive and symmetric. Let $x, y, z \in X$ be such that xR_ty and yR_tz, then

$$\inf_{V\in\mathcal{V}(t)}\sup_{s\in V}|x(s)-y(s)| = 0 = \inf_{V\in\mathcal{V}(t)}\sup_{s\in V}|y(s)-z(s)|.$$

Given $\epsilon > 0$, there exists an open neighborhood V of t such that $|x(s) - y(s)| < \epsilon/2$ and $|y(s) - z(s)| < \epsilon/2$, for every $s \in V$, it follows that $\sup_{s \in V} |x(s) - z(s)| \leqslant \epsilon$, therefore $xR_t z$. □

An alternative, and now justified, notation for the relation R_t defined in X is $\equiv_t$.

Definition 6. For every $t \in T$, denote by E_t the quotient set of X module the equivalence relation $\equiv_t$, that is, $E_t = X/R_t = X/\equiv_t$ and for each $x \in X$, denote by $[x]_t$ the equivalence class of x module $\equiv_t$.

Proposition 3. The relation $\equiv_t$ is compatible with operations in X, in the following sense:

(a) If $x, y, u, v \in X$ are such that $x \equiv_t u$ and $y \equiv_t v$, then $x + y$ and $u + v$ belong to X and $x + y \equiv_t u + v$.

(b) If $x, u \in X$ are such that $x \equiv_t u$ and if f is a non-negative bounded continuous real valued function defined in T, then fx and fu belong to X and $fx \equiv_t fu$.

Proof. Let $x, y, u, v \in X$, $x \equiv_t u$, and $y \equiv_t v$, then the functions $x - u$ and $y - v$ are continuous at t and $x(t) - u(t) = y(t) - v(t) = 0$, thus $(x + y) - (u + v) = (x - u) + (y - v)$ is also a continuous function at t and $x(t) + y(t) = u(t) + v(t)$, hence $x + y \equiv_t u + v$. Let $s \in T$ and $\epsilon > 0$ be given, $\delta > 0$ be such that $\delta(\delta + f(s) + x(s)) < \epsilon$ and V be an open neighborhood of s such that $x(r) < x(s) + \delta$ and $|f(r) - f(s)| < \delta$, for each $r \in V$, then $f(r)x(r) < (f(s) + \delta)(x(s) + \delta) = f(s)x(s) + \delta(\delta + f(s) + x(s)) < f(s)x(s) + \epsilon$, thus $fx \in X$. It is apparent that $fx - fu = f(x - u)$ is also continuous at t and that $f(t)x(t) = f(t)u(t)$, hence $fx \equiv_t fu$. □

Definition 7. Let $x,\ y \in X$, $\alpha \geq 0$ be a non negative real number and f be a non negative bounded continuous real valued function defined in T. The following define operations between equivalent classes module $\equiv_t$.

a) $[x]_t + [y]_t := [x + y]_t$,
b) $\alpha[x]_t := [\alpha x]_t$ and
c) $f[x]_t := [fx]_t$.

By the preceding proposition, these operations, between equivalent classes module $\equiv_t$, are indeed well defined.

Proposition 4. Let $t \in T$, $x \in X$, and f be a continuous bounded non negative function, then $[fx]_t = [f(t)x]_t = f(t)[x]_t$.

Proof. Since $f(t) \geq 0$, by definition, $[f(t)x]_t = f(t)[x]_t$. It remains to verify that $f(t)x \equiv_t fx$, indeed, since f is continuous at the point t, given $\epsilon > 0$, there exists an open neighborhood W of t such that $|f(s) - f(t)| < \epsilon$, for every $s \in W$, then $0 \leqslant \sup_{s \in V} |f(s) - f(t)| \leqslant \epsilon$. Hence

$$\overline{|f(t)x - fx|}(t) = \inf_{V \in \mathcal{V}(t)} \sup_{s \in V} |f(t) - f(s)||x(s)| \leqslant$$
$$\epsilon \inf_{V \in \mathcal{V}(t)} \sup_{s \in V} |x(s)| = \epsilon \overline{|x|}(t).$$

It follows that $\overline{|f(t)x - fx|}(t) = 0$, therefore $f(t)x \equiv_t fx$. □

Proposition 5. Let T be a topological space, $t \in T$ and E_t be the quotient of X module $\equiv_t$, as defined above, then the function $d_t : E_t \times E_t \longrightarrow \mathbb{R}$ defined by $d_t([x]_t, [y]_t) = \overline{|x - y|}(t)$ is a distance function, that is, the fiber E_t is a metric space with respect to d_t.

Proof. To show that d_t is well defined, take $[x]_t, [y]_t \in E_t$ and $v \in [y]_t$. Then

$$d_t([x]_t, [y]_t) = \inf_{V \in \mathcal{V}(t)} \sup_{s \in V} |x(s) - y(s)| \leqslant$$
$$\inf_{V \in \mathcal{V}(t)} \sup_{s \in V} (|x(s) - v(s)| + |v(s) - y(s)|) \leqslant$$
$$\inf_{V \in \mathcal{V}(t)} \sup_{s \in V} |x(s) - v(s)| + \inf_{V \in \mathcal{V}(t)} \sup_{s \in V} |v(s) - y(s)| =$$
$$d_t([x]_t, [v]_t) + d_t([v]_t, [y]_t) = d_t([x]_t, [v]_t).$$

By interchanging the roles of y and v, it follows that $d_t([x]_t, [y]_t) \geq d_t([x]_t, [v]_t)$. This proves that the definition of d_t is independent of the representatives chosen in the classes $[x]_t$ and $[y]_t$. It is obvious that d_t is non negative and symmetric. From the definition of d_t, it follows that $d_t([x]_t, [y]_t) = 0$, if and only if, $x \equiv_t y$, if and only if, $[x]_t = [y]_t$, hence d_t is reflexive. The triangle inequality is a direct consequence of the triangle inequality of the absolute value. □

Remark 1. For every pair $x, y \in X$ the function $\widetilde{d} : T \longrightarrow \mathbb{R}$ defined by $\widetilde{d}(t) = d_t([x]_t, [y]_t) = \overline{|x - y|}(t)$ is a bounded upper semicontinuous function, that is, $\widetilde{d} \in X$.

3. The Representation Theorem

As an application of Theorem 1, a metric bundle is contructed whose global sections represent the bounded upper semicontinous functions defined in the topological space T.

Theorem 2. Let $E = \coprod_{t\in T} E_t$ be the disjoint union of the family $(E_t)_{t\in T}$, $p : E \longrightarrow T$ be the function defined by $p(u) = t$, if $u \in E_t$, $\Sigma := \{\widehat{x} : x \in X\}$, where $\widehat{x} : T \longrightarrow E, t \longmapsto \widehat{x}(t) = [x]_t$ and $d^* : E \times E \longrightarrow [0, +\infty]$ be the function defined by $d^*(u, v) = +\infty$, if $p(u) \neq p(v)$, and $d^*(u, v) = d_t(u, v)$ if $p(u) = p(v) = t$. Then (E, p, T) is a metric bundle, Σ is a full set of global sections for p and the family of ϵ-tubes around $\widehat{x} \restriction_V$, where $\epsilon > 0$, $\widehat{x}$ runs throughout Σ and V throughout the collection of non empty open subsets of T, is a base for the topology of E. Even more, the Gelfand morphism, defined by $\widehat{\ }: X \longrightarrow \Gamma(p),\ x \longmapsto \widehat{x}$, is an isometry from the space X of all bounded upper semicontinuous functions onto the space Σ , with respect to the corresponding sup metrics.

Proof. Since $p(u) \neq p(v)$, if and only if, $d^*(u, v) = +\infty$, and $d_t = d^* \restriction_{E_t\times E_t}$, it follows that d^* is a metric for p and since, for every $x \in X$, $p(\widehat{x}(t)) = p([x]_t) = t$, Σ is a family of selections for p. On the other hand, if $u \in E$ and $t \in T$ is such that $u \in E_t$, there exist $x \in X$ such that $u = [x]_t$. Hence, for every $\epsilon > 0$, $u \in \mathcal{T}_\epsilon(\widehat{x})$. Now let $\widehat{x},\ \widehat{y} \in \Sigma$ and consider $\Delta : T \longrightarrow [0, +\infty]$, defined by $\Delta(t) = d^*(\widehat{x}(t), \widehat{y}(t))$. For every $t \in T$, $p(\widehat{x}(t)) = p(\widehat{y}(t))$, hence $\Delta(t) \neq +\infty$, for every $t \in T$, and $\Delta(t) = d^*(\widehat{x}(t), \widehat{y}(t)) = d_t(\widehat{x}(t), \widehat{y}(t)) = d_t([x]_t, [y]_t) = \overline{|x - y|}(t)$, thus Δ is an upper semicontinuous function. By Theorem 1, it follows that E can be equipped with a topology $\mathfrak{T}$, with the properties specified in the statement of the theorem, that is,

(1) the family of subsets of E of the form $\mathcal{T}_\epsilon(\widehat{x}_Q)$ is a base of $\mathfrak{T}$, where $\epsilon > 0$, Q runs throughout of open subsets of T, $\widehat{x}$ throughout Σ and $\widehat{x}_Q$ denotes the restriction of α to Q.
(2) every $\widehat{x} \in \Sigma$ is a section and
(3) (E, p, T) is a bundle of metric spaces.

Let $x, y \in X$, then $|x(t) - y(t)| \leqslant d(x, y)$, for every $t \in T$, thus $\sup_{s\in V} |x(s) - y(s)| \leqslant d(x, y)$, for every open neighborhood V of t, therefore $d_t(\widehat{x}(t), \widehat{y}(t)) = \overline{|x - y|}(t) = \inf_{V\in\mathcal{V}(t)} \sup_{s\in V} |x(s) - y(s)| \leqslant d(x, y)$. It follows that

$\sup_{t\in T} d_t(\widehat{x}(t), \widehat{y}(t)) \leqslant d(x, y)$. The converse inequality is immediate, since for every $t \in T$, $|x(t) - y(t)| \leqslant \overline{|x - y|}(t) = d_t(\widehat{x}(t), \widehat{y}(t))$. □

Definition 8. The bundle (E, p, T) given by Theorem 2, will be called the *Bundle of Upper Semicontinuous Functions*.

Lemma 1. Let (F, q, T) be a bundle of metric spaces and $\Gamma(q)$ its set of all global sections, let $u \in F$, $\beta \in \Gamma(q)$ and $t \in T$ such that $\beta(t) = u$, then the family $\{\mathcal{T}_\epsilon(\beta \restriction_W) : \epsilon > 0 \text{ and } W \in \mathcal{V}(t)\}$ is a fundamental system of neighborhoods of $u = \beta(t)$.

Proof. Let $\mathcal{T}_\epsilon(\alpha \restriction_V)$ be an arbitrary basic neighborhood of $u = \beta(t)$, then $t \in V$. To be shown that there exists $\varrho > 0$ and $W \in \mathcal{V}(t)$ such that $u \in \mathcal{T}_\varrho(\beta \restriction_W) \subset \mathcal{T}_\epsilon(\alpha \restriction_V)$. Following C. M. Neira [4], Lemma 1, chapter II, page 15, let $\delta > 0$ be such that $d(\beta(t), \alpha(t)) < \delta < \epsilon$, then there exists $W \in \mathcal{V}(t)$ such that $W \subset V$ and $d(\beta(s), \alpha(s)) < \delta$, for each $s \in W$. Let $\varrho = \epsilon - \delta$ and $v \in \mathcal{T}_\varrho(\beta_W)$, then $d(v, \beta(p(v)) < \varrho$, hence $d(v, \alpha(p(v))) < \epsilon$. □

The image $\widehat{X}$, of the space X of upper semicontinuous functions under the Gelfand morphism, $x \longmapsto \widehat{x}$, is closed under addition and non negative function multiplication, as it is made precise in the following proposition.

Proposition 6. Let $t \in T$, $x,\ y \in X$, $\alpha \geq 0$ be a non negative real number and f be a bounded continuous non negative real valued function, then

(a) $(\widehat{x} + \widehat{y})(t) = (x + y)\widehat{\ }(t)$
(b) $\alpha\widehat{x}(t) = (\alpha x)\widehat{\ }(t)$ and
(c) $(f\widehat{x})(t) = (fx)\widehat{\ }(t)$

Proof. Directly, from the definition of pointwise addition and multiplication, Definition 7 and Proposition 4, it follows that $(\widehat{x} + \widehat{y})(t) = \widehat{x}(t) + \widehat{y}(t) = [x]_t + [y]_t = [x + y]_t = (x + y)\widehat{\ }(t)$, $\alpha\widehat{x}(t) = \alpha[x]_t = [\alpha x]_t = (\alpha x)\widehat{\ }(t)$ and $(f\widehat{x})(t) = f(t)\widehat{x}(t) = f(t)[x]_t = f[x]_t = [fx]_t = (fx)\widehat{\ }(t)$. □

The property depicted in item (c) of the above proposition, is extended to arbitrary global sections in $\Gamma(p)$, in the next lemma.

Lemma 2. Let $\Gamma(p)$ be the set of all global sections of the Bundle of Upper Semicontinuous Functions (E, p, T), let f be a non negative continuous real valued function defined in T and $\sigma \in \Gamma(p)$, then $f\sigma \in \Gamma(p)$, where $(f\sigma)(t) = f(t)\sigma(t) = (f(t)\sigma)(t)$, for each $t \in T$.

Proof. Let $t \in T$ and let $x \in X$ be such that $\sigma(t) = \widehat{x}(t)$, then $\sigma(t) \in \mathcal{T}_\epsilon(\widehat{x})$, $(f\sigma)(t) = f(t)\sigma(t) = f(t)\widehat{x}(t) = f(t)[x]_t = [fx]_t$ and $p((f\sigma)(t)) = t$. Since σ is continuous at the point t, given $\epsilon > 0$ and a neighborhood $W \in \mathcal{V}(t)$, there exists a neighborhood $V \in \mathcal{V}(t)$ such that $\sigma(s) \in \mathcal{T}_\delta(\widehat{x} \restriction_W)$ for each $s \in V$, where δ is chosen such that $(f(t) + \delta)\delta < \epsilon$. Assume that $|f(s) - f(t)| < \delta$, for each $s \in V$. Then, for every $s \in V$, $d_s((f\sigma)(s), (fx)\widehat{\ }(s)) = d_s((f\sigma)(s), (f\widehat{x})(s)) = d_s(f(s)\sigma(s), f(s)\widehat{x}(s)) = f(s)d_s(\sigma(s), \widehat{x}(s)) < (f(t) + \delta)\delta < \epsilon$. Thus $(f\sigma)(s) \in \mathcal{T}_\epsilon((fx)\widehat{\ } \restriction_W)$, whenever $s \in V$, since by Lemma 1 the family $\{\mathcal{T}_\epsilon((fx)\widehat{\ } \restriction_W) : \epsilon > 0 \text{ and } W \in \mathcal{V}(t)\}$ is a fundamental system of neighborhoods for $(fx)\widehat{\ }(t)$, it follows that $f\sigma$ is continuous at t. □

Theorem 3. Let T be a compact and completely regular space (non necessarily Hausdorff), then $\Sigma = \Gamma(p)$, that is, every global section, of the Bundle of Upper Semicontinuous Functions (E, p, T), is the image by the Gelfand Map $\widehat{\ }$ of a bounded upper semicontinuous function defined in T.

Proof. Let $\sigma \in \Gamma(p)$. An upper semicontinuous function $x \in X$, such that $\widehat{x} = \sigma$, ought to be found. For each $t \in T$, there exists $x_t \in X$ such that $\sigma(t) = \widehat{x_t}(t)$. Define $x : T \longrightarrow \mathbb{R}$ by $x(t) = x_t(t)$, for each $t \in T$. This function is well defined since two upper semicontinuous functions equivalent module $\equiv_t$ coincide at the point t. We claim that x is upper semicontinuous. Let $\epsilon > 0$, by the upper semicontinuity of x_t at t, there exists an open neighborhood $V \in \mathcal{V}(t)$ of t such that $x_t(s) < x_t(t) + \epsilon/2$, for each $s \in V$, and by the continuity of the section σ at t, there exists an open neighborhood $Q \in \mathcal{V}(t)$ such that $\sigma(s) \in \mathcal{T}_{\epsilon/2}(\widehat{x_t})$, for each $s \in Q$, that is, $d_s(\sigma(s), \widehat{x_t}(s)) < \epsilon/2$, for each $s \in Q$. Let $P = Q \cap V$ and $s \in P$, then

$$|x_s(s) - x_t(s)| \leqslant \inf_{W \in \mathcal{V}(s)} \sup_{r \in W} |x_s(r) - x_t(r)| = d_s(\sigma(s), \widehat{x_t}(s)) < \epsilon/2,$$

consequently $x_s(s) - x_t(t) < \epsilon/2$, it follows that, for each $s \in P$, $x_s(s) < x_t(t) + \epsilon$, hence, for each $s \in P$, $x(s) < x(t) + \epsilon$. This shows that the function x is upper semicontinuous. It remains to prove that the function x is bounded. An identical argument does it. Again by the continuity of σ at t, we can find an open neighborhood $U_t \in \mathcal{V}(t)$ of t such that, $s \in U_t$ implies that

$|x_s(s)-x_t(s)| < 1$, it follows that for each $s \in U_t$, $|x_s(s)| < 1+|x_t(s)| \leqslant 1+M_t$, where M_t is an upper bound of $|x_t|$. By compactess of T, find $U_{t_1}, U_{t_2}, \ldots, U_{t_p}$, open subsets covering T, it follows that, for each $s \in T$, $|x_s(s)| < 1+\max\{M_{t_k} : k+1, 2, \ldots, p\}$, that is, x is a bounded upper semicontinuous function. Thus $\widehat{x} = \sigma$ and $\Sigma = \Gamma(p)$. □

4. properties of the fibers of the Bundle of Upper Semicontinuous Functions

We now take up the study of the properties of the fibers of the Bundle of Upper Semicontinuous Functions. They will be shown to be complete and connected spaces. We begin by recalling a well known property of the spaces of upper semicontinuous function, namely:

Proposition 7. The spaces X of all bounded upper semicontinuous functions defined in the space T and X_t of all bounded functious defined in T that are upper semicontinuous at the point $t \in T$, with the corresponding sup metrics, are complete metric spaces.

Proof. It is a consequence of the following lemma. □

Lemma 3. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of elements of X_t, converging uniformly to a function x, in an open neighborhood Q of the point t, then x is upper semicontinuous at t.

Proof. Let $a \in \mathbb{R}$ and suppose that $x(t) < a$. Choose $\epsilon > 0$, such that $x(t) < a-2\epsilon$, then there exists $p \in \mathbb{N}$ such that, $|x_n(s)-x(s)| < \epsilon$, if $s \in Q$ and $n \geq p$. In particular, $x_n(t) < x(t)+\epsilon < a-2\epsilon+\epsilon = a-\epsilon$, for all $n \geq p$, it follows that, for every $n \geq p$, there exists $Q_n \in \mathcal{V}(t)$, $Q_n \subset Q$, such that $x_n(s) < a-\epsilon$, for each $s \in Q_n$. In particular, $x_p(s) < a-\epsilon$, for every $s \in Q_p \subset Q$. It follows that $x(s) = x(s)-x_p(s)+x_p(s) < \epsilon+(a-\epsilon) = a$, for each $s \in Q_p$, therefore the function x is upper semicontinuous at t. □

The next lemma plays a crucial role in establishing the completeness of the the fibers of the Bundle of Upper Semicontinuous Functions.

Lemma 4. Let (E, p, T) be the Bundle of Upper Semicontinuous Functions and assume that its base space T is completely regular. Let $\epsilon > 0$ and let $[x]_t$,

$[y]_t \in E_t$ be such that $d_t([x]_t,[y]_t) < \epsilon$, then there exists a function $v \in X$ such that $v \in [y]_t$ and $d(x,v) < \epsilon$.

Proof. Let $x, y \in X$, $\epsilon > 0$. Suppose that $d_t([x]_t,[y]_t) < \epsilon$. Since $d_t([x]_t,[y]_t) = \overline{|x-y|}(t) = \limsup_{s\to t}|x(s)-y(s)|$, there exists an open neighborhood $V \in \mathcal{V}(t)$, of the point t, such that $\sup_{s\in V}|x(s)-y(s)| < \epsilon$. Since T is assumed to be completly regular, there exists a continuous function $f : T \longrightarrow [0,1]$ such that $f(t) = 1$ and $f(r) = 0$, for every $r \notin V$. Let $v = (1-f)x + fy$, then v is upper semicontinuous and $\sup_{s\in T}|x(s)-v(s)| = \sup_{s\in T}|f(s)(x(s)-y(s))| = \sup_{s\in V}|f(s)(x(s)-y(s))| \leqslant \sup_{s\in V}|x(s)-y(s)| < \epsilon$, then $d(x,v) < \epsilon$. On the other hand, $v(t) = x(t)-f(t)x(t)+f(t)y(t) = y(t)$ and by the continuity of f at t, given $\delta > 0$, there exists $W \in \mathcal{V}(t)$ such that if $s \in W$ then $|1-f(s)| < \delta/\epsilon$. Let $s \in V \cap W$, then

$$\begin{aligned}
|v(s)-y(s)| &= |x(s)-f(s)x(s)+f(s)y(s)-y(s)| \\
&= |(1-f(s))(x(s)-(1-f(s))y(s)| \\
&= |1-f(s)||x(s)-y(s)| < \delta.
\end{aligned}$$

This proves the continuity of $v - y$ at the point t. By Proposition 1, it follows that $v \in [y]_t$.

□

Theorem 4. Let T be a completely regular space and $t \in T$, then the fiber E_t, of the Bundle of Upper Semicontinuous Functions (E,p,T), is a complete metric space.

Proof. Let $([x_n]_t)_n$ be a Cauchy sequence in E_t, then there exists a subsequence $([y_n]_t)_n$ of $([x_n]_t)_n$ such that $d_t([y_n]_t,[y_{n+1}]_t) < \dfrac{1}{2^n}$, for every $n \in \mathbb{N}$. Let $z_1 = y_1$, since $d_t([y_1]_t,[y_2]_t) < \dfrac{1}{2}$, the preceding lemma secures the existence of a representative $z_2 \in X$ of $[y_2]_t$, such that $d(z_1,z_2) = d(y_1,z_2) < \dfrac{1}{2}$. Since $d_t([z_2]_t,[y_3]_t) = d_t([y_2],[y_3]_t) < \dfrac{1}{2^2}$, there exists a representative $z_3 \in X$ of $[y_3]_t$, such that $d(z_2,z_3) < \dfrac{1}{2^2}$. The procedure is continued by assuming that $z_1, z_2, ..., z_{n-1}, z_n \in X$ are representatives of $[y_1]_t, [y_2]_t, ..., [y_{n-1}]_t$, $[y_n]_t$ respectively, such that $d(z_k,z_{k+1}) < \dfrac{1}{2^k}$, for $k = 1,2,..,n-1$, then there exists a $z_{n+1} \in X$ such that $z_{n+1} \in [y_{n+1}]_t$ and $d(z_n, z_{n+1}) < \dfrac{1}{2^n}$. If $m > n$, then

$$d(z_n, z_m) \leqslant \sum_{k=n}^{m-1} d(z_k, z_{k+1}) \leqslant \sum_{k=n}^{m-1} \frac{1}{2^k} \leqslant \frac{1}{2^n} \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n-1}}.$$

Since X is a complete metric space, there exists $z \in X$ such that $(z_n)_n$ tends to z, as n tends to infinity. Since $d_t([y_n]_t, [z]_t) \leqslant d(z_n, z)$, then $[y_n]_t \to [z]_t$ in E_t. This proves that E_t is a complete metric space. □

This completeness property of the fibers is an indication that the bundle (E, p, T) is a suitable candidate to be *the real numbers object* in the category of metric bundles, thus, in a certain sense, it could be viewed as some sort of semicontinuous continuum. It also enjoys other desirable properties that are next examined.

Proposition 8. For every $t \in T$, the fiber E_t of the Bundle of Upper Semicontinuous Functions (E, p, T) is arcwise connected.

Proof. Given $[x]_t, [y]_t \in E_t$, the map $\varphi : [0,1] \longrightarrow E_t$, such that $\varphi(\xi) = \xi[x]_t + (1-\xi)[y]_t$ is continuous, indeed,

$$d_t(\varphi(\xi), \varphi(\xi_o)) = d_t(\xi[x]_t + (1-\xi)[y]_t, \xi_o[x]_t + (1-\xi_o)[y]_t) \leqslant$$

$$d_t(\xi[x]_t+(1-\xi)[y]_t, \xi_o[x]_t+(1-\xi)[y]_t)+d_t(\xi_o[x]_t+(1-\xi)[y]_t, \xi_o[x]_t+(1-\xi_o)[y]_t) =$$

$$d_t([\xi x + (1-\xi)y]_t, [\xi_o x + (1-\xi)y]_t) + d_t([\xi_o x + (1-\xi)y]_t, [\xi_o x + (1-\xi_o)y]_t) =$$

$$\overline{|(\xi x + (1-\xi)y) - (\xi_o x + (1-\xi)y)|}(t)+\overline{|(\xi_o x + (1-\xi)y) - (\xi_o x + (1-\xi_o)y)|}(t)$$

$$= \overline{|(\xi-\xi_o)x|}(t) + \overline{|(\xi-\xi_o)y|}(t) = |\xi-\xi_o|(\overline{|x|}(t) + \overline{|y|}(t)).$$

□

Remark 2. Let $K_t := \{[\alpha]_t : \alpha \in \mathbb{R}\} \subset E_t$ be the subset of equivalence classes of constant functions. Note that K_t coincides with the subset of equivalent classes of continuous functions at t, in fact, if $x : T \longrightarrow \mathbb{R}$ is a continuous function in t, then $[x]_t = [x(t)]_t$ and therefore $[x]_t \in K_t$. Since the function $f : \mathbb{R} \longrightarrow K_t \subseteq E_t$ defined by $f(\alpha) = [\alpha]_t$ is an isometry, then K_t is a connected subspace of E_t.

The local compactness property of the Real Line does not have a a corresponding counterpart in fibers of the Bundle of Upper Semicontinuous Functions, as is shown in the following counterexample.

Example 1. Take $T = [0,1]$ and $t = 0$. For each $m \in \mathbb{N}$, the sequence $(a_{mk})_{k\in\mathbb{N}}$, defined by

$$a_{mk} = \frac{1}{k+1} + \frac{1}{2^m}\left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{1}{2^m k} + \frac{2^m - 1}{2^m(k+1)},$$

tends to zero, when $k \to \infty$.

For each $m \in \mathbb{N}$, let $A_m = \{a_{mk} : k \in \mathbb{N}\} \cup \{0\}$ and let χ_{A_m} be the characteristic function of A_m.

Let $\epsilon > 0$ and $V_\epsilon = \{[x]_t \in E_t : d_t([0]_t, [x]_t) \leqslant \epsilon\}$ be the closed ball of radius ϵ around $[0]_t$ (class of the function 0, module $\equiv_t$) in the fiber E_t above $t = 0$, then $\big[\frac{\epsilon}{2}\chi_{A_m}\big]_0 \in V_\epsilon$, for every $m \in \mathbb{N}$. Since $A_n \cap A_m = \{0\}$, when $n \neq m$, it follows that $d_t\left(\big[\frac{\epsilon}{2}\chi_{A_n}\big]_t, \big[\frac{\epsilon}{2}\chi_{A_m}\big]_t\right) = \frac{\epsilon}{2}$, thus $\left(\big[\frac{\epsilon}{2}\chi_{A_n}\big]_0\right)_{n\in\mathbb{N}}$ is a sequence in V_ϵ that doesn't have a convergent subsequence, hence V_ϵ is not a compact set for any $\epsilon > 0$, therefore E_t is not a locally compact space.

On the other hand, the fibers E_t, of the Bundle of Upper Semicontinuous Functions, can be equipped with a partial order relation:

Definition 9. Let $[x]_t$, $[y]_t \in E_t$. If for every $\epsilon > 0$, there exists an open neighborhood $V \in \mathcal{V}(t)$ such that $x(s) \leqslant y(s) + \epsilon$, for each $s \in V$, we say that "$[x]_t$ *is less or equal to* $[y]_t$", in symbols, $[x]_t \leqslant_t [y]_t$. By $[x]_t <_t [y]_t$, is meant, $[x]_t \leqslant_t [y]_t$ but $[x]_t \neq [y]_t$.

Proposition 9. The fiber E_t, endowed with the relation $\leqslant_t$, defined above, is a partially ordered set.

Proof. The relation $[x]_t \leqslant_t [y]_t$ does not depend on the representatives of the classes, in fact, take $u, v \in X$, such that $x \equiv_t u$, $y \equiv_t v$ and let $\epsilon > 0$, then there exists $Q \in \mathcal{V}(t)$ such that $x(s) \leqslant y(s) + \epsilon$, for every $s \in Q$, and there exist functions $h, k : T \longrightarrow \mathbb{R}$, continuous at the point t, such that $h(t) = k(t) = 0$, $x - u = h$ and $y - v = k$, then $u(s) = x(s) - h(s) \leqslant y(s) - h(s) + \epsilon = v(s) - h(s) + k(s) + \epsilon \leqslant v(s) + 3\epsilon$, if $s \in P$, for a small enough open neighborhood P of t, thus the relation $\leqslant_t$ is a well defined relation in E_t. Reflexivity and transitivity of $\leqslant_t$ follow from the definition. Assume that $[x]_t \leqslant_t [y]_t$ and $[y]_t \leqslant_t [x]_t$ and let $\epsilon > 0$, then there exists an open neighborhoods V of t such that $x(s) \leqslant y(s) + \epsilon$ and $y(s) \leqslant x(s) + \epsilon$, for every $s \in V$, thus $|x(s) - y(s)| \leqslant \epsilon$, for every $s \in V$, hence $x - y$ is continuous at t and $x(t) = y(t)$, therefore $[x]_t = [y]_t$. Hence $\leqslant_t$ is a partial order in E_t. $\square$

Remark 3. In general $\leqslant_t$ is not a total order in E_t, for instance, let $T = [-1, 1]$ and $t = 0$, then the class of the function $\vartheta : T \longrightarrow \mathbb{R}$, such that $\vartheta(s) = -1$, if $s < 0$, and $\vartheta(s) = 1$, if $0 \leqslant s$, and the class of the constant function 0 are not comparable.

Let $x, y, z, w \in X$ be such that $[x]_t \leqslant_t [y]_t$ and $[z]_t \leqslant_t [w]_t$, then, obviously, $[x + z]_t \leqslant_t [y + w]_t$, also if $\alpha \geq 0$ is a non negative real number, then $\alpha[x]_t \leqslant_t \alpha[y]_t$.

Given $x,\ y \in X$, by defining $x \leqslant y$, if and only if, for each $s \in T$, $x(s) \leqslant y(s)$, it readily follows that the canonical map, $x \longmapsto [x]_t : X_t \longrightarrow E$, is increasing.

Remark 4. Let $a \in X$ be a bounded upper semicontinuous function in T and let $\epsilon > 0$. Denote by $S_\epsilon([a]_t)$ the open ball with center $[a]_t$ and radius ϵ, then $S_\epsilon([a]_t) \subset ([a-\epsilon]_t, [a+\epsilon]_t)$, in fact: $d_t([x]_t, [a]_t) < \epsilon$, if and only if, there exists $\delta > 0$ such that $d_t([x]_t, [a]_t) \leqslant \epsilon - \delta$, if and only if, $\overline{|x - a|}(t) \leqslant \epsilon - \delta$, if and only if, $\inf_{V \in \mathcal{V}(t)} \sup_{s \in V} |x(s) - a(s)| \leqslant \epsilon - \delta$, hence there exists $V \in \mathcal{V}(t)$ such that $a(s) - \epsilon + \delta \leqslant x(s) \leqslant a(s) + \epsilon - \delta$, for all $s \in V$, then $[a - \epsilon]_t <_t [a - \epsilon + \delta]_t \leqslant_t [x]_t \leqslant_t [a + \epsilon - \delta]_t <_t [a + \epsilon]_t$.
However, the converse inclusion does not hold in general, indeed, let $T = [-1, 1]$, $t = 0$ and $\vartheta : T \longrightarrow \mathbb{R}$ be as in the previous remark, let $a = 0$ be the constant function zero, and take $\epsilon = 1$, then $([a - \epsilon]_t, [a + \epsilon]_t) \nsubseteq S_\epsilon([a]_t)$, because $[\vartheta]_t \in ([a - \epsilon]_t, [a + \epsilon]_t)$, but $[\vartheta]_t \notin S_\epsilon([a]_t)$, since $d_t([\vartheta]_t, [a]_t) = 1$.

Nevertheless by turning on closed balls and closed intervals, the identity of the two sets is obtained.

Lemma 5. Let T be a topological space, (E, p, T) be the Bundle of Upper Semicontinuous Functions over T, $a \in X$ be a bounded upper semicontinuous function at a point $t \in T$ and let $\epsilon > 0$, then the closed ball, with center a and radius ϵ, relative to the metric d_t of the fiber E_t, equals the closed interval with endpoints $[a - \epsilon]_t$, $[a + \epsilon]_t$, in symbols: $S_\epsilon\big[[a]_t\big] = \big[[a - \epsilon]_t, [a + \epsilon]_t\big]$

Proof.
$$[x]_t \in S_\epsilon\big[[a]_t\big] \iff d_t([x]_t, [a]_t) \leqslant \epsilon \iff$$
$$\overline{|x - a|} \leqslant \epsilon \iff \inf_{V \in \mathcal{V}(t)} \sup_{s \in V} |x(s) - a(s)| \leqslant \epsilon \iff$$
$$(\forall \delta > 0)(\inf_{V \in \mathcal{V}(t)} \sup_{s \in V} |x(s) - a(s)| < \epsilon + \delta) \iff$$

$$(\forall\delta>0)(\exists V_\delta\in\mathcal{V}(t))(\sup_{s\in V_\delta}|x(s)-a(s)|\leqslant\epsilon+\delta)\iff$$
$$(\forall\delta>0)(\exists V_\delta\in\mathcal{V}(t))(\forall s\in V_\delta)(a(s)-\epsilon-\delta\leqslant x(s)\leqslant a(s)+\epsilon+\delta)\iff$$
$$[a-\epsilon]_t\leqslant_t[x]_t\leqslant_t[a+\epsilon]_t\iff[x]_t\in\big[[a-\epsilon]_t,[a+\epsilon]_t\big].$$

Then $S_\epsilon\big[[a]_t\big]=\big[[a-\epsilon]_t,[a+\epsilon]_t\big]$. □

Proposition 10. In the fibers of the Bundle of Upper Semicontinuous Functions (E,p,T) the order topology is strictly finer than the metric topology.

Proof. Let $a\in X$ be a bounded upper semicontinuous function in T and $\epsilon>0$, then the open ball $S_\epsilon([a]_t)$ is open in the order topology, indeed, let $[b]_t\in S_\epsilon([a]_t)$ and let $\zeta>0$ be such that $d_t([a]_t,[b]_t)<\epsilon-\zeta$, then $\big[[b-\zeta]_t,[b+\zeta]_t\big]=S_\zeta\big[[b]_t\big]\subset S_\epsilon([a]_t)$, thus $([b-\zeta]_t,[b+\zeta]_t)\subset S_\epsilon([a]_t)$. Hence , in the fiber E_t, the order topology is finer than the metric topology.
On the other hand, let $T=[-1,1]$ be the closed real interval with endpoints -1, 1, let $t=0$ and $\vartheta\in X$ be as in Remark 3, then it is apparent that $[\vartheta]_t$ belongs to the open interval $([-c]_t,[c]_t)$, where $c\in X$ is the constant function 1, but, for each $\epsilon>0$, $S_\epsilon([\vartheta]_t)\nsubseteq([-c]_t,[c]_t)$, that is, $[\vartheta]_t$ is not an interior point of $([-c]_t,[c]_t)$ for the metric d_t, then $([-c]_t,[c]_t)$ is not an open subset with respect to the metric topology, therefore, in each fiber E_t, the order topology is strictly finer than the metric topology. □

Lemma 6. Let $a,b,c\in X$ such that $[a]_t\leqslant_t[b]_t\leqslant_t[c]_t$, then $\overline{|b-a|}(t)\leqslant\overline{|c-a|}(t)$.

Proof. By contradiction, assume that $\overline{|c-a|}(t)<\overline{|b-a|}(t)$. Take $\epsilon>0$ such that $\overline{|c-a|}(t)<\overline{|b-a|}(t)-\epsilon$, then

$$\inf_{V\in\mathcal{V}(t)}\sup_{s\in V}|c(s)-a(s)|<\overline{|b-a|}(t)-\epsilon,\ \text{then}$$
$$\text{there exists } V_o\in\mathcal{V}(t)\text{ such that }\sup_{s\in V_o}|c(s)-a(s)|<\overline{|b-a|}(t)-\epsilon\implies$$
$$(\exists V_o\in\mathcal{V}(t))(\forall s\in V_o)(|c(s)-a(s)|<\overline{|b-a|}(t)-\epsilon)\implies$$
$$(\exists V_o\in\mathcal{V}(t))(\forall s\in V_o)(|c(s)-a(s)|<\inf_{W\in\mathcal{V}(t)}\sup_{r\in W}|b(r)-a(r)|-\epsilon)\implies$$
$$(\exists V_o\in\mathcal{V}(t))(\forall s\in V_o)(\forall W\in\mathcal{V}(t))(|c(s)-a(s)|<\sup_{r\in W}|b(r)-a(r)|-\epsilon)\implies$$
$$(\exists V_o\in\mathcal{V}(t))(\forall s\in V_o)(\forall W\in\mathcal{V}(t))(\exists w\in W)$$

$$(|c(s) - a(s)| < |b(w) - a(w)| - \epsilon) \Longrightarrow$$
$$(\forall W \in \mathcal{V}(t))(\exists w \in W)(|c(w) - a(w)| < |b(w) - a(w)| - \epsilon) \Longrightarrow$$
$$(\forall W \in \mathcal{V}(t))(\exists w \in W)(c(w) - a(w) < |b(w) - a(w)| - \epsilon) \Longrightarrow$$
$$(\forall W \in \mathcal{V}(t))(\exists w \in W)(\text{ either } c(w) < b(w) - \epsilon \text{ or } b(w) + c(w) < 2a(w) - \epsilon),$$

by taking W small enough, one contradicts $[b]_t \leqslant [c]_t$ or $[2a]_t \leqslant [b+c]_t$. □

The following result deals with a partial completeness property of the fibers of the Bundle of Upper Semicontinuous Functions

Proposition 11. Let A be a non empty linearly ordered subset of E_t having an upper bound $[b]_t \in E_t$, then A has a least upper bound in E_t.

Proof. Let $[a]_t \in A$. Define the closed interval $[[a_o]_t, [b_o]_t] := [[a]_t, [b]_t]$, and if $\left[[\frac{a+b}{2}]_t, [b]_t\right] \cap A \neq \emptyset$, define $F_1 = [[a_1]_t, [b_1]_t] := [[\frac{a+b}{2}]_t, [b]_t]$, or else, $F_1 := [[a]_t, [\frac{a+b}{2}]_t]$. If $[[\frac{a_1+b_1}{2}]_t, [b_1]_t] \cap A \neq \emptyset$, take $F_2 = [[a_2]_t, [b_2]_t] := [[\frac{a_1+b_1}{2}]_t, [b_1]_t]$, or else, $F_2 := [[a_1]_t, [\frac{a_1+b_1}{2}]_t]$, and so on. The sequence of diameters $d_t([a_n]_t, [b_n]_t) = \frac{1}{2^n} d_t([a]_t, [b]_t)$ tends to zero, as n tends to infinity, then $\bigcap\{F_n : n = 1, 2, \dots\}$ reduces to a single point $[x]_t$ (obtained as the limit of any sequence $([x_n]_t)_n$, satisfying the condition $x_n \in F_n$, for all n, which secures that the sequence is a Cauchy sequence and also that $x \in \overline{F_n} = F_n$, for each n.) Since for each n, $[b_n]$ is an upper bound of A, so is $[x]_t$, indeed, assume that is not the case, then there is $[\tilde{a}]_t \in A$, such that $[x]_t <_t [\tilde{a}]_t$, then $d_t([x]_t, [\tilde{a}]_t) > 0$, take $\tilde{\epsilon} = d_t([x]_t, [\tilde{a}]_t) > 0$ and $\tilde{n}$ such that $d_t([b_{\tilde{n}}]_t, [x]_t) < \tilde{\epsilon}$, by the preceding lemma, $[x]_t <_t [\tilde{a}]_t \leqslant_t [b_m]_t$ implies that $d_t([b_m]_t, [x]_t) \geq d_t([\tilde{a}], [x])$, for each m, which is a contradiction. Furthermore, for each $\epsilon > 0$, there exists $F_n \subset ([x-\epsilon]_t, [x+\epsilon]_t)$, therefore $[x]_t = \sup A$. □

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Personal note of the third author. No doubt Jairo Charris was one of the best Colombian mathematicians of the 20^{th} century. His legacy is vast and precious. We remember him not only for his talent, diligence and passion but also for his noble character. He was extremely generous, compassionate and tolerant, still he did abhor those "flagrant inconsistencies" and the abuse of power.

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