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Two–Parametric Liftings of Toeplitz Forms

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The parametrization problem of the minimal unitary extensions of an isometric operator allows its application, through the spectral theorem, to the case of the Fourier representations of a bounded Hankel form with respect to the norms $(\int |f|^2 d\mu_1)^{1/2}$ and $(\int |f|^2 d\mu_2)^{1/2}$ where μ_1, μ_2 are positive finite measures in $\mathbb{T} \sim [0,2\pi[$ (see [1]). In this work we develop a similar procedure for the two–parametric case, where μ_1, μ_2 are positive measures defined in $\mathbb{T}^2 \sim [0,2\pi[\times[0,2\pi[$. With this purpose, we define the generalized Toeplitz forms on the space of the two–variable trigonometric polynomials and use the lifting existence theorems due to Cotlar and Sadosky [3]. We provide a parametrization formula which is also valid to the special case of the Nehari problem.

In the whole work, V will stand for the vector space of all two–parametric trigonometric polynomials:

$$V = \{f: \mathbb{T}^2 \longrightarrow \mathbb{C} : f(s,t) = \Sigma_{m,n} \hat{f}(m,n)\, e_{m,n}(s,t) \text{ with } \hat{f}(m,n) \text{ finitely supported}\}$$

If we decompose the plane $\mathbb{Z}^2$ in two halfplanes $\mathbb{F}_1$ and $\mathbb{F}_2$ as follows: $\mathbb{F}_1 = \{(m_1,m_2) \in \mathbb{Z}^2 : m_2 \geqslant 0\} = \mathbb{Z} \times \mathbb{Z}_1$, $\mathbb{F}_2 = \mathbb{Z}^2 \setminus \mathbb{F}_1 = \{(m_1,m_2) \in \mathbb{Z}^2 : m_2 < 0\}$, we can denote the subspaces of the analytic and anti–analytic polynomials in respects to the given partition as : $W_1 = \{f \in V : \hat{f}(m,n) = 0 \text{ if } (m,n) \in \mathbb{F}_2\}$, $W_2 = \{f \in V : \hat{f}(m,n) = 0 \text{ if } (m,n) \in \mathbb{F}_1\}$. In V, the shifts $\sigma f(s,t) = e^{is} f(s,t)$ and $\tau f(s,t) = e^{it} f(s,t)$, are unitary operators and verify: i) $\sigma W_1 \subset W_1$, $\sigma^{-1} W_2 \subset W_2$, ii) $\tau W_1 \subset W_1$, $\tau^{-1} W_2 \subset W_2$, iii) $\sigma\tau = \tau\sigma$.

Remark. The set $(V, W_1, W_2, \sigma, \tau)$ formed by a vector space V, two closed subspaces of V, W_1 and W_2, and two linear isomorphisms $\sigma, \tau : V \longrightarrow V$ satisfying i), ii) and iii) is called a *discrete algebraic scattering system.*

DEFINITION. a) A sesquilinear form $B : V \times V \longrightarrow \mathbb{C}$ is called a *Toeplitz form* if $B(\sigma f, \sigma g) = B(\tau f, \tau g) = B(f,g)$, $\forall (f,g) \in V \times V$.

b) Given three Toeplitz forms, B_0, B_1, B_2, we say that B_0 is *bounded* by B_1 and B_2 [and write $B_0 \leqslant (B_1, B_2)$], if $B_1, B_2 \geqslant 0$, $|B_0(f,g)|^2 \leqslant B_1(f,f) B_2(g,g)$, $\forall (f,g) \in V \times V$.

c) B_0 is *weakly bounded* by B_1 and B_2 [$B_0 \prec (B_1, B_2)$], if $B_1, B_2 \geqslant 0$, and $|B_0(f,g)|^2 \leqslant B_1(f,f) B_2(g,g)$, $\forall (f,g) \in W_1 \times W_2$.

The next lifting theorem, proved in [3] will be the object of our parametrization. We use the next notation: If μ, μ_1 and μ_2 are three finite complex measures in $\mathbb{T}^2$, we write $\mu \leqslant (\mu_1, \mu_2)$ when $\mu_1 \geqslant 0, \mu_2 \geqslant 0, |\mu(\Delta)|^2 \leqslant \mu_1(\Delta) \mu_2(\Delta)$, for all Borel set Δ in $\mathbb{T}^2$.

THEOREM. *Given three Toeplitz forms B_0, B_1, B_2 in $V \times V$, such that $B_0 \prec (B_1, B_2)$, then: a) (Lifting Property) There exists a Toeplitz form B_0' such that $B_0' \prec (B_1, B_2)$ and $B_0'|_{W_1 \times W_2} = B_0$. b) (Generalized Bochner Theorem) There exist μ, μ_1, μ_2 finite measures in $\mathbb{T}^2$ such that $\mu \leqslant (\mu_1, \mu_2)$, and*

$$B_i(f,g) = \int\int f \bar{g} \, \mathrm{d}\mu_i, \ (i=1,2), f,g \in V, \ B_0(f,g) = \int\int f\bar{g} \, \mathrm{d}\mu, \ (f,g) \in W_1 \times W_2.$$

Here μ_1, μ_2 are unique, due to Bochner theorem, but μ is not, and the set of all the solutions can be parametrized throughout their Stieltjes transform by the same method of the one–parametric case [1].

A parametrization formula describing all the liftings B_0' will develop next.

In the space $E = W_1 \times W_2$ we define the following inner product:

$$\langle (f_1, g_1), (f_2, g_2) \rangle = B_1(f_1, f_2) + B_2(g_1, g_2) + B_0(f_1, g_2) + \overline{B_0(f_2, g_1)} \, .$$

As $B_0 \prec (B_1, B_2)$, then $(\overline{E}, \langle \cdot, \cdot \rangle)$ is a Hilbert space and we can suppose that $W_1 \sim [W_1, 0]$, $W_2 \sim [0, W_2]$ are closed subspaces of $\overline{E}$.

If we define $\tau : \overline{E} \longrightarrow \overline{E}$ as $\tau[f,g] = [\tau f, \tau g]$, it results that τ is an isometric operator with domain $W_1 \times \tau^{-1} W_2$ and range $\tau W_1 \times W_2$. On the other hand $\sigma : \overline{E} \longrightarrow \overline{E}$, defined as $\sigma[f,g] = [\sigma f, \sigma g]$, is a unitary operator.

Thus, the space $\overline{E}$ is generated by $[W_1, 0] \sim \{\tau^k e_{n,0} : k \geqslant 0\}$ and $[0, W_2] \sim \{\tau^k e_{m,-1} : k \leqslant 0\}$, $m, n \in \mathbb{Z}$.

Our next goal is to describe the minimal commuting unitary extensions (m.c.u.e.) of (σ, τ). A whole description which provides existence and unicity conditions for the minimal commuting unitary extensions is due to Morán [4]. In particular, a necessary and sufficient condition for the existence of such extensions is $\langle \sigma^n \tau f, \tau f' \rangle = \langle \sigma^n f, f' \rangle, \forall f, f' \in D_\tau$, $n = 1, 2, \ldots$ and here this condition is trivial, because σ is a unitary operator and $\sigma(D_\tau)=D_\tau$, $\sigma(\Delta_\tau)=\Delta_\tau$, $\sigma\tau=\tau\sigma_{|D_\tau}$.

Using a formula stated by Chumakin [2], we can express the generalized resolvent R_z of τ as $R_z=\Sigma_{n\geqslant 0}z^n T_z^n$, if $|z|<1$ where $T_z=\tau\oplus\Phi_z$ and $\Phi_z:\overline{E}\ominus D_\tau \longrightarrow \overline{E}\ominus\Delta_\tau$ is for each z in the unit circle, a non–expanding operator.

As $\overline{E}=D_\tau\oplus(\overline{E}\ominus D_\tau)$ if $\{u_{i,-1}\}_{i\in\mathbb{Z}}$, $\{u_{j,0}\}_{j\in\mathbb{Z}}$ are two orthonormal basis of $\overline{E}\ominus D_\tau$ and $\overline{E}\ominus\Delta_\tau$, respectively, we can write:

$$e_{m,-1}=\Sigma_{i\in\mathbb{Z}}c^0_{mi}u_{i,-1}+v^0_m,\quad u_{j,0}=\Sigma_{i\in\mathbb{Z}}d^0_{ji}u_{i,-1}+w^0_j,\quad v^0_m,w^0_j\in D_\tau$$

and

$$\Phi_z(u_{i,-1})=\Sigma_{j\in\mathbb{Z}}\varphi_{i,j}(z)u_{j,0},\quad \forall i\in\mathbb{Z}.$$

If we write $\Phi=[\varphi_{ij}]_{i,j\in\mathbb{Z}}$ the associated matrix of the operator Φ_z, then $\|\Phi_z\|\leqslant 1$. We define the sequences $\{v^p_m\}_{p\geqslant 0}\subset D_\tau$ and $\{w^p_j\}_{p\geqslant 0}\subset D_\tau$ by recurrence as:

(1) $\tau v^p_m=\Sigma_{i\in\mathbb{Z}}c^{p+1}_{mi}u_{i,-1}+v^{p+1}_m$ $(p\geqslant 0)$ and $\tau w^p_j=\Sigma_{i\in\mathbb{Z}}d^{p+1}_{ji}u_{i,-1}+v^{p+1}_j$ $(p\geqslant 0)$

and the polynomial sequence $\{P^p_m\}$:

$$(2)\qquad \begin{cases} P^0_j(\Phi)=(\ldots,c^0_{j0},c^0_{j1},\ldots,c^0_{jn},\ldots)\equiv c^0_j \\ P^p_j(\Phi)=P^{p-1}_j(\Phi)\Phi D^0+P^{p-2}_j(\Phi)\Phi D^1+\ldots+P^0_j(\Phi)\Phi D^{p-1}+c^p_j,\ (p\geqslant 1)\end{cases}$$

where $c^p_j\equiv(\ldots,c^p_{j0},c^p_{j1},\ldots,c^p_{jn},\ldots)$ and $D^k\equiv(d^k_{nj})_{n,j\in\mathbb{Z}}$.

So the following general result can be stated

THEOREM. *Let τ,σ be the given operators and R_z the generalized resolvent of τ. From the vectorial sequences $\{P^p_m\}$, $\{v^p_m\}$, $\{w_p\}$, the m.c.u.e. of (σ,τ) are parametrizated by the matrix $\{\langle R_z e_{j,-1},e_{n,0}\rangle; j,n\in\mathbb{Z}\}$, with $|z|<1$, whose element j,n has the form*

$$(3)\quad \langle R_z e_{j,-1},e_{n,0}\rangle=\textstyle\sum_{m\geqslant 1}z^m\left[\sum_{k=1}^m P^{k-1}_j(\Phi)\Phi\langle w^{m-k},e_{n,0}\rangle\right]+\sum_{m\geqslant 0}z^m\langle v^m_j,e_{n,0}\rangle$$

and where the matrix $\Phi=[\varphi_{i,n}]_{i,n\in\mathbb{Z}}$ is the one associated to Φ_z such that

$$(4)\qquad \{\Phi_z:\overline{E}\ominus D_\tau\longrightarrow\overline{E}\ominus\Delta_\tau:\|\Phi_z\|\leqslant 1,\ \sigma\Phi_z=\Phi_z\sigma_{|\overline{E}\ominus D_\tau}\}.$$

In order to see how it produces the desired lifting, we proceed as follows:

Having into account that $V=\vee_{n\in\mathbb{Z}}\{\tau^n W_1\}=\vee_{n\in\mathbb{Z}}\{\tau^n W_2\}$, we define the form $B':V\times V\longrightarrow\mathbb{C}$ as $B'(\tau^m w_1,\tau^n w_2)=\langle U^{m-n}[w_1,0],[0,w_2]\rangle$ where U is a unitary extension of τ that satisfies (4). Thus, it is easy to prove that B' is a sesquilinear form, τ and σ–invariant and for each $(f,g)\in V\times V$, $|B'(f,g)|^2\leqslant B_1(f,f)B_2(g,g)$. Moreover B' extends to B_0 and is uniquely determinated by U

because it suffices to calculate $B'(\tau^m e_{j,-1}, e_{k,0})$, $j,k \in \mathbb{Z}$ in order to determine the lifting.

At last, each form B' defines a measure μ' by the formula

$$B'(f,g) = \int\int f\bar{g}\,d\mu'.$$

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