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CATS, THE DOWNWARD LÖWENHEIM-SKOLEM-TARSKI THEOREM AND THE DISJOINT AMALGAMATION PROPERTY

PEDRO HERNÁN ZAMBRANO RAMÍREZ (∗)

Abstract. In this paper we prove that cats (compact abstract theories, see [Be03]) satisfy a version of Tarski-Vaught test (Theorem 3.1), a version of DLST (downward Löwenheim-Skolem Tarski) theorem using density character instead of cardinality (Theorem 3.3) and the DAP property (disjoint amalgamation property, Theorem 4.3).

Resumen. En este artículo demostramos que las cats (teorías abstractas compactas, ver [Be03]) satisfacen una versión del test de Tarski-Vaught (Teorema 3.1), una versión del Teorema DLST (Lowenheim-Skolem Tarski descendente) usando el carácter de densidad en lugar de la cardinalidad (Teorema 3.3) y la propiedad DAP (propiedad de amalgamación disyunta, Teorema 4.3).

Key words and phrases. Model theory, compact abstract theories, Tarski-Vaught test, downward Löwenheim-Skolem-Tarski theorem, disjoint amalgamation property.

Palabras claves. Teoría de Modelos, teorías abstractas compactas, test de Tarski-Vaught, Teorema de Lowenheim-Skolem-Tarski descendente, propiedad de amalgamación disyunta.

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(∗) Pedro Hernán Zambrano Ramírez. Departamento de Matemáticas Universidad Nacional de Colombia, Bogotá.
E-mail: phzambranor@unal.edu.co phzambranor@gmail.com
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1. Introduction

Itay Ben-Yaacov studied existentially closed (ec) models working with the positive fragment of $L_{\omega\omega}$ called Δ_0. This fragment is the positive closure of atomic formulas and change of variables in these formulas (see [Be03]).
Cats correspond to a very interesting categorical generalization of the notion of abstract elementary class (aec). Some of the abstract elementary class (aec) axioms do not directly hold in this context. In fact, the closure under the union of $\prec$-chains does not hold in some cat examples (a very known example of cat is the category of Banach spaces, see [Be03]). We can regard cats as the class of existentially closed models of a positive Robinson theory (see [Be03]), so we consider them in this way throughout this paper.

In the cat context we work with the density character notion instead of the cardinality notion, in a way similar to the positive bounded formulas context (see [HeIo02]). Although a cat does not satisfy some of the aec axioms, in this paper we prove a version of the downward Löwenheim-Skolem-Tarski theorem. In our version of this theorem, we work with the density character notion instead of the cardinality notion.

Ben-Yaacov has proved some interesting properties of cats (for example, he proved that cats satisfy a version of the categoricity Morley theorem but working with density character instead of cardinality and with a countable positive Robinson theory, see [Be05]).

We prove in this paper the disjoint amalgamation property (DAP) in the cat context.

2. Positive Robinson theories

Cats were introduced by I. Ben-Yaacov as a possible model theoretic generalization of Banach space structures. Ben-Yaacov centers his attention in density character instead of cardinality as in the context of positive bounded formulas (see [HeIo02]). Ben-Yaacov proved a version of the Morley theorem working with density character. In our proof of the downward Löwenheim-Skolem-Tarski theorem we focus on the density character notion.

Although Ben-Yaacov defined the density character notion as in the context of positive bounded formulas (using topological notions), he constructed a metric (in the topological sense) using the language.
C. W. Henson and J. Iovino based their work on abstract metrics.

The framework of cats is given in [Be03]. However, for the sake of completeness we enunciate some basic definitions and facts which are given in [Be03] and [Be05] which are necessary for defining the density character notion given in [Be05]

Definition 2.1 (change of variables)**.** Let $\varphi(x_{\in I})$ be a formula (with I a finite set), $J \subseteq \omega$ finite and $f : I \to J$ a mapping. We define the semantics of $f^*(\varphi(x_{\in I}))$ in the following way: for $a_{\in J} := \langle a_j : j \in J\rangle \in M^J$ we define $f^*(a_{\in J}) := \langle a_{f(i)} : i \in I\rangle \in M^I$ and $f_*(A) := (f^{*-1})(A)$ for $A \subseteq M^I$. Then, we define $M \models f^*(\varphi(a_{\in I}))$ if and only if $M \models \varphi(f^*(a_{\in J}))$

Definition 2.2. An *almost atomic formula* is the result of a change of variables on an atomic formula.

Definition 2.3. $\Delta \subseteq L_{\omega\omega}$ is called a *positive fragment* if every atomic formula belongs to Δ and if Δ is closed under subformulas, change of variables and positive combinations. Given Δ a positive fragment, $\Sigma(\Delta)$ denotes the closure of Δ under existential quantification (we can easyly prove that $\Sigma(\Delta)$ is a positive fragment). Also, we define $\Pi(\Delta) := \{\neg\varphi \mid \varphi \in \Sigma(\Delta)\}$.
We can write Σ and Π instead of $\Sigma(\Delta)$ and $\Pi(\Delta)$ respectively, if it is not ambiguous.

Definition 2.4. Let $\Delta \subseteq L_{\omega\omega}$ be a positive fragment. A mapping $f : M \to N$ (where M, N are first order L-structures) is called Δ-*homomorphism* if and only if $M \models \varphi[\overline{a}]$ implies $N \models \varphi[f(\overline{a})]$ for every $\varphi(\overline{x}) \in \Delta$ and every tuple $\overline{a}$ in M.
A mapping $f : M \to N$ (where M, N are first order L-structures) is called Δ-*embedding* if and only if $M \models \varphi[\overline{a}] \Leftrightarrow N \models \varphi[f(\overline{a})]$ for every $\varphi(\overline{x}) \in \Delta$ and every tuple $\overline{a}$ in M.

Definition 2.5. Let Δ be a positive fragment. If T is a Π-theory, we say that $M \models T$ is *existentially closed* (ec) if and only if every Δ-homomorphism $f : M \to N$ (where $N \models T$) is a $\Sigma(\Delta)$-embedding. The category of ec models of T is denoted by $\mathcal{M}(T)$.

Definition 2.6. A Π-theory T is called *positive Robinson theory* if given $M_i \in \mathcal{M}(T)$ and $a_i \in M_i^I$ $(i < 2)$ such that $tp_\Delta^{M_0}(a_0) \subseteq tp_\Delta^{M_1}(a_1)$, then $tp^{M_0}(a_0) = tp^{M_1}(a_1)$.

We can not work with syntactic types in some non-elementary classes because the class may not have a logic associated to it. However, Shelah gave a very interesting generalization of the syntactic type notion called Galois-type (see [Sh394]). These notions of types coincide in the first order context. Although cats are not abstract elementary classes, we can consider the Galois-type notion in the cat context. As we have a logic associated to cats (see [Be03]) then we can consider the syntactic type notion. In this context, the syntactic type notion (without parameters) is equivalent to the Galois-type notion (without parameters) as in the first order case.

Fact 2.7. *Given $M_i \in \mathcal{M} := \mathcal{M}(T)$ and $a_i \in M_i^I$ ($i < 2$), the following are equivalent:*

(1) *There exist $N \in \mathcal{M}$ and morphisms $f_i : M_i \to N$ ($i < 2$) such that $f_0(a_0) = f_1(a_1)$.*
(2) $tp^{M_0}(a_0) = tp^{M_1}(a_1)$.
(3) $tp^{M_0}(a_0) \subseteq tp^{M_1}(a_1)$.

We stress the following fact because a model $M \models T$ (where T is a Π-theory) may not be existentially closed, however the model can be extended to an existentially closed model.

Fact 2.8. *If $M \models T$, then there exists $N \in \mathcal{M}(T)$ and a Δ-homomorphism $f : M \to N$.*

Definition 2.9. A *pre-model* of T is a subset M of an universal domain of T (or a subset of an ec model of T) such that for every $n < \omega$ the realized n-types are denses in $S_n(M)$.

The following definitions are taken from [Be05]. We stress them because the density character notion is based on them.

Definition 2.10. A *distance* is a reflexive type-definable relation $\epsilon(x, y)$, without parameters. If $\epsilon(x, y)$ is a distance, we define $\overleftarrow{\epsilon}(x, y) := \epsilon(y, x)$. It is called *symmetric* if $\overleftarrow{\epsilon} = \epsilon$. Otherwise, we define $\overleftrightarrow{\epsilon} := \epsilon \wedge \overleftarrow{\epsilon}$.

The minimal distance is $\{x = y\}$ and the maximal one is $\{T\}$, where T is the true formula (denoting them 0 and ∞, respectively).

Definition 2.11. We say that $\epsilon \leq \epsilon'$ if $\epsilon \vdash \epsilon'$. Also, we say that $\epsilon < \epsilon'$ if $\epsilon \subseteq (\epsilon')^\circ$, where we interpret ϵ and ϵ' as subsets of $S_2(T)$ and $(\epsilon')^\circ$ is the interior of ϵ' there (we can interpret δ as $\delta^t := \{tp(a, b) \mid a, b \models \delta\}$, with $\delta \in \{\epsilon, \epsilon'\}$).

Definition 2.12. A distance ϵ is called *positive* if $\epsilon > 0$. We say that $d(a,b) \leq \epsilon$ if $tp(a,b) \in \epsilon$ and $d(a,b) < \epsilon$ if $tp(a,b) \in \epsilon^{\circ}$.

Definition 2.13. Fix a sort and X a set of elements in that sort, we say $F \subseteq X$ is *closed* in the *logic topology* if it is defined in X by some partial type with parameters. Thus, an open set is one defined by the negation of some partial type.

Definition 2.14. For a distance $\epsilon > 0$ and an element b, we define $B^X(b,\epsilon) := \{c \in X \mid d(b,c) \leq \epsilon\}$.

Fact 2.15. *A subset $U \subseteq X$ is open if and only if for every $a \in U$ there is $\epsilon > 0$ such that $B^X(a,\epsilon) \subseteq U$.*

Definition 2.16. $\varepsilon^{>0} := \{\epsilon \mid \epsilon > 0\}$ is the set of all positive distances. A *base of positive distances* is a family $\varepsilon \subseteq \varepsilon^{>0}$ which is co-final in the sense that for every $\epsilon > 0$ there exists $\epsilon' \in \varepsilon$ such that $\epsilon > \epsilon'$. $cfdist(T)$ is the minimal cardinality of a base of positive distances. If $cfdist(T) \leq \omega$, we say that T is *metric.*

Definition 2.17. Assume that T is metric. Let $(\epsilon_q \mid q \in \mathbb{Q} \cap [0,1])$ be a sequence of distances such that $q > r$ implies $\epsilon_q > \epsilon_r$ and $\bigwedge \epsilon_{1/n} = 0$. Define $h(a,b) := \inf\{q : d(a,b) \leq \epsilon_q\} = \sup\{q : d(a,b) \geq \epsilon_q\}$ (taking $\inf \emptyset := 1$ and $\sup \emptyset := 0$). That sequence can be constructed. For more details, see [Be05].

We have h is not a metric (in the topological sense), however Ben-Yaacov discovered a technical way for constructing a metric based on h.

Notation 2.18. Let D denote the set of dyadic numbers in $[0,1]$

Fact 2.19. *Let $g : [0,1]^2 \to [0,1]$ be a symmetric, non-decreasing mapping that satisfies $g(0,t) = t$ for every $t \in [0,1]$ and if $g(u,w) < t$ then there exists $u < v \leq 1$ such that $g(v,w) < t$ for every $u,w,t \in [0,1]$. Then there exists a function $f : D \to [0,1]$ such that:*

(1) *f is strictly increasing*
(2) *For all n we have $f(\frac{1}{2^n}) \leq \frac{1}{2^n}$*
(3) *For every $t,u \in D \cap [0,1]$ we have $t + u \leq 1$ implies $g(f(t),f(u)) < f(t+u)$.*

Fact 2.20. *Define* $g : [0,1]^2 \to [0,1]$ *by* $g(t,,u) := \sup\{h(a,b) \mid$ *there exists* c *such that* $h(a,c) \leq t$ *and* $h(c,b) \leq u\}$. *Then* g *satisfies the assumptions of fact 2.19.*

It follows that there exists a function f as the conclusion of fact 2.19.

Fact 2.21. *Define* $d(a,b) := \inf\{t \mid h(a,b) < f(t)\} = \sup\{t \mid h(a,b) > f(t)\}$. *Then* d *is a metric (in the topological sense).*

Definition 2.22. A *complete model* is the closure $\overline{M}$ of a pre-model M.

Fact 2.23. *Every ec model of* T *is a pre-model.*

Definition 2.24. Let M be a complete model. We define the *density character* of M (we denote it by $dc(M)$) by the least cardinal of a dense subset of M. Let A be a subset, we define

$$dc(A) := \min\{dc(M) \mid A \subseteq dcl(M) \text{ and } M \text{ is a complete model}\}$$

3. DLST in cats

The following facts hold in every positive fragment, though we are interested in the fragment Δ_0. We will assume that T is a positive Robinson theory in a positive fragment Δ.

First, we will prove a version of the Tarski-Vaught test.

Theorem 3.1 (Tarski-Vaught test)**.** *Taking* $\mathfrak{C} := \mathcal{M}(T)$*, the following are equivalent:*

1. $M \preceq_{\mathfrak{C}} N$.
2. $M \subseteq_{\mathcal{L}} N$ *and given* $\varphi \in \Sigma(\Delta)$ *and* $a \in |M|$*, if there exists* $b \in |N|$ *such that* $N \models \varphi(b,a)$ *then there exists* $b' \in |M|$ *such that* $N \models \varphi(b',a)$.

Proof. (2. $\Rightarrow$ 1.) We have $M \preceq_{\mathfrak{C}} N$ is equivalent to $M \models \varphi(a) \Leftrightarrow N \models \varphi(a)$ for every $\varphi \in \Sigma := \Sigma(\Delta)$ and every $a \in |M|$. We prove this fact by induction over formulas.

1. If R is a $\mathcal{L}$-relational symbol and $\overline{a} \in |M|^{arity(R)}$, we have $M \models R(a) \Leftrightarrow N \models R(\overline{a})$ because $M \subseteq_{\mathcal{L}} N$.

(2) If $\varphi = \psi_1 \wedge \psi_2$, $M \models \varphi(a) \Leftrightarrow M \models \psi_1(a) \wedge \psi_2(a) \Leftrightarrow M \models \psi_1(a)$ and $M \models \psi_2(a) \Leftrightarrow N \models \psi_1(a)$ and $N \models \psi_2(a)$ (inductive hypothesis) $\Leftrightarrow N \models \psi_1(a) \wedge \psi_2(a) \Leftrightarrow N \models \varphi(a)$

(3) If $\varphi = \psi_1 \vee \psi_2$, we can prove it in a way similar to the previous case.

(4) If $\varphi = \neg\psi \in \Sigma$ we have $\psi \in \Sigma$ because Σ is closed over subformulas. So $M \models \varphi(a) \Leftrightarrow M \models \neg\varphi(a) \Leftrightarrow M \not\models \psi(a) \Leftrightarrow N \not\models \psi(a)$ (inductive hypothesis) $\Leftrightarrow N \not\models \psi(a) \Leftrightarrow N \models \varphi(a)$

(5) If $\varphi = \exists x\psi \in \Sigma$, we have $\psi \in \Sigma$. So $N \models \varphi(a)$ implies there exists $b \in |N|$ such that $N \models \psi(b, a)$. By hypothesis, there exists $b' \in |M|$ such that $N \models \psi(b', a)$ and by inductive hypothesis we have $M \models \psi(b', a)$, so $M \models \varphi(a)$. In the other way, if $M \models \varphi(a)$ then there exists $c \in |M|$ such that $M \models \psi(c, a)$ and by inductive hypothesis we have $N \models \psi(c, a)$. We have $c \in |M| \subseteq |N|$, so $M \models \varphi(a)$.

(1. $\Rightarrow$ 2.) We have $M \preceq_{\mathfrak{C}} N$, so $M \subseteq_{\mathcal{L}} N$ (because Σ has every atomic formula). Let $\varphi \in \Sigma$ and $a \in |M|$. Assume there exists $b \in |N|$ such that $N \models \varphi(b, a)$, therefore $N \models \exists x\varphi(x, a)$. $\exists x\varphi(x, a) \in \Sigma$, then by hypothesis $M \models \exists x\varphi(x, a)$. If $b' \in |M|$ is a witness of the previous claim, then $M \models \varphi(b', a)$; so by hypothesis we have $N \models \varphi(b', a)$. □

We can prove a version of the downward Löwenheim-Skolem-Tarski theorem using our version of Tarski-Vaught test. We must prove that the model we construct is also existentially closed, a step that was not needed in the first order case. First, we will prove a version of this theorem working with cardinality.

Theorem 3.2. *If $M \in \mathfrak{C} := \mathcal{M}(T)$ and $A \subseteq |M|$, there exists $N \preceq_{\mathfrak{C}} M$ such that $A \subseteq |N|$ and $\|N\| \leq |A| + |L| + \aleph_0$.*

Proof. Consider $\Sigma := \Sigma(\Delta)$. We obtain an ec model $N \subseteq_{\mathcal{L}} M$ such that $\|N\| \leq |A| + |L| + \aleph_0$ and $A \subseteq |N|$, where $i : N \hookrightarrow M$ is a Σ-embedding. Fix $<$ a well-order in $|M|$ and denote by b_0 its first element. For every $\varphi(x, \overline{y}) \in \Sigma$ and every $\overline{a} \in |a|$ we define:

$$G_\varphi(\overline{a}) := \begin{cases} \min \{b \in |M| : M \models \varphi(b, \overline{a})\} & \text{if } M \models \exists x\varphi(x, \overline{a}) \\ b_0 & \text{otherwise} \end{cases}$$

Also, we construct $A_n \subseteq |M|$ $(n < \omega)$ such that

(1) $A_0 = A$

(2) For every $n < \omega$, $A_n \subseteq A_{n+1}$

(3) For every $n < \omega$, $|A_n| \leq |L| + |A| + \aleph_0$
(4) For every $n < \omega$, every formula $\varphi(x, \overline{y}) \in \Sigma$ and every $\overline{a} \in A_n$, we have $G_\varphi(\overline{a}) \in A_{n+1}$.

Taking $S := \bigcup_{n<\omega} A_n$, notice that S is the domain of an L-structure (where we denote it by N) and $N \preceq_{\mathfrak{C}} M$ (if $\varphi(x, \overline{y}) \in \Sigma$, $b \in |M|$ and $\overline{a} \in S$ (therefore exists $N < \omega$ such that $\overline{a} \in A_N$) are such that $M \models \varphi(b, \overline{a})$, so $M \models \exists x \varphi(x, \overline{a})$ and therefore $G_\varphi(\overline{a}) \in A_{N+1} \subseteq S$ and $M \models \varphi(G_\varphi(\overline{a}), \overline{a})$, by theorem 3.1 we have $N \preceq_{\mathfrak{C}} M$).

We can construct this sequence, taking $A_0 := A$ and $A_{n+1} := A_n \cup \{G_\varphi(\overline{a}) : \overline{a} \in A_n, \varphi \in \Sigma\}$, where by construction we have $\|N\| \leq |A| + |L| + \aleph_0$.

We will prove that N is existentially closed. Let $M' \models T$ (wlog we can assume it is existentially closed, by fact 2.8) and $f : N \to M'$ a Δ-homomorphism such that $M' \models \varphi(f(\overline{a}))$, where $\varphi(\overline{x}) \in \Sigma$ and $\overline{a} \in |N|$. We have $tp^M(\overline{a}) \subseteq tp^{M'}(f(\overline{a}))$: if $\psi(\overline{x}) \in tp^M(\overline{a})$ then $M \models \psi(\overline{a})$, as $i : N \hookrightarrow M$ is a Σ-embedding then $N \models \psi(\overline{a})$, as $f : N \to M'$ is a Δ-homomorphism then it is a Σ-homomorphism and $M' \models \psi(f(\overline{a}))$; i.e., $\psi(\overline{x}) \in tp^{M'}(f(\overline{a}))$. By fact 2.7 there exists $N' \in \mathcal{M}(T)$ and $f_0 : M' \to N'$, $f_1 : M \to N'$ Δ-homomorphisms such that $f_0(f(\overline{a})) = f_1(\overline{a})$. So, $N' \models \varphi(f_0(f(\overline{a})))$, as $f_0(f(\overline{a})) = f_1(\overline{a})$ and M is existentially closed (as $M \in \mathcal{M}(T)$) then $M \models \varphi(\overline{a})$. As $i : N \hookrightarrow M$ is a Σ-embedding, we have $N \models \varphi(\overline{a})$. Therefore $N \in \mathcal{M}(T)$. □

Theorem 3.3 (DLST in cats)**.** *If $M \in \mathfrak{C} := \mathcal{M}(T)$ and $A \subseteq |M|$, there exists $N \preceq_{\mathfrak{C}} M$ such that $dc(N) \leq (|A| + |L| + \aleph_0)^\omega$ and $A \subseteq |N|$.*

Proof. By theorem 3.2, we obtain an existentially closed model N such that $A \subseteq |N|$ and $N \preceq_{\mathfrak{C}} M$. By fact 2.23 we have N is pre-model, therefore $\overline{N}$ is a complete model such that $dc(\overline{N}) \leq |\overline{N}| \leq |N|^\omega \leq (|A|+|L|+\aleph_0)^\omega$ and $N \subseteq \overline{N} \subseteq dcl(\overline{N})$. So, $dc(N) = \min\{dc(M) \mid N \subseteq dcl(M) \text{ and } M \text{ is a complete model}\} \leq (|A| + |L| + \aleph_0)^\omega$. □

4. Cats and DAP

In this section, we will work with the fragment Δ_0 because we can not work with negations in the cat context. But it is not a problem because we work with that fragment in the cat context. We assume that T is a positive Robinson theory in a fragment Δ_0.

Notation 4.1.

$$DC(M) := \{\varphi(\overline{a}) : \varphi(\overline{x}) \text{ is almost atomic, } \overline{a} \in |M| \text{ and } \quad M \models \varphi(\overline{a})\}.$$

For proving $N \preceq_{\mathfrak{C}} M$ with $M, N \in \mathfrak{C}$, we only need to prove $M \models DC(N)$ (modulo renaming; intuitively, renaming is a technique for obtaining an isomorphic model to a given model M that contains a set which is embedded into the original model M).

Lemma 4.2. *Let $M, N \in \mathfrak{C} := \mathcal{M}(T)$ such that $M \models DC(N)$. Then $N \preceq_{\mathfrak{C}} M$ (using renaming in M).*

Proof. We will prove for all $\varphi(\overline{x}) \in \Delta_0$ and for all $\overline{a} \in |N|$, $N \models \varphi(\overline{a})$ implies $M \models \varphi(\overline{a}')$, where $\overline{a}' := \overline{a}^M$.

(1) If φ is almost atomic, obviously $N \models \varphi(\overline{a})$ implies $M \models \varphi(\overline{a}')$, as $M \models DC(N)$.
(2) If $\varphi = \psi_0 \wedge \psi_1$, suppose the result holds for ψ_i $(i \in \{0,1\})$. If $N \models \varphi(\overline{a})$ then $N \models (\psi_0 \wedge \psi_1)(\overline{a})$. Therefore $N \models \psi_0(\overline{a})$ and $N \models \psi_1(\overline{a})$; by inductive hypothesis we have $M \models \psi_0(\overline{a}')$ and $M \models \psi_1(\overline{a}')$; i.e., $M \models (\psi_0 \wedge \psi_1)(\overline{a}')$.
(3) If $\varphi = \psi_0 \vee \psi_1$, suppose the result holds for ψ_i $(i \in \{0,1\})$. If $N \models \varphi(\overline{a})$ then $N \models (\psi_0 \vee \psi_1)(\overline{a})$. Therefore $N \models \psi_0(\overline{a})$ or $N \models \psi_1(\overline{a})$; by inductive hypothesis we have $M \models \psi_0(\overline{a}')$ or $M \models \psi_1(\overline{a}')$; i.e., $M \models (\psi_0 \vee \psi_1)(\overline{a}')$.

Therefore, $f : N \to M$ defined by $f(a) := a'$ is a Δ_0-homomorphism. As N is existentially closed (as $N \in \mathcal{M}(T)$) we have f is a $\Sigma(\Delta_0)$-embedding. Renaming M, we can assume $N \preceq_{\mathfrak{C}} M$. □

In the following lines, we will give a proof of the disjoint amalgamation property in the cat context. We need to prove that the model which we construct is also existentially closed. We did not need to prove this step in the first order case.

Theorem 4.3 (DAP in cats)**.** *Let $M, N_0, N_1 \in \mathfrak{C} := \mathcal{M}(T)$ such that $M \preceq_{\mathfrak{C}} N_0$ and $M \preceq_{\mathfrak{C}} N_1$. Then there exists $N \in \mathfrak{C}$ such that $N_0 \preceq_{\mathfrak{C}} N$ and $g : N_1 \to N$ an elementary embedding such that $g(|N_1|) \cap |N_0| = |M|$.*

Proof. Wlog, we can assume $|N_0| \cap |N_1| = |M|$. Let $T_1 := DC(N_0) \cup DC(N_1) \cup \{\neg\varphi(\overline{b}, \overline{d}) : \varphi \in \Delta_0, \overline{b} \in N_0, \overline{d} \in N_1, N_0 \not\models \varphi(\overline{b}, \overline{a}) \text{ for all } \overline{a} \in M\} \cup T$. Assume T_1 does not have a model. We can use the *first order compactness theorem* because T_1 is a first order theory. By first orden compactness theorem, there exists $\overline{a} \in M$, $\overline{d} \in N_1 - M$, $\overline{b} \in N_0$, $\theta(\overline{a}, \overline{d}) \in DC(N_1)$, $\varphi \in T$ and $\neg\varphi_i(\overline{b}, \overline{a}, \overline{d})$ ($i < k$ with fixed $k < \omega$ and $N_0 \not\models \varphi_i(\overline{b}, \overline{a}, \overline{a}')$ for all $\overline{a}, \overline{a}' \in M$) such that $DC(N_0) \cup \{\theta(\overline{a}, \overline{d}), \varphi\} \models \bigvee_{i<k} \varphi_i(\overline{b}, \overline{a}, \overline{d})$. As $N_1 \models \theta(\overline{a}, \overline{d})$, so $N_1 \models \exists\overline{y}\theta(\overline{a}, \overline{y})$, and as $M \preceq_{\mathfrak{C}} N_1$ then there exists $\overline{a}'' \in M$ such that $M \models \theta(\overline{a}, \overline{a}'')$. We have $M \preceq_{\mathfrak{C}} N_0$, so $N_0 \models \theta(\overline{a}, \overline{a}'')$. Also $N_0 \models \varphi$, as $N_0 \models T$, then there exists $i < k$ such that $N_0 \models \varphi_i(\overline{b}, \overline{a}, \overline{a}'')$ (contradiction). So, there exists $N \models T_1$ (therefore $N \models T$). Wlog we can assume N is e.c. by fact 2.8 (however, the required intersection is not going to grow because T_1 is codifying that fact). As $N \models DC(N_0) \cup DC(N_1)$ we have $N_0 \preceq_{\mathfrak{C}} N$ and there exists $g : N_1 \to N$ an elementary embedding such that $g(\overline{c}) = \overline{c}^N$ for all $\overline{c} \in N_1$. Obviously $|M| \subseteq g(|N_1|) \cap |N_0|$. If $b \in N_0$ is taken such that $N \models b = g(c)$ for some $c \in N_1$, then there exists $a \in M$ such that $N_0 \models b = a$ (otherwise taking the formula $x = y$ we have $N_0 \models \neg(b = a)$ for every $a \in M$, so $N \models \neg(b = g(c))$ as $\neg(b = g(c)) \in T_1$, contradiction), therefore $|M| \supseteq g(|N_1|) \cap |N_0|$ □

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