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Pöschl-Teller potentials based solution to Hilbert's tenth problem[1]

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Resumen

Los hipercomputadores computan funciones o números, o en general solucionan problemas que no pueden ser computados o solucionados por una máquina de Turing. Se presenta una adaptación del algoritmo cuántico hipercomputacional propuesto por Tien D. Kieu, al álgebra dinámica $\mathfrak{su}(1,1)$ realizada en los potenciales Pöschl-Teller. El problema clásicamente incomputable que se resuelve con este algoritmo hipercomputacional es el décimo problema de Hilbert. Se señala que una condición matemática fundamental para estos algoritmos es la existencia de una representación unitaria infinito dimensional irreducible de álgebras de baja dimensión que admitan la construcción de estados coherentes del tipo Barut-Girardello. Adicionalmente se presenta como caso límite del algoritmo propuesto sobre los potenciales Pöschl-Teller, el algoritmo hipercomputacional sobre la caja de potencial infinita construido previamente por los autores.

Palabras claves: hipercomputación, computación cuántica adiabática, décimo problema de Hilbert.

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Abstract
Hypercomputers compute functions or numbers, or more generally solve problems or carry out tasks, that cannot be computed or solved by a Turing machine. An adaptation of Tien D. Kieu's quantum hypercomputational algorithm is carried out for the dynamical algebra $\mathfrak{su}(1,1)$ of the Pöschl-Teller potentials. The classically incomputable problem that is resolved with this hypercomputational algorithm is Hilbert's tenth problem. We indicated that an essential mathematical condition of these algorithms is the existence of infinite-dimensional unitary irreducible representations of low dimensional dynamical algebras that allow the construction of coherent states of the Barut-Girardello type. In addition, we presented as a particular case of our hypercomputational algorithm on Pöschl-Teller potentials, the hypercomputational algorithm on an infinite square well presented previously by the authors.

Key words: hypercomputation, adiabatic quantum computation, Hilbert's tenth problem.

1 Introduction

In a memorable international mathematics congress in Paris in 1900, David Hilbert proposed a series of twenty three problems, which according to their purpose, would mark the future of mathematics in the 20th century. The tenth problem consisted in finding an effective procedure, which would determine whether or not a Diophantine equation had a non-negative integer solution. The negative response to that problem was found 70 years later by Yury V. Matiyasevich [2], who resolved Hilbert's tenth problem, establishing its equivalence with the Halting problem (the Turing machine incomputable problem by antonomasia).

The hypercomputers compute functions or numbers, or more generally solve problems or carry out tasks, that cannot be computed or solved by a Turing machine (TM) [3, 4]. Since 2001, Tien D. Kieu has proposed that even though Hilbert's tenth problem is not computable in the realm of Turing machines, it is possible after all to compute it [5, 6, 7, 8], but within a new computation paradigm named quantum adiabatic hypercomputation [9, 10]. Kieu's proposal has generated much controversy because it is based on arguments that are polemic on their own or are at least not unanimously accepted, such as those that stem from hypercomputation [11, 12, 13, 14]. However, until now, no one has established well-founded arguments that show any fallacy in Kieu's hypercomputational construction [8, 15, 16]. In this sense, Kieu's idea continues to be valid.

The purpose of this work is to present an algorithm of quantum hypercomputation *à la* Kieu for Hilbert's tenth problem, which contributes important considerations about the role that infinite-dimensional unitary irreducible representations (UIR) of dynamical algebras play in the hypercomputational context. Since our algorithm solves a Turing machine incomputable problem, it is not possible to make a comparison in terms of algorithmic complexity between our algorithm and a possible Turing machine computable algorithm. The way to proceed is to substitute Weyl-Heisenberg algebra $\mathfrak{g}_{\text{W-H}}$ carried out in the quantum harmonic oscillator for non-compact Lie algebra $\mathfrak{su}(1,1)$ carried out in the Pöschl-Teller potentials [17]. This work is a generalization of previous work done by the authors about the hypercomputational algorithm on an infinite square well [18, 19]. Here the hypercomputational algorithm on an infinite square well is obtained as a particular case of our hypercomputational algorithm on Pöschl-Teller potentials..

2 Hilbert's Tenth Problem and Kieu's Algorithm

Kieu's idea is essentially to transform Hilbert's tenth problem in the realm of the theory of numbers, into a quantum problem in the realm of the spectral theory and to resolve this problem using the adiabatic theorem of quantum mechanics [20]. Kieu's proposal consists in codifying a Diophantine polynomial $D(x_1, x_2, .., x_k)$ via a certain quantum operator named H_D which represents Hamiltonian's role for a given quantum system. The issue of determining if the Diophantine equation $D(x_1, x_2, .., x_k) = 0$ has a non-negative integers solution, is now reconsidered in determining if the quantum operator H_D has a zero-energy eigenvalue.

This new problem cannot be resolved directly and Kieu proposes to resolve it in an indirect fashion by using the adiabatic theorem. This theorem affirms that it is possible to control the instantaneous spectrum of a time dependent operator, if the initial spectrum is known and if the operator involved is the corresponding Hamiltonian of a Schrödinger equation. In other words, the theorem suggests an understanding of something regarding the H_D spectrum, through interpolation from the known spectrum of a certain operator H_I; considering the interpolating Hamiltonian of the form $H_A(t) = (1-s)H_I + sH_D$, with $0 \leq s \leq 1$. Since the zero-eigenvalue of H_D is justly the question, it is natural to begin the quantum evolution with an eigenstate of H_I associated with the zero-energy eigenvalue and to solve the Schrödinger equation $i\hbar\frac{\partial}{\partial t}\,|\,\Psi(t)\rangle = H_A(t)\,|\,\Psi(t)\rangle$ with the initial condition $|\,\Psi(0)\rangle = |\,ini\rangle$ such that $H_I\,|\,ini\rangle = 0$. Then, measuring the quantum state in $t = T$, corresponding to $|\,\Psi(T)\rangle = |\,final\rangle$ will obtain information on the lowest eigenvalue of H_D given that according to the adiabatic theorem, the state $|\,final\rangle$ is an eigenvector of H_D corresponding to the lowest energy eigenvalue.

Kieu's algorithm incorporates the following elements: (i) A physical quantum referent, (ii) An algebraic structure carried out on the physical quantum referent, it is to say certain dynamical algebra, (iii) A coding scheme of the Diophantine equation, (iv) The initiation of the quantum system, (v) A quantum adiabatic evolution process, (vi) A measuring procedure of quantum states, (vii) A halting criteria and (viii) A decoding scheme to determine the solution to the Diophantine equation. The (one-dimensional) simple harmonic oscillator (SHO) is the physical quantum referent selected by Kieu for his algorithm, which represents the quantum extension of the classic harmonic oscillator of mechanics [21]. In contrast to classical mechanics, in quantum mechanics the fundamental notions are not Newton's forces and Newton's equation, instead, they are Schrödinger's energy and Schrödinger's equation. In quantum mechanics, energy is represented by an operator that operates in a functional space of wave functions, and the spectral properties of this operator determine the properties of the physical observables. The energy operator is named Hamiltonian and is considered the engine of the quantum evolution of systems. For the SHO, the Hamiltonian operator is defined as

$$H = a^\dagger a + 1/2\,,$$

where $a^\dagger$ represents the creation operator and a denotes the annihilation operator. These operators act upon the space of quantum states $|\,n\rangle$ of the SHO defined by

$$\{|\,n\rangle \mid n \in \mathbb{N} = \{0, 1, 2, \dots\}\}\,. \tag{1}$$

The action of the annihilation operator on the vacuum and actions of the creation and annihilation operators on a general state of (1) have the form

$$a\,|\,0\rangle = 0, \qquad a\,|\,n\rangle = \sqrt{n}\,|\,n-1\rangle\,, \qquad a^\dagger\,|\,n\rangle = \sqrt{n+1}\,|\,n+1\rangle\,, \tag{2}$$

and the commutation relations between the creation and annihilation operators which define the Lie algebra carried on the SHO are

$$[a, a^\dagger] = 1\,, \qquad [a, a] = [a^\dagger, a^\dagger] = 0\,. \tag{3}$$

The number operator defined by way of the creation and annihilation operators of the SHO is given as

$$N = a^\dagger a\,, \qquad N\,|\,n\rangle = n\,|\,n\rangle\,, \tag{4}$$

and the coherent state, eigenvector of the annihilation operator of the algebra is

$$|\,\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|\,n\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger}\,|\,0\rangle \quad \alpha \in \mathbb{C}\,,$$

where a$|\,\alpha\rangle = \alpha\,|\,\alpha\rangle$ and the probability distribution of the discrete random variable n of this coherent state is

$$P_n(\alpha) = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}\,.$$

Given a Diophantine equation with k unknowns,

$$D(x_1, \ldots, x_k) = 0\,, \tag{5}$$

Kieu provides the following quantum algorithm to decide whether this equation has any non-negative integer solution or not:

1. Construct a physical process in which a system initially begins from a state that is a direct product of k coherent states

$$|\,\psi(0)\rangle = \bigotimes_{i=1}^{k} |\,\alpha_i\rangle\,, \tag{6}$$

and from which the system is submitted to the action of a Hamiltonian $H_A(t)$ dependent on time over the interval $[0, T]$, for a time T

$$H_A(t) = \left(1 - \frac{t}{T}\right) H_I + \left(\frac{t}{T}\right) H_D\,, \tag{7}$$

with the initial Hamiltonian

$$H_I = \sum_{i=1}^{k} \left(a_i^{\dagger} - \alpha_i^{*}\right)(a_i - \alpha_i)\,, \tag{8}$$

and the final Hamiltonian

$$H_D = (D(N_1, \ldots, N_k))^2\,. \tag{9}$$

2. Measure or estimate (using the Schrödinger equation with the Hamiltonian $H_A(t)$) the maximum probability of finding the system in a particular multi-particle state in the chosen time T

$$\begin{aligned} P_{max}(T) &= \max_{\mid\{n\}\rangle} \mid \langle \psi(T) \mid \{n\}\rangle \mid^2 \\ &= |\langle \psi(T) \mid \{n\}_0\rangle|^2\,, \end{aligned}$$

where $\mid\{n\}_0\rangle$ (which is a direct product of k particular states, $\bigotimes_{i=1}^{k} \left| n_i^0 \right\rangle$) possesses the maximum probability among the rest of the multi-particle states.

3. If $P_{\max}(T) \leq 1/2$, increase T and repeat the previous steps.

4. If

$$P_{\max}(T) > 1/2 \tag{10}$$

then $\mid\{n\}_0\rangle$ is the fundamental state of H_D (it is assumed that there is no spectral degeneration) and the following conclusion is obtained: $H_D \mid \{n\}_0\rangle = 0$, if and only if, (5) has a non-negative integer solution.

The Halting criteria (10) is established by the peak of maximum probability corresponding to the initial state (6), where, for any n and α

$$|\langle \alpha \mid n\rangle|^2 = P_n(\alpha) < 1/2\,,$$

and verifying that the probability that any excited state is not greater than 1/2 at any time.

In spite of the fact that Kieu does not explicitly mention it, the dynamical algebra associated with the SHO is Weyl-Heisenberg $\mathfrak{g}_{\text{W-H}}$ algebra, established by the commutation relations (3). This algebra has infinite-dimensional UIR given by actions (2) over base (1).

3 Algorithm *à la* Kieu on the Pöschl-Teller Potentials

In this section we present an adaptation of Kieu's algorithm, which consists in replacing Weyl-Heisenberg dynamical algebra corresponding to the SHO by the dynamical algebra $\mathfrak{su}(1,1)$ related to the Pöschl-Teller (PT) potentials [17].

The problem of finding the energy spectrum and the wave functions of a particle of mass m confined inside the infinite square well $0 \leq x \leq \pi l$, submitted to a barrier of infinite potential at both frontiers, has commonly been one of the most elemental and illustrative problems of quantum mechanics. The energy levels result being quantized and the wave functions correspond to sinusoidal functions that satisfy the conditions imposed at the frontiers [17]. The previous situation is generalized for a situation in which the potential to which the particle is submitted, is given by a family of continuously indexed potentials of the PT type [17]

$$V^{PT}_{\lambda,\kappa}(x) = \frac{1}{2}V_0\left(\frac{\lambda(\lambda-1)}{\cos^2 x/2l} + \frac{\kappa(\kappa-1)}{\sin^2 x/2l}\right), \qquad 0 \leq x \leq \pi l\,, \tag{11}$$

where the continuous parameters $\lambda, \kappa > 1$ and the coupling constant $V_0 > 0$.

The Hamiltonian

$$H^{\mathrm{PT}} = i^2\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{\hbar^2}{8ml^2}\left(\frac{\lambda(\lambda-1)}{\cos^2 x/2l} + \frac{\kappa(\kappa-1)}{\sin^2 x/2l}\right) - \frac{\hbar^2}{8ma^2}(\lambda+\kappa)^2, \tag{12}$$

where $V_0 = \hbar^2/4ml^2$ corresponds to a particle of mass m subject to the interaction of the PT potentials [17]. The constant term that appears in the Hamiltonian (12) depends on the particular choice of zero in the energy scale.

The spectrum of the energy values and their corresponding eigenstates are the solutions of the Schrödinger's equation $H^{\mathrm{PT}}\psi(x) = E\psi(x)$ for the Hamiltonian independent of time (12), subject to boundary conditions $\psi(0) = \psi(\pi l) = 0$. The normalized eigenstates of the Schrödinger's equation are given by [17]

$$\Psi_n(x) = [c_n(\lambda,\kappa)]^{-1/2}\left(\cos\frac{x}{2l}\right)^{\lambda}\left(\sin\frac{x}{2l}\right)^{\kappa} {}_2F_1\left(-n, n+\lambda+\kappa; \kappa+\frac{1}{2}; \sin^2\frac{x}{2l}\right),$$

where $[c_n(\lambda,\kappa)]^{-1/2}$ is a normalization factor given analytically when λ, κ are positive integers, and ${}_2F_1$ is a particular case of the generalized hypergeometric function.

Now the wave function $\Psi_n(x)$ is rewritten

$$\Psi_n(x) \equiv \langle x \mid \eta/2, n\rangle \,, \qquad 0 \leq x \leq \pi l \,,$$

where $\eta = \lambda + \kappa + 1$ and Hamiltonian's action over its normalized eigenvalues is

$$H^{\mathrm{PT}} \mid \eta/2, n\rangle = E_n^{\mathrm{PT}} \mid \eta/2, n\rangle \,. \tag{13}$$

The spectrum of values of the energy corresponding to the Hamiltonian of the particle crucially depends on parameters λ, κ

$$E_n^{\mathrm{PT}} = \hbar\omega e_n(\lambda, \kappa) \,, \tag{14}$$

where $\omega = \hbar/2ml^2$ and $e_n(\lambda, \kappa) = n(n + \lambda + \kappa)$.

Given the spectral structure of the PT potentials, it is possible to associate a dynamical algebra based on a similar structure to the one developed for the creation and annihilation operators in the case of the SHO. The generators of the dynamical algebra are constructed based on the PT potentials, having as a starting point the spectral structure defined in (14) and according to the following criteria [17]

$$K_+ \mid \eta/2, n\rangle = \sqrt{e_{n+1}(\lambda, \kappa)} \mid \eta/2, n+1\rangle \,, \tag{15}$$

$$K_- \mid \eta/2, n\rangle = \sqrt{e_n(\lambda, \kappa)} \mid \eta/2, n-1\rangle \,, \tag{16}$$

$$K_3 \mid \eta/2, n\rangle = [e_{n+1}(\lambda, \kappa) - e_n(\lambda, \kappa)] \mid \eta/2, n\rangle \,, \tag{17}$$

where

$$\begin{aligned} e_{n+1}(\lambda, \kappa) &= (n+1)(n+\eta) \,, \\ e_n(\lambda, \kappa) &= n(n+\eta-1) \,, \\ e_{n+1}(\lambda, \kappa) - e_n(\lambda, \kappa) &= (2n+\eta) \,. \end{aligned} \tag{18}$$

The operators K_+, K_- and K_3 are called creation, annihilation and Cartan operators respectively, in analogy with the SHO. Those operators satisfy the commutation relations of Lie algebra $\mathfrak{su}(1,1)$ given by

$$[K_\pm, K_3] = \mp 2K_\pm \,, \qquad [K_-, K_+] = K_3 \,,$$

which admits an infinite-dimensional UIR, given by (15), (16), and (17) that is obtained from the actions of creation, annihilation and Cartan operators over the base states defined in (13).

Based on the spectrum of the values of the energy defined in (14) and (13), (15) and (16), the Hamiltonian (12) could be rewritten in the following way

$$H^{\mathrm{PT}} = \hbar\omega K_{+} K_{-} \,.$$

From (17) and (18) a new number operator is constructed given as

$$N^{\mathrm{PT}} \,|\, \eta/2, n\rangle = n \,|\, \eta/2, n\rangle \,, \qquad N^{\mathrm{PT}} = (1/2)(K_3 - \eta) \,, \tag{19}$$

where the eigenstates of the number operator N^{PT} constitute an orthonormal base for the Fock space

$$\mathcal{H} = \{|\, \eta/2, n\rangle \,| n \in \mathbb{N}\} \,,$$

where the ket $|\, \eta/2, 0\rangle$ is called the vacuum normalized state given that it satisfies

$$K_{-} \,|\, \eta/2, 0\rangle = \mathbf{0} \,, \qquad \langle \eta/2, 0 \mid \eta/2, 0\rangle = 1 \,.$$

The existence of the dynamical algebra $\mathfrak{su}(1,1)$ associated to the PT potentials, permits the construction of the generalized coherent states of Barut-Girardello type. These states are the eigenvectors of the annihilation operator, it is to say, $K_{-} \,|\, \eta/2, z\rangle = z \,|\, \eta/2, z\rangle$, where $z \in \mathbb{C}$, and η is a positive integer. The explicit form of these states is [22]

$$|\, \eta/2, z\rangle = \left\{ \Gamma(\eta) \,|z|^{-(\eta-1)} \, I_{\eta-1}(2\,|z|) \right\}^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!\,(\eta)_n}} \,|\, \eta/2, n\rangle \,, \tag{20}$$

where $(\eta)_n$ is Pochammer symbol $(\eta)_n = \eta(\eta+1)\ldots(\eta+n-1)$, and I_ν is the modified Bessel function of first class. The probability distribution of the discrete random variable n related to coherent state (20) is

$$P_n^{\mathrm{PT}}(\eta/2, z) = \left\{ \Gamma(\eta) \,|z|^{-(\eta-1)} \, I_{\eta-1}(2\,|z|) \right\}^{-1} \frac{|z|^{2n}}{n!\,(\eta)_n} \,. \tag{21}$$

From these elements, we can construct an algorithm *à la* Kieu with the dynamical algebra $\mathfrak{su}(1,1)$, in the following way: instead of replacing each of the variables of the Diophantine equation (5) using (4) to construct the Hamiltonian (9); these could be replaced by (19) to obtain

$$H_D^{\mathrm{PT}} = \left(D\left(N_1^{\mathrm{PT}}, \ldots, N_k^{\mathrm{PT}}\right)\right)^2 .$$

In accordance with (8), we construct the initial Hamiltonian H_I^{PT} parting from the creation and annihilation operators of $\mathfrak{su}(1,1)$ given in (15) and (16)

$$H_I^{\mathrm{PT}} = \sum_{i=1}^{k} (K_{+_i} - z_i^*)(K_{-_i} - z_i) ,$$

as an associated fundamental state of the zero eigenvalue, the coherent multiparticle disentangled sate, written as $\left|\,\psi^{\mathrm{PT}}(0)\right\rangle$, constructed by means of tensorial product from the one-particle coherent states given in (20)

$$\left|\,\psi^{\mathrm{PT}}(0)\right\rangle = \bigotimes_{i=1}^{k} \left|\,\eta_i/2, z_i\right\rangle .$$

Finally, in accordance with (7), the new Hamiltonian for the evolution takes the form

$$H_A^{\mathrm{PT}}(t) = \left(1 - \frac{t}{T}\right) H_I^{\mathrm{PT}} + \left(\frac{t}{T}\right) H_D^{\mathrm{PT}} . \tag{22}$$

All of the Hamiltonians involved in Kieu's algorithm and in our algorithm on Pöschl-Teller potentials, are unbounded operators and is therefore necessary in such hypercomputational quantum algorithms to use a version of the adiabatic theorem for unbounded operators [23]. However, this technical aspect is not very significant given that such algorithms fundamentally operate in the infrared section (low energies) of the energy spectrum and not in the ultraviolet section (high energies). On other hand, recently a proof was given that quantum adiabatic computation is equivalent to standard quantum computation [24]. This equivalence generates no contradiction between the hypercomputability of quantum adiabatic computation and the Turing machine computability of standard quantum computation due to such proof of equivalence is only valid for finite Hilbert spaces.

A fundamental hypothesis to apply the adiabatic theorem is that the interpolating Hamiltonian must have a well-defined spectral gap during the entire evolution. In other words, the evolution lines of their eigenvalues must not cross each other, at least for the lowest eigenvalues. In the case of Kieu's algorithm, he has shown the existence of a spectral gap for the interpolating Hamiltonian, therefore the adiabatic theorem could be applied even in the case of unbounded infinite-dimensional operators [6]. It is possible to see that Kieu's demonstration of the existence of the spectral band does not depend strongly on the type of dynamical algebra involved, and therefore, this demonstration continues to be valid for the case of the dynamical algebra $\mathfrak{su}(1,1)$ of the Pöschl-Teller potentials. It means that we can use the adiabatic theorem on the Hamiltonian (22).

Finally, in order to satisfy the Halting criteria (10) and due to $P_0^{\mathrm{PT}}(\eta/2, z) > P_1^{\mathrm{PT}}(\eta/2, z) > P_2^{\mathrm{PT}}(\eta/2, z) \ldots$, for the probability distribution (21), it is necessary to select z_i and η_i values such that

$$|\langle \eta_i/2, z_i \mid n\rangle|^2 = P_0^{\mathrm{PT}}(\eta_i/2, z_i) < 1/2\,,$$

which is valid for any value of η_i and $z_i > 1.1$.

4 A particular case: the infinite square well

As a particular case of our hypercomputational algorithm *à la* Kieu on the Pöschl-Teller potentials, we obtain our hypercomputational algorithm on the infinite square well (ISW) which was previously given in [18, 19].

For a particle with mass m trapped inside the infinite well $0 \leq x \leq \pi l$, the potential is obtained using a smooth approach wherein the parameters $\lambda, \kappa \to 1^+$ over interval $[0, \pi l]$ on potential (11), and the Hamiltonian is

$$H^{\mathrm{ISW}} = i^2 \frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \frac{\hbar^2}{2ml^2}\,, \tag{23}$$

could be derived from the Hamiltonian for the PT potentials (12), taking the limit [17]

$$\lim_{\lambda,\kappa \to 1^+} H^{PT} = H^{\mathrm{ISW}}\,.$$

The spectrum of the values of the energy is obtained, likewise, taking the limit

$$\lim_{\lambda,\kappa\to 1^+} E_n^{PT} = E_n^{\mathrm{ISW}} = \hbar\omega e_n(1,1)\,,$$

where $e_n(1,1) = n(n+2)$. The eigenstates of the Hamiltonian (23) that defines the ISW are also obtained as a particular case of the eigenvalues of the defined Hamiltonian for the PT potentials according to

$$\lim_{\lambda,\kappa\to 1^+} |\,\eta/2, n\rangle = |\,3/2, n\rangle\,.$$

The explicit infinite-dimensional UIR of the dynamical algebra $\mathfrak{su}(1,1)$, of the Fock space associated to the UIR and of the actions of the creation and annihilation operators over the states of the Fock space, as well as the number operator and the Barut-Girardello coherent state corresponding to the ISW, are obtained by replacing $\eta = 3$ in the respective expressions of the PT potentials. This way we obtain our algorithm *à la* Kieu on the infinite square well.

5 Conclusions

It is inferred from what has been exposed that Kieu's algorithm consists of four basic parts: (i) Coding of the instance to solve Hilbert's tenth problem, (ii) Establishment of initial conditions, (iii) Evolution from an initial state to a final stage, (iv) Setting of Halting criteria. Part (i) is founded upon a dynamical algebra associated with the physical referent applied in the description of the algorithm. Part (ii) is established based on the coherent states and the ladder operators associated with the dynamical algebra of the physical system. Part (iii) is based upon an adiabatic quantum computation regarding unbounded Hamiltonians. Part (iv) requires certain properties from the initial state based on the distribution of the probability of the coherent states associated with the dynamical algebra.

In the present work we have carried out a variation on the parts (i), (ii) and (iv) with respect to Kieu's algorithm. The coding of a Diophantine equation $D(x_1, x_2, .., x_k) = 0$ consists in substituting the k unknowns by certain operators called number operators, represented by infinite-dimensional matrices, whose spectrum is exactly the set of non-negative integers. Kieu's idea is to take these number operators from the dynamical algebra Weyl-Heisenberg $\mathfrak{g}_{\text{W-H}}$ of the quantum harmonic oscillator. Our idea has been to take these number operators from the dynamical algebra $\mathfrak{su}(1,1)$ of the Pöschl-Teller potentials. Similarly, Kieu's idea consists in taking the initial state as a coherent state of the oscillator. And the Halting criteria is based on the probability distribution for this state. Our idea has been to consider the initial state as the Barut-Girardello coherent state for $\mathfrak{su}(1,1)$ and the Halting criteria based on its probability distribution.

Therefore, it is possible to construct other versions of Kieu's algorithm, supported by other quantum physical referents different to the quantum harmonic oscillator, with different dynamical algebras, given that these new algebras allow infinite-dimensional unitary irreducible representations from which to construct Hamiltonian coders of Diophantine equations. A possible implementation of Kieu's algorithm and of our algorithm implies a great challenge in engineering and quantum technology that is not currently possible. There is however the possibility of running numeric simulations of these hypercomputing algorithms but in (classic) computers derived from Turing machines. Such simulations imply truncations of the matrices originally infinite-dimensional, which generates a weakening of the hypercomputational power of the algorithms.

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