EXTRACTA MATHEMATICAE Vol. **16**, Núm. 3, 303 – 341 (2001)

Qualitative Study of Nonlinear Parabolic Equations: an Introduction

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(Expository paper presented by W. Okrasiǹski)

AMS *Subject Class.* (2000): 35K10, 35K57 *Received March* 23, 2001

1. Introduction: the problem model

Given Ω, open bounded regular set of $\mathbb{R}$, $N \geq 1$, we consider the *model problem*

$$(P)\begin{cases} b(u)_t - \operatorname{div} \mathbf{A}(x,u,\nabla u) + g(x,u) = f(t,x), & t>0, \quad x \in \Omega, \\ u = h, & t>0, \quad x \in \partial\Omega, \\ b\left(u(0,x)\right) = b\left(u_0(x)\right), & x \in \Omega. \end{cases}$$

Before making explicit the *structural assumptions* on the data $b, \mathbf{A}, f, h$ and u_0 let us mention some important special examples. Perhaps the simpler example is*the linear heat equation*

$$u_t - \Delta u = f. \tag{1}$$

So, $b(s) = s, \mathbf{A}(x,u,\xi) = \xi$ and $g \equiv 0$. This is a typical example of linear partial differential equation of parabolic type usually considered in undergraduate courses (see, e.g., John [31]). A *modern* treatment starts by introducing the notion of weak solution or by its reformulation as an abstract Cauchy problem on a Banach space

$$\begin{cases} \dfrac{du}{dt}(t) + Au(t) = f(t), \\ u(0) = u_0, \end{cases}$$

*Partially sponsored by the DGES (Spain), project REN2000-0766.

(see, e.g., Brezis [17]). It is well known, that one of the main results of the stabilization theory is that if

$$\left.\begin{array}{l} f(t,x) \longrightarrow f_\infty(x) \\ h(t,x) \longrightarrow h_\infty(x) \end{array}\right\} \quad \text{as} \quad t \longrightarrow \infty$$

in some *suitable* sense then the solution of the linear heat equation $u(t,x)$ verifies that

$$u(t,x) \longrightarrow u_\infty(x) \quad \text{as} \quad t \longrightarrow \infty$$

in *some* functional space, with u_∞ satisfying the *associated stationary problem*

$$\left\{\begin{array}{lcll} -\Delta u_\infty & = & f_\infty(x) & \text{in} \quad \Omega, \\ u_\infty & = & h_\infty & \text{on} \quad \partial\Omega, \end{array}\right. \tag{2}$$

(*the linear diffusion equation*). Notice that problem (2) is also included in the formulation (P) by making $b \equiv 0$, $\mathbf{A}(x,u,\xi) = \xi$, $g \equiv 0, f = f_\infty$ and $h = h_\infty$. More in general, *given a choice of* $b, \mathbf{A}, g, f, h$ *and* u_0 *leading to a special formulation of* (P), *the choice of choice of* $b \equiv 0, \mathbf{A}$ *and* g *as before leads to the formulation of the associated stationary problem.* In this way (P) include also stationary problems. In order to present some *nonlinear examples*, it is useful *to read* (P) as a *balance of different phenomena*

$$\underbrace{b(u)_t}_{\text{(I)}} \quad \underbrace{-\operatorname{div} \mathbf{A}}_{\text{(II)}} + \underbrace{g(x,u) - f(x,t)}_{\text{(III)}} = 0.$$

Let us make some comments on the *accumulation term* (I). It arises, for instance, in thermal processes when the *heat capacity* of the medium depends on the temperature. This is the case, e.g., when water and ice are simultaneously present and then $b(u)$ is a strictly increasing function having a discontinuity at $u = 0$. This special case (called *Stefan problem*) requires a delicate mathematical treatment.

In fact, as a general rule, the assumption $b : \mathbb{R} \to \mathbb{R}$ nondecreasing is absolutely fundamental to formulate (P) in the class of problems of parabolic type since otherwise the problem becomes *ill posed* (as, for instance, $-u_t - \Delta u = f$; *the backward heat equation*).

This type of *accumulation term* (I) also arises in the theory of filtration of a fluid in a porous media. In that case

$$b \in C^0(\mathbb{R}), \ b \text{ nondecreasing},$$

(see, e.g., Bear [10]). Now $u(t,x)$ is not a temperature but the *humidity of the soil*. Different choices are possible: in the study of *unsaturated soils* b is assumed to be strictly increasing, as, for example, $b(u) = |u|^{\alpha-1}u$. In the case of *partially saturated soils*, $b(u)$ is not strictly increasing but becomes constant for $u > u^1$, for some $u^1 > 0$. Notice that, in this physical framework, $u \geq 0$ and so the values of b on $\mathbb{R}^-$ are not relevant. The, so called *dam problem*, corresponds to a limit case in which b is the Heaviside function. This choice of b also arises in problems of a different physical context, as, for instance, the *Hele-Shaw problem* or some problems arising in *lubrication theory* (see, e.g., Bayada and Chambat [9] where many other references can be found).

Let us refer now to the *diffusion* and *convection* terms involved in (II). The dependence of $\mathbf{A}(x,u,\nabla u)$ with respect to ∇u (resp. u) leads to diffusion terms (resp. convection terms). Some examples of relevance in the applications are commented in the following. The, so called, *nonlinear heat equation* arises when the *Fourier law* fails and the *thermal conductivity* depends on the temperature (case of many gases, lubricating fluids, etc). Then the diffusion of heat leads to the expression

$$\operatorname{div}(k(u)\nabla u) = \Delta\beta(u) \quad \text{with} \quad \beta(s) := \int_0^s k(\sigma)d\sigma.$$

In most of the cases $\beta(u)$ grows like a power

$$\beta(u) = |u|^{m-1}u \quad \text{with} \quad m > 0.$$

The above second order operator (sometimes written as $-\Delta u^m$) also arises in the study of *filtration in porous media* (D'Arcy law) with $m > 1$ and in *plasma physics* when $0 < m < 1$.

A different class of examples of nonlinear terms $\mathbf{A}(x,u,\nabla u)$ arises in the study of *non-Newtonian fluids*. The study of one-directional flows of some special fluids (as, for instance, polymer melts, suspensions, paints, animal blood, honey, shampoo, etc.) leads to nonlinear diffusion operators of the type

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right), \text{ (denoted by } \Delta_p u), \quad \text{for some} \quad p > 1.$$

Notice that if $p = 2$ then $\Delta_2 = \Delta$ (the linear Laplacian operator, arising in the study of Newtonian fluids). The case $1 < p < 2$ corresponds to *pseudo-plastic fluids* (as, e.g., gasoline, lubricating oil, etc.) and $p > 2$ arises in the consideration of *dilatant fluids* (as, for instance, the polar ice and glaciers, volcano lava, etc.).

The above two operators may become *degenerate* since

$$\Delta u^m = \text{div } \left(mu^{m-1}\nabla u\right) = mu^{m-1}\Delta u + m(m-1)u^{m-2}|\nabla u|^2.$$

So, if $m > 1$ the coefficient of ∇u vanishes on the set $\{(t,x)\colon\ u(t,x) = 0\}$. Analogously,

$$\Delta_p u = \text{div } \left(|\nabla u|^{p-2}\nabla u\right) = |\nabla u|^{p-2}\Delta u + \nabla u \cdot \nabla \left(|\nabla u|^{p-2}\right)$$

and when $p > 2$ the coefficient of ∇u vanishes on the set $\{(t,x)\colon\ \nabla u(t,x) = 0\}$. Due to this reason the *qualitative behavior* of solutions of (P) may be very different (according the assumptions on the data $b, \mathbf{A}(x,u,\nabla u)$ and g) to the one of the solution of the linear heat equation. In fact, to show such kind of differences is one of the main goals of these notes.

We also mention that another relevant choice of nonlinear terms $\mathbf{A}(x,u,\nabla u)$ arises in the study of *transient minimal surfaces*, in which case the second order diffusion operator is given by

$$\text{div } \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right).$$

Concerning the *transport* or *convection* terms, we mention that they arise very often in *Fluid Mechanics*. Usually they appear formulated in terms of an *additive term*, as, for instance, in the case of the temperature in a fluid

$$\underbrace{-\Delta\beta(u)}_{\text{diffusion}} + \underbrace{\mathbf{w}\cdot\nabla u}_{\text{convection}} .$$

If the fluid is *incompressible* (case of *liquids*) then $\text{div}\,\mathbf{w} = 0$ and so we get

$$-\text{div}\,(k(u)\nabla u - u\mathbf{w}),\ \text{i.e.,}\ \mathbf{A}(x,u,\xi) = k(u)\xi + u\mathbf{w}.$$

Nevertheless, sometimes the convection term is not an additive term but appears in a different form.

$$\text{div } (\Phi(\nabla u + K(b(u)\mathbf{e}))$$

where

$$\Phi(\xi) = |\xi|^{p-2}\ \xi, \mathbf{e} \in \mathbb{R}^N \text{ and } K \in C^1(\mathbb{R}:\mathbb{R}).$$

This situation arises, for instance, in the study of *turbulent flow of a fluid through a porous medium* (with $\mathbf{e}$ the vector indicating the *main filtration direction*). For a general exposition on different examples of diffusion-convection

operators, containing many other references see Díaz [20] and Díaz and de Thelin [25].

The expression (III) represents the *absorption/forcing term*. The presence of the term $g(x,u)-f(t,x)$ is very typical of many problems arising in reaction-diffusion problems in Biology, Chemistry and other contexts. By writing

$$g(x,u) = g_1(x,u) - g_2(x,u),$$

with g_1 and g_2 nondecreasing functions, we can distinguish the term of *absorption* $g_1(x,u)$ (which contributes to make $|u|$ smaller than if $g_1 = 0$) from the one of *forcing* $g_2(x,u)$ (which contributes to make $|u|$ bigger than if $g_2 = 0$).

In most of the cases

$$g_1(x,u) = \lambda |u|^{q-1}u, \qquad \lambda > 0,$$

with $q > 0$ (the *order of the reaction*). Notice that if $0 < q < 1$, g_1 is *not* a Lipschitz function.

Returning to the *structural assumptions* on the data, in the rest of the exposition, we shall always assume that

$$b : \mathbb{R} \to \mathbb{R} \text{ is continuous and nondecreasing, } b(0) = 0, \tag{3}$$

$$\begin{cases} \mathbf{A} : \Omega \times \mathbb{R} \times \mathbb{R}^N \text{ is a Caratheodory function} \\ \text{(i.e., measurable in } x \text{ and continuous in } (u,\xi)), \\ \exists p > 1 \text{ such that } |\mathbf{A}(x,u,\xi)| \le C(|u|^{\frac{p^*}{p'}} + |\xi|^{p-1}), \quad \forall u \in \mathbb{R}, \\ \forall \xi \in \mathbb{R}^N \text{ with } p' = \frac{p}{p-1}, \quad p^* = \frac{Np}{N-p} \text{ and} \\ (\mathbf{A}(x,u,\xi) - \mathbf{A}(x,u,\xi^*)) \cdot (\xi - \xi^*) > 0, \forall \xi, \xi^* \in \mathbb{R}^N, \xi \ne \xi^*, \end{cases} \tag{4}$$

$$\begin{cases} g \text{ is Caratheodory function and} \\ |g(x,u)| \le \gamma(|u|)(1 + d(x)),\ d \in L^1(\Omega) \text{ and } \gamma \text{ strictly increasing,} \end{cases} \tag{5}$$

$$f = f_1 + f_2,\ f_1 \in L^{p'}(0,T;\ W^{-1,p'}(\Omega)),\ f_2 \in L^1((0,T)\times\Omega), \forall T > 0, \tag{6}$$

$$h \in L^p(0,T;\ W^{1,p}(\Omega)) \cap L^\infty((0,T)\times\Omega), \forall T > 0, \tag{7}$$

$$u_0 \in L^\infty(\Omega).$$

For the sake of simplicity in the exposition, we shall deal merely with *bounded (weak) solutions*

DEFINITION 1. We say that u is a *bounded weak solution* of (P) if $u-h \in L^p(0,T:W^{1,p}(\Omega)) \cap L^\infty((0,T)\times\Omega)$, $\forall T>0$, and we have:

$$(i)\begin{cases} b(u)_t \in L^{p'}(0,T:W^{-1,p'}(\Omega)) \text{ and} \\ \int_0^T \langle b(u)_t, v\rangle_{W^{-1,p'}\times W_0^{1,p}} dt + \int_0^T \int_\Omega (b(u)-b(u_0))v_t dxdt = 0 \\ \forall v \in L^p(0,T: W_0^{1,p}(\Omega)) \cap W^{1,1}(0,T:L^1(\Omega)) \text{ with } \ v(T,\cdot)\equiv 0, \end{cases}$$

and

$$(ii)\begin{cases} \int_0^T \langle b(u)_t, v\rangle dt + \int_0^T \int_\Omega \mathbf{A}(x,u,\nabla u)\cdot\nabla v dxdt + \int_0^T \int_\Omega g(x,u)v dxdt \\ \quad = \int_0^T \langle f_1, v\rangle dt + \int_0^T \int_\Omega f_2 v dxdt, \\ \forall v \in L^p(0,T: W_0^{1,p}(\Omega)) \cap L^\infty((0,T)\times\Omega), \quad \forall T>0. \end{cases}$$

The above definition is adapted from Alt and Luckhaus [2].

In the rest of this exposition we shall consider different qualitative properties of solutions of (P) arising according the nature of the nonlinear terms $b(u)$, $\mathbf{A}(x,u,\nabla u)$ and $g(x,u)$. Our plan is the following: Section 2 will be devoted to two *comparison principles* which will be important tools in our study. Two qualitative properties are presented in the rest of the exposition: *the finite extinction time property* (Section 3) and *the finite speed of propagation property* (Section 4). In both of the above sections we shall apply the two comparison principles as well as some *energy methods*.

It is clear that the above presentation is far to be exhaustive. Problems like (P) have attracted the attention of many specialists in the last forty years (perhaps the earliest mathematical paper on this subject was [38]). In consequence, many other very interesting qualitative properties are today available in the literature. The present notes only pretend to be an elementary introduction.

2. TWO USEFUL TOOLS

2.1. INTRODUCTION. The study of several qualitative properties for solutions of model problem (P) will be carried out thanks to some useful tool: *the comparison principles*.

The most popular comparison principle has a pointwise nature and usually holds for elliptic and parabolic second order equations (as well as for first order hyperbolic equations). A first statement of such a principle is the following:

THEOREM 1. (Pointwise comparison principle) *Let* (f, h, u_0) *and* $(\widehat{f}, \widehat{h}, \widehat{u}_0)$ *be two set of ordered data, i.e., such that*

$$f \leq \widehat{f}, \quad h \leq \widehat{h} \quad \text{and} \quad u_0 \leq \widehat{u}_0,$$

in their respective domains of definition. Let u *and* $\widehat{u}$ *be (any) solutions of* (P) *corresponding to* (f, h, u_0) *and* $(\widehat{f}, \widehat{h}, \widehat{u}_0)$ *respectively. Then*

$$u(t, x) \leq \widehat{u}(t, x), \quad \text{for any } t > 0 \text{ and a.e. } x \in \Omega.$$

In the case of *linear* problems, this property is a trivial consequence of the *maximum principle* (in fact, it suffices to assume $(\widehat{f}, \widehat{h}, \widehat{u}_0) \equiv (0, 0, 0)$ and so $\widehat{u} \equiv 0$). The first (general) result for linear equations seems to be due to Paraf in 1892 (later generalizations where due to Picard, Lichtenstein and, finally, Hopf (in 1927) (see details in the book Gilbarg and Trudinger [30]).

It is clear that for the *nonlinear case* some conditions on b, $\mathbf{A}$ and g are needed (notice that the pointwise comparison principle implies the uniqueness of solutions). This topic is still under investigation (see the series of works by Ph. Benilan, J. Carrillo and others). Here we shall *recall* a particular result (of a *short* proof) stated in terms of an estimate for a suitable expression.

The second tool refers to another comparison principle, but this time, of a different nature. We can call it as the *symmetrized mass comparison principle*. The process of *symmetrization* need to be carefully presented. We start by the *symmetrization of the domain* Ω: Given Ω, an open bounded set of $\mathbb{R}^N$, the symmetrized version of Ω is the ball centered at the origin having the same measure than Ω. Let us call Ω^* to this ball. The condition $m(\Omega) = m(\Omega^*)$ has a relation with the *isoperimetric inequality*

$$L \geq N \omega_N^{\frac{1}{N}} A^{\frac{N-1}{N}} \tag{8}$$

where L is the *lenght* of $\partial\Omega$ (or $m(\partial\Omega)$), A is the *area* of Ω (or $m(\Omega)$) and

$$\omega_N \text{ is the } \textit{area} \text{ of the unit ball of } \mathbb{R}^N (\text{i.e., } \omega_N = m(S^{n-1})).$$

In (8) the equality holds if and only if Ω is a ball. This was a first noted by Dido de Cartago (850 B.C.) (*in* $\mathbb{R}^2$ *the circles are the domains with fixed area having a longer perimeter*). Rigorous proofs of (8) are due to Steiner (1882), Schwarz (1890) and Schmidt (1939).

The second step of the process of symmetrization consists in the *symmetrization of data* f and u_0. We shall use the notion of the *decreasing symmetric rearrangement* of a function introduced by H.A. Schwarz in

1890: Given a function $h : \Omega \to \mathbb{R}$, $h \in L^1(\Omega)$, we define the *decreasing symmetric rearrangement of* h, h^*, as the (unique) function $h^* : \Omega^* \to \mathbb{R}$ such that h^* is symmetric (i.e., $h^*(x) = h^*(\widehat{x})$ if $|x| = |\widehat{x}|$), h^* decreases if $|x|$ decreases and the level sets of h and h^* are equimeasurables (i.e., $m(\{x \in \Omega : h(x) > \theta\}) = m(\{x \in \Omega^* : h^*(x) > \theta\}), \forall \theta \in \mathbb{R}$). A more systematic definition of h^* can be introduced as follows: we first define the *distribution function of* h by

$$\mu : \mathbb{R} \to \mathbb{R},\ \mu(\theta) := m\{x \in \Omega : h(x) > \theta\}.$$

Then we define the *scalar decreasing rearrangement of* h by

$$\widetilde{h} : (0, m(\Omega)] \to \mathbb{R},\ \widetilde{h}(s) := \inf\{\theta \in \mathbb{R} : \mu(\theta) \leq s\}$$

(notice that $\widetilde{h}(s) \sim \mu^{-1}(s)$). Finally, we define the *symmetric decreasing rearrangement of* h, by

$$h^* : \Omega^* \to \mathbb{R},\ h^*(x) := \widetilde{h}(\omega_N |x|^N).$$

Notice that, since h^* is symmetric, we can write $h^*(x) = H(|x|)$ with $H : \mathbb{R} \to \mathbb{R}$. Nevertheless $H \neq \widetilde{h}$ since $H(r) = \widetilde{h}(\omega_N r^N)$. Notice, also, that assumed $h \geq 0$, by construction, we have that

$$h \in L^1(\Omega) \text{ implies that } h^* \in L^1(\Omega^*) \text{ and}$$
$$\int_\Omega h(x)dx = \int_{\Omega^*} h^*(x)dx \text{ (the Cavalieri Principle)}$$

and that

$$h \in L^\infty(\Omega) \text{ implies that } h^* \in L^\infty(\Omega^*) \text{ and}$$
$$\operatorname{ess\,sup}_{x\in\Omega} h(x) = \operatorname{ess\,sup}_{x\in\Omega^*} h^*(x).$$

The third step of the process is the *symmetrization of the second order operator*. We must replace the diffusion operator $\operatorname{div} \mathbf{A}(x, u, \nabla u)$ by another *isotropic* diffusion operator, i.e., with the same behavior in any direction x_i. Several possibilities arise. Here we shall consider, merely, a special case. Assume that condition (4) holds and that, *in addition,*

$$\mathbf{A}(x, u, \xi) \cdot \xi \geq |\xi|^p \quad \forall \xi \in \mathbb{R}^N.$$

Then we shall define as *symmetrized operator of* $\operatorname{div}\mathbf{A}(x,u,\nabla u)$ the one given by

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

(notice than if we take $\mathbf{A}^*(x,u,\xi) = |\xi|^{p-2}$ then condition (4) holds with the equality sign instead the inequality one).

We also must introduce an *isotropic absorption* by assuming (besides (5)) the condition

$$\begin{cases} g(x,u)u \geq \widehat{g}(u)u \qquad \text{a.e.} \quad x \in \Omega, \\ \text{for some continuous function} \quad \widehat{g} : \mathbb{R} \to \mathbb{R}. \end{cases} \tag{9}$$

Summarizing, we say that the *symmetrized problem of* (P) is the following one:

PROBLEM. (P^*): Find $U : [0,\infty) \times \Omega^* \to \mathbb{R}$ such that

$$(P^*) \begin{cases} b(U)_t - \Delta_p U + \widehat{g}(U) = f^*(t,x), & t > 0, \quad x \in \Omega^*, \\ U = h^*, & t > 0, \quad x \in \partial\Omega^*, \\ b\left(U(0,x)\right) = b\left(u_0^*(x)\right), & x \in \Omega^*. \end{cases}$$

Here $f^*(t,\cdot)$ and $u_0^*(\cdot)$ are the decreasing symmetric rearrangements of $f(t,\cdot)$ and u_0, respectively. For the sake of simplicity in the exposition we shall assume *now* that

$$h = h^* = 0. \tag{10}$$

Let us make some remarks on the statement of the *symmetrized mass comparison principle*. The first one is that some pioneer authors finding different relations between u and U where Saint-Venant (1856), Poya and Szego (1951) and Weimberger (1962). The inequality

$$u^*(x) \leq U(x) \ , \ x \in \Omega^*, \tag{11}$$

was first proved by G. Talenti, in 1976, for the case of the stationary problem without absorption term (i.e., $b \equiv 0$ and $g \equiv 0$). Unfortunately, this (pointwise) comparison *fails* to be true for parabolic problems (i.e., $b \neq 0$) or/and for problems in presence of absorption terms ($g \neq 0$). In those cases we only can compare the *distribution of the mass* of u and U

THEOREM 2. (Symmetrized Mass Comparison Principle (SMCP))

$$\int_{B(0,r)} u^*(t,x)dx \leq \int_{B(0,r)} U(t,x)dx, \forall t > 0, \forall r \in [0,R],$$

assumed that $\Omega^* = B(0,R)$.

Notice that this comparison can be, equivalently, expressed in terms of scalar decreasing rearrangement as

$$\int_0^s \widetilde{u}(t,\sigma)d\sigma \le \int_0^s \widetilde{U}(t,\sigma)d\sigma,\ \forall t>0,\ \forall s\in[0,m(\Omega)].$$

The SMCP has many applications (as we shall see in other sections). The main philosophy of the applications is that function U can be easily estimated in many cases and thus, thanks to the SMCP, properties for U can be extended in similar properties for u. Some books dealing with the symmetrization process are the ones by Bandle [6], Mossino [35] and Kawohl [33]. The proof we shall present here follows the memoir Díaz [21] (see also Díaz [22]). A different (and very original) approach is due to Abourjail and Benilan [1]. The first result in the literature for degenerate parabolic problems was Vázquez [41].

2.2. Proof of the two comparison principles.

On the pointwise comparison principle. We present here a particular version of this principle (more general results will be indicated later) for the special case of the diffusion-convection operator arising in the study of turbulent flow of a fluid through a porous medium. More precisely, we consider the problem

$$(P_{\phi,K})\begin{cases} b(u)_t - \text{div}\ (\phi(\nabla u + \mathbf{e}K(b(u)))) + g(x,u) = f(x,t), & t>0, x\in\Omega,\\ u = h, & t>0, x\in\partial\Omega,\\ b\,(u(0,x)) = b\,(u_0(x)) & x\in\Omega, \end{cases}$$

where $\phi(\xi)=|\xi|^{p-2}\xi,\quad p>1$, $\mathbf{e}\in\mathbb{R}^N$ and $K\in C^0(\mathbb{R},\mathbb{R})$. Besides the conditions made explicit in Section 1 we shall made some extra assumptions:

$$(H_{g,b})\begin{cases} \text{there exists}\quad C^*\ge 0\quad \text{such that}\\ g(\cdot,\eta)-g(\cdot,\widehat{\eta})\ge -C^*\,(b(\eta)-b(\widehat{\eta}))\,,\quad \forall\eta>\widehat{\eta},\ \eta,\widehat{\eta}\in\mathbb{R}, \end{cases}$$

(notice that $(H_{g,b})$ trivially holds if, for instance, $g(\cdot,\eta)$ is nondecreasing in η or if $g(\cdot,\eta):=\widetilde{g}(\cdot,b(\eta))$ with $\widetilde{g}(\cdot,s)$ Lipschitz continuous in s),

$$(H_K)\begin{cases} K(b(\eta)) \text{ is Hölder continuous in } \eta \text{ of exponent } \gamma\ge\frac{1}{p} \text{ if } 1<p<2\\ \text{and } \gamma\ge\frac{1}{p'}\ (\frac{1}{p}+\frac{1}{p'}=1) \text{ if } p\ge 2\ ,\\ |K(b(\eta))-K(b(\widehat{\eta}))|\le C|\eta-\widehat{\eta}|^{\gamma},\quad \forall\eta,\widehat{\eta}\in\mathbb{R}, \end{cases}$$

(notice that condition (4) is now trivially satisfied).

THEOREM 3. *Let $(f,h,u_0),(\widehat{f},\widehat{h},\widehat{u}_0)$ be such that $f\le\widehat{f},h\le\widehat{h}$ and $u_0\le\widehat{u}_0$ on their respective domains. Let $u,\widehat{u}$ be two bounded weak solutions of $(P_{\phi,K})$ associated to (f,h,u_0) and $(\widehat{f},\widehat{h},\widehat{u}_0)$, respectively. Assume, in addition, that u and $\widehat{u}$ are* strong solutions, *i.e.,*

$$b(u)_t, b(\widehat{u})_t \in L^1\left((0,T)\times\Omega\right), \quad \forall T>0. \tag{12}$$

Then $u\le\widehat{u}$ on $(0,T)\times\Omega$. More in general, if we replace the ordered data assumption by the simpler condition $h\le\widehat{h}$ and $f_1\le\widehat{f_1}$ then

$$\begin{aligned}||[b(u(t,\cdot))-b(\widehat{u}(t,\cdot))]_+||_{L^1(\Omega)} &\le e^{C^*t}||[b(u_0)-b(\widehat{u}_0)]_+||_{L^1(\Omega)}\\ &\quad+\int_0^t e^{C^*\tau}||[f_2(\tau,\cdot)-\widehat{f_2}(\tau,\cdot)]_+||_{L^1(\Omega)}d\tau\end{aligned} \tag{13}$$

for any $t>0$ (C^* given in $(H_{g,b})$), where $\varphi_+=\max(\varphi,0)$.

Proof. We take as test function the following approximation of the $sign_0^+(u-\widehat{u})$ function: we start by defining $\Psi_\delta(\eta):=\min(1,\max(0,\frac{\eta}{\delta}))$, for $\delta>0$ small. Then we define $v=\Psi_\delta(u-\widehat{u})$. Notice that $v\in L^p(0,T:W_0^{1,p}(\Omega))\cap L^\infty\left((0,T)\times\Omega\right),\forall T>0$, and that

$$\nabla v=\begin{cases}\nabla\frac{1}{\delta}(u-\widehat{u}) & \text{if } 0<u-\widehat{u}<\delta,\\ 0 & \text{otherwise.}\end{cases}$$

Then, since $f_1\le f_2$, defining the set

$$A_\delta:=\{(t,x)\in(0,T)\times\Omega:0<u(t,x)-\widehat{u}(t,x)<\delta\}$$

we get

$$\begin{aligned}&\int_0^T\int_\Omega\left(b(u)_t-b(\widehat{u})_t\right)\Psi_\delta(u-\widehat{u})dxdt+I_1(\delta)+I_2(\delta)\\ &+\int_0^T\int_\Omega(g(x,u)-g(x,\widehat{u}))\Psi_\delta(u-\widehat{u})dxdt\\ \le\ &\int_0^T\int_\Omega\left(f_2-\widehat{f_2}\right)\Psi_\delta(u-\widehat{u})dxdt,\end{aligned}$$

where

$$\begin{aligned}I_1(\delta) &= \frac{1}{\delta}\int_0^T\int_{A_\delta}\left\{\phi\left(\nabla u+K(b(u))\mathbf{e}\right)-\phi\left(\nabla\widehat{u}+K(b(\widehat{u}))\mathbf{e}\right)\right\}\cdot\\ &\quad\cdot\{\nabla u+K(b(u))\mathbf{e}-\nabla\widehat{u}-K(b(\widehat{u}))\mathbf{e}\}dxdt,\end{aligned}$$

$$I_2(\delta) = \frac{1}{\delta}\int_0^T\int_{A_\delta}\{\phi\left(\nabla u + K(b(u))\mathbf{e}\right) - \phi\left(\nabla\widehat{u} + K(b(\widehat{u}))\mathbf{e}\right)\}\cdot$$
$$\cdot\{-K(b(u))\mathbf{e} + K(b(\widehat{u}))\mathbf{e}\}\,dxdt$$

(here T is arbitrary but fixed, $T > 0$). Applying the Young inequality, $\alpha\beta \leq C(\epsilon)p^{-1}\alpha^p + \epsilon p'^{-1}\beta^{p'}$, we see that

$$|I_2(\delta)| \leq \frac{\epsilon}{\delta p'}\int_0^T\int_{A_\delta}|\phi\left(\nabla u + K(b(u))\mathbf{e}\right) - \phi\left(\nabla\widehat{u} + K(b(\widehat{u}))\mathbf{e}\right)|^{p'}\,dxdt$$
$$+\frac{C(\epsilon)}{\delta p}\int_0^T\int_{A_\delta}|K(b(u)) - K(b(\widehat{u}))^p|\,dxdt := I_2^a + I_2^b.$$

We shall only consider the case of $p \in (1,2)$ (the case $p > 2$ is similar and, even, easier). We need an algebraic inequality

LEMMA 1. (see, e.g., Díaz and de Thelin [25]) *Let $\phi(\xi) := |\xi|^{p-2}\xi$ with $p > 1$. Then, there exists $C > 0$ such that*

$$C\left|\phi(\xi) - \phi(\widehat{\xi})\right|^{p'} \leq \left\{(\phi(\xi) - \phi(\widehat{\xi}))\cdot(\xi - \widehat{\xi})\right\}^{\frac{\alpha}{2}}\left\{|\phi(\xi)|^{p'} + |\phi(\widehat{\xi})|^{p'}\right\}^{1-\frac{\alpha}{2}}$$

with $\alpha = 2$ if $1 < p < 2$ and $\alpha = p'$ if $p \geq 2$.

Using Lemma 1 we obtain that

$$|I_2^a| \leq \epsilon\widetilde{C}I_1(\delta),$$

for some $\widetilde{C}$ independent of δ. Moreover

$$I_2^b \leq \frac{C(\epsilon)}{\delta p}\int_{A_\delta}(C|u - \widehat{u}|)^p dxdt \leq \widetilde{C}(\epsilon)m(A_\delta)\delta^{\delta p-1}$$

for some $\widetilde{C}(\epsilon) > 0$ independent of δ. Then

$$I_1(\delta) + I_2(\delta) \geq I_1(\delta) - |I_2(\delta)| \geq (1 - \epsilon\widetilde{C})I_1(\delta) - \widetilde{C}(\epsilon)m(A_\delta)\delta^{\delta p-1}.$$

Taking ϵ small enough (so that $1 - \epsilon\widetilde{C} > 0$) and using that $I_1(\delta) \geq 0$ we have that

$$\lim_{\delta\downarrow 0}(I_1(\delta) + I_2(\delta)) \geq 0$$

and so

$$\int_{u>\widehat{u}}(b(u) - b(\widehat{u}))_t dxdt + \int_{u>\widehat{u}}(g(x,u) - g(x,\widehat{u})) \leq 0.$$

From assumption $(H_{g,b})$ we deduce that

$$\int_{u>\widehat{u}} (b(u)-b(\widehat{u}))_t dxdt \le \int_{u>\widehat{u}} (b(u)-b(\widehat{u}))dxdt,$$

so that

$$\int_0^T \int_\Omega \max\{b(u)-b(\widehat{u}),0\}_t dxdt \le \int_0^T \int_\Omega \max\{(b(u)-b(\widehat{u})),0\}dxdt,$$

and, finally

$$\int_\Omega \max\{b(u(T,x))-b(\widehat{u}(T,x)),0\}dxdt \le \int_0^T \int_\Omega \max\{(b(u)-b(\widehat{u})),0\}dxdt.$$

Then, by Gronwall inequality

$$b(u) \le b(\widehat{u}) \qquad \text{a.e} \quad (t,x) \in (0,T)\times\Omega.$$

If b is strictly increasing this implies that $u \le \widehat{u}$ and the proof of the first conclusion ends. In the general case (i.e., when b is merely nondecreasing) it remains the consideration of the case in which $A_\delta \subset \{b(u)=b(\widehat{u})\}$, for any δ small, (since otherwise the above arguments apply). In that case $I_2(\delta)\equiv 0$ implies that $I_1(\delta)\equiv 0$. But from Lemma 1

$$I_1(\delta) \ge C\delta \int_0^T \int_\Omega \frac{|\nabla \Psi_\delta(u-\widehat{u})|^2 \quad dxdt}{\{|\nabla u + K(b(u))\mathbf{e}|^p + |\nabla \widehat{u} + K(b(\widehat{u}))\mathbf{e}|^p\}^{\frac{2-p}{p}}} \ge 0.$$

So, $\Psi(u-\widehat{u})=0$ a.e. on $(0,T)\times\Omega$ which implies that $u\le\widehat{u}$ on this set. The proof of the case $p>2$ and inequality (13) follows the same type of arguments. ∎

Remark 1. It can be proved (see Díaz and de Thelin [25]) that if b is a *Lipschitz function* and u_0 is regular enough then any bounded weak solution is a strong solution (i.e., $b(u)_t \in L^1(Q_T)$, $Q_T := (0,T)\times\Omega$). The proof of the existence of strong solutions under more general conditions on b is a delicate task (see the *recent* results by Benilan and Gariepy [13]).

Remark 2. The (pointwise) comparison principle can be obtained for weaker solutions by using more complicated arguments and other selected notions of solutions (*entropy solutions, renormalized solutions, good solutions*,...). See the works by Benilan and Touré, Benilan and Wittbold, Carrillo, Otto,...

Remark 3. The *quantitative* inequality (13) is a typical consequence of the application of abstract results (*the T-accretiveness of the operator*). An illustration of how this theory can be applied to the concrete case of problem $(P_{\phi,K})$ (when $h \equiv 0$) is due to Bouhsiss [16]

On the symmetrized mass comparison principle. We recall that this time we assume the additional conditions

$$\mathbf{A}(x,u,\xi)\cdot\xi \geq |\xi|^p, \tag{14}$$

$$g(x,u)u \geq \widehat{g}(u)u \quad \text{for some} \quad \widehat{g} \in C(\mathbb{R}:\mathbb{R}), \tag{15}$$

and, for simplicity, (10). Here we also assume that

$$f = f_2 \in L^1_{loc}(0,\infty : L^1(\Omega)).$$

We shall only consider (for simplicity) the case in which u and U are nonnegative functions.

Theorem 4. *Assume that $\widehat{g}$ is nondecreasing or locally Lipschitz and that the function*

$$\varphi(\eta) := \widehat{g}(b^{-1}(\eta))$$

is well defined and can be decomposed as

$$\varphi = \varphi_1 + \varphi_2 \tag{16}$$

with φ_1 convex and φ_2 concave. Then

$$\int_0^s b(\widetilde{u}(t,\sigma))d\sigma \leq \int_0^s b(\widetilde{U}(t,\sigma))d\sigma \quad \forall s \in [0, m(\Omega)], \forall t \in [0,\infty). \tag{17}$$

Idea of the proof. First of all we point out that conclusion (17) is stable by approximations of the data (f, u_0, b and $\mathbf{A}$) leading to the convergence of solutions in $L^1(0,T : L^1(\Omega))$. Due to that, we can assume the data regular enough (and, in particular, that u and U are strong solutions $b(u)_t \in L^1(Q_T)$, $b(U)_t \in L^1(Q_T^*), Q_T^* := (0,T)\times\Omega$ and that b is strictly increasing.

Step 1. The radially symmetric problem. We define

$$K(t,s) = \int_0^s b(\widetilde{U}(t,\sigma))d\sigma$$

where $\widetilde{U}(t,\cdot)$ is the scalar decreasing rearrangement of $U(t,\cdot)$. First of all, let us prove that $U(t,x)$ decreases when $|x|$ increases. By the symmetry of the data (and the uniqueness of solutions, implicitly assumed) we deduce that $U(t,x)=U(t,|x|)$. Moreover $U_r := \frac{\partial}{\partial r}U(t,r)$, $r=|x|$ verifies that

$$\begin{cases} \frac{\partial}{\partial t}\left(b'(U)U_r\right) - \frac{\partial^2}{\partial r^2}\left(|U_r|^{p-2}U_r\right) + \widehat{g}'(U)U_r = F_r & \text{in } (0,T)\times(0,R), \\ U_r(t,0)=0, \quad U_r(t,R)\le 0, & t\in(0,T), \\ U_r(0,r)=U_{0,r}(r) & r\in(0,R), \end{cases}$$

where $\Omega^* = B(0,R)$, $U_0(r)=\widetilde{u}_0(\omega_N r^N)$ and $F(t,r)=\widetilde{f}(t,\omega_N r^N)$. Then by the maximum principle (here is possible to apply classical results since U_r can be assumed to be smooth), as $F_r(t,\cdot)\le 0$ and $U_{0,r}(\cdot)\le 0$, we deduce that $U_r(t,\cdot)\le 0$, i.e., $U(t,r)$ decreases when r increases. In consequence, $U(t,\cdot)=U^*(t,\cdot)$ (the function coincides with its decreasing symmetric rearrangement), and so

$$U(t,x)=\widetilde{U}(t,\omega_N r^N), \quad r=|x|.$$

Making

$$s=\omega_N r^N \qquad (s\in(0,m(\Omega)))$$

we get that

$$\frac{\partial K}{\partial s}(t,s)=b(\widetilde{U}(t,s)), \quad \frac{\partial U}{\partial r}=N\omega_N^{\frac{1}{N}}s^{\frac{N-1}{N}}\frac{\partial \widetilde{U}}{\partial s}.$$

We deduce that K satisfies the parabolic (fully non-linear) problem

$$(FN^*)\begin{cases} \dfrac{\partial K}{\partial t} - a(s)\left|\dfrac{\partial}{\partial s}b^{-1}(\dfrac{\partial K}{\partial s})\right|^{p-2}\dfrac{\partial}{\partial s}b^{-1}(\dfrac{\partial K}{\partial s}) + \\ \quad \displaystyle\int_0^s \widehat{g}\left(b^{-1}\left(\frac{\partial K}{\partial s}(t,\sigma)\right)\right)d\sigma = \int_0^s \widetilde{f}(t,\sigma)d\sigma, & s\in(0,m(\Omega)) \\ & t\in(0,T), \\ K(t,0)=0, \quad K(t,m(\Omega))=0, & t\in(0,T), \\ K(0,s)=\displaystyle\int_0^s b(\widetilde{u}_0(\sigma))d\sigma & s\in(0,m(\Omega)), \end{cases}$$

where

$$a(s) := \left[N\omega_N^{1/N}s^{(n-1)/n}\right]^p.$$

Step 2. Study of the rearrangement of u. Given $\widetilde{u}(t,\cdot)$ (the scalar decreasing rearrangement of the solution u of (P)), we define

$$k(t,s)=\int_0^s b(\widetilde{u}(t,\sigma))d\sigma.$$

The main goal of this second step is to prove that $k(t,s)$ is a subsolution of (FN^*) in the sense that it verifies all the conditions but replacing the fully nonlinear equation by the inequality

$$\frac{\partial k}{\partial t} - a(s)\left|\frac{\partial}{\partial s}b^{-1}\left(\frac{\partial k}{\partial s}\right)\right|^{p-2}\frac{\partial}{\partial s}b^{-1}\left(\frac{\partial k}{\partial s}\right) + \int_0^s \widehat{g}\left(b^{-1}\left(\frac{\partial k}{\partial s}(t,\sigma)\right)\right)d\sigma \le \int_0^s \widetilde{f}(t,\sigma)d\sigma, \tag{18}$$

$s \in (0, m(\Omega)), t \in (0,T)$. The proof of this inequality is quite long and technical. This process can be also divided in several steps:

(i) Define the function $T_{\tau,h} : \mathbb{R}^+ \to \mathbb{R}^+$ given by

$$\begin{array}{ll} T_{\tau,h}(s) = 0 & \text{if } 0 \le s \le t, \\ T_{\tau,h}(s) = s - t & \text{if } t < s \le t + h, \\ T_{\tau,h}(s) = h & \text{if } s > t + h. \end{array}$$

We take $v = T_{\tau,h}(u)$, as test function. Passing to the limit, as $h \downarrow 0$, we deduce that

$$-\frac{\partial}{\partial\theta}\int_{u>\theta}|\nabla u|^p dx \le \int_0^{\mu(\theta)}\widetilde{f}(t,s)ds - \int_0^{\mu(\theta)}\widehat{g}(\widetilde{u}(t,s))ds - \int_{u>\theta}\frac{\partial b(u)}{\partial t}dx$$

where we used the assumptions (14) and (9) and where $\mu(\theta)$ denotes the distribution function of $u(t,\cdot)$.

(ii) We have that

$$N\omega_N^{1/N}\mu(\theta)^{(N-1)/N} \le \left(-\mu'(\theta)\right)^{1/p'}\left(-\frac{\partial}{\partial\theta}\int_{u>\theta}|\nabla u|^p dx\right)^{1/p}$$

(this is a classical result in the rearrangement theory: the proof uses the, so called, *Fleming-Rishel formula*, the *isoperimetric inequality* and the notion of *perimeter in the Giorgi sense*).

(iii) the following identity holds

$$\int_{u>\theta}\frac{\partial b(u)}{\partial t}dx = \int_0^{\mu(\theta)}\frac{\partial b(\widetilde{u}(t,\sigma))}{\partial t}d\sigma = \frac{\partial k}{\partial t}(t,\mu(\theta))$$

(although a first proof of this formula already appears in the book by Bandle [6] a more general, and rigorous, proof is due to Mossino and Rakotoson [36]). An easy manipulation of (i), (ii), (iii) leads to the wanted inequality for k.

Step 3. Comparison using the fully nonlinear equation. First of all, notice that the comparison

$$k(t,s) \le K(t,s) \qquad \forall t \in [0,T], \quad \forall s \in (0, m(\Omega)),$$

coincides with the conclusion of the theorem. The main difficulty now is not associated to the very *complicated* diffusion operator but with the *nonlocal nature* of the zero order perturbation term. The key idea to obtain the result is that, by assumption (16),

$$\varphi(r) - \varphi(\widehat{r}) \le \left(\varphi_1'(r) + \varphi_2'(\widehat{r})\right)(r - r) \qquad \forall r, \widehat{r} \in \mathbb{R}$$

(use for instance, Taylor formula, the convexity of φ_1 and the concavity of φ_2). Then

$$\begin{aligned}\int_0^s \left[\widehat{g}(\widetilde{U}(t,\sigma) - \widehat{g}(\widetilde{u}(t,\sigma)\right] d\sigma \;&\le\; \int_0^s \left[\varphi_1'(b(\widetilde{U}(t,\sigma))) + \varphi_2'(b(\widetilde{u}(t,\sigma)))\right] \cdot \\ &\quad \cdot \left[b(\widetilde{U}(t,\sigma)) + b(\widetilde{u}(t,\sigma))\right] d\sigma \\ &\le\; C_1 \left|k(t,s) - K(t,s)\right| \\ &\quad + C_2 \max_{\tau\in[0,T],\sigma\in[0,s]} \left|k(\tau,\sigma) - K(\tau,\sigma)\right|,\end{aligned}$$

for some positive constants C_1 and C_2. The comparison is now a consequence of the classical pointwise comparison principle also related to the T-accretiveness of the *complicated* operator, but this time in the space $C^0(\overline{\Omega})$, (details can be found in Díaz [21]: see also other references indicated at the Introduction of this section).

Remark 4. Thanks to a result due to Hardy, Littlewood and Polya in 1929 (see, e.g., [6]), the comparison

$$\int_0^s b\left(\widetilde{u}(t,\sigma)\right) d\sigma \le \int_0^s b(\widetilde{U}(t,\sigma)) d\sigma \qquad \forall s \in [0, m(\Omega)], \forall t \in [0,\infty),$$

implies that

$$\int_0^s \Phi\left(\left(\widetilde{u}(t,\sigma)\right)\right) d\sigma \le \int_0^s (\Phi(b(\widetilde{U}(t,\sigma))) d\sigma \qquad \forall s \in [0, m(\Omega)], \forall t \in [0,\infty)$$

for any convex nondecreasing function Φ. In particular, if

$$b \text{ is a concave function}$$

we get

$$\int_0^s \widetilde{u}(t,\sigma)d\sigma \leq \int_0^s \widetilde{U}(t,\sigma)d\sigma \qquad \forall s \in [0, m(\Omega)], \forall t \in [0,\infty),$$

which is the conclusion presented at the Introduction of this section. Notice that a different application of the above result by Hardy, Littlewood and Polya is that

$$||b(u(t,\cdot))||_{L^q(\Omega)} \leq ||b(U(t,\cdot))||_{L^q(\Omega^*)}$$

for any $q \in [1,\infty)$. Indeed, it suffices to use $\Phi(r) = |r|^q$ and that

$$\int_0^{m(\Omega)} |b(\widetilde{u}(t,\sigma))|^q d\sigma = \int_{\Omega^*} |b(u^*(t,x))|^q dx = \int_\Omega |b(u(t,x))|^q dx.$$

3. The finite extinction time property

3.1. Introduction. One of the most natural questions concerning problem (P) is the *stabilization of solutions*: Assumed that

$$f(t,\cdot) \longrightarrow f_\infty(\cdot) \text{ and } h(t,\cdot) \longrightarrow h_\infty(\cdot) \quad \text{as} \quad t \to +\infty$$

in *suitable* functional spaces then $u(t,\cdot) \longrightarrow u_\infty(\cdot)$ as $t \to +\infty$ (in some *suitable* sense) with $u_\infty(\cdot)$ solution of the associated stationary problem

$$(P_\infty) \begin{cases} -\text{div}\, \mathbf{A}(x, u_\infty, \nabla u_\infty) + g(x, u_\infty) = f_\infty(x), & x \in \Omega, \\ u_\infty = h_\infty, & \text{on } \partial\Omega. \end{cases}$$

A general result, stated in terms of the *omega limit set*

$$\omega(u) := \{u_\infty \in W^{1,p}(\Omega) \colon \exists\ t_n \to \infty \text{ such that } u(t_n,\cdot) \to u_\infty \text{ in } L^p(\Omega), \text{ as } n \to \infty\}$$

jointly with stronger convergence results (but for different particular cases) can be found in Díaz and de Thelin [25]. For stronger convergence results for one-dimensional particular equations see Feireisel and Simondon [28] and their references.

Very often $f_\infty \equiv 0$, $h_\infty \equiv 0$ and $\mathbf{A}$ and g are such that $u_\infty \equiv 0$ is the unique solution to problem (P_∞). In several applications (case of models in plasma physics and also in some chemical reactions) it is observed that there is a very strong stabilization in the following sense: *there exists a finite time* $T_0 > 0$ *such that* $u(t,x) \equiv 0$, $\forall\ t \geq T_0$ *and a.e.* $x \in \Omega$. This property is called as

the *finite extinction time property* and has been considered by many authors in the literature. The main goal of this section is to illustrate the application of the above two comparison principles to the study of this property. A third method (using *energy arguments* and so applicable to higher order parabolic problems and systems) will be also presented.

3.2. The finite extinction time via the pointwise comparison principle. A first result proving the occurrence of this property for some special formulation of problem (P) is the following

Theorem 5. *Let u satisfying*

$$(P_{\alpha,p}) \begin{cases} \left(|u|^{\alpha-1}u\right)_t - \triangle_p u = 0, & t \in (0,\infty), x \in \Omega, \\ u = 0, & t \in (0,\infty), x \in \partial\Omega, \\ u(0,x) = u_0(x) & x \in \Omega, \end{cases}$$

with

$$u_0 \in C_c(\Omega), \text{ i.e., with } \operatorname{supp} u_0 \text{ a compact subset of } \Omega. \tag{19}$$

Assume that

$$(p-1) < \alpha. \tag{20}$$

Then the finite extinction time property holds.

Proof. We assume u in the class of solutions in which the pointwise comparison principle holds (due to the special formulation of $(P_{\alpha,p})$ it can be shown (Benilan [11]) that this is our case for any $\alpha > 0$ and $p > 1$). Then if $\overline{u}$ (resp. $\underline{u}$) is a supersolution of problem $(P_{\alpha,p})$ (resp. subsolution) then

$$\underline{u} \le u \le \overline{u}. \tag{21}$$

So, if we are able to construct $\overline{u}$ (resp. $\underline{u}$) vanishing after a finite time this property also holds for u. Inspired in a pioneering paper (Sabinina [39]) we shall construct $\overline{u}$ as a *separable* supersolution, i.e., $\overline{u}(t,x) = \Phi(t)w(x)$. Since we want to have $\Phi \ge 0$ and $w \ge 0$, we define

$$N\overline{u} :\equiv \left(|\overline{u}|^{\alpha-1}\overline{u}\right)_t - \triangle_p \overline{u} = \left(\Phi^\alpha\right)_t w^\alpha - \Phi^{p-1}\triangle_p w.$$

We take Φ such that

$$\begin{cases} \left(\Phi^\alpha\right)_t = -\lambda\Phi^{p-1}, \ t \in (0,\infty), \\ \Phi(0) = M, \end{cases} \tag{22}$$

with $\lambda > 0$ and $M > 0$ to be determined. Due to the crucial assumption (20) the solution of (22) vanishes after a finite $T_\Phi > 0$ (notice that $\Psi := \Phi^\alpha$ verifies an ODE with a term which is not Lipschitz $\Psi_t + \lambda \Psi^{\frac{p-1}{\alpha}} = 0$). Notice also that (22) is integrable since it is a first order ordinary equation of separable variables. Then

$$N\overline{u} = \Phi^{(p-1)} \left(-\lambda w^\alpha - \Delta_p w\right).$$

In consequence we choose, as w, the solution of the *first eigenvalue problem for the* Δ_p operator, i.e., $\lambda = \lambda_1 > 0$ and

$$\left\{ \begin{array}{ll} -\Delta w = \lambda_1 w^{p-1} & \text{on} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega, \end{array} \right. \tag{23}$$

(the existence of a unique function w satisfying that $w > 0$ on Ω and $||w||_{L^\infty(\Omega)} = 1$ was due to Anane [3] and Barles [8]). Then

$$\begin{array}{rcl} N\overline{u} & = & \Phi^{p-1} \left(-\lambda_1 w^\alpha + \lambda_1 w^{p-1}\right) \\ & = & \lambda_1 \Phi^{p-1} w^{p-1} \left(1 - w^{\alpha-(p-1)}\right) \geq 0 \end{array}$$

since $0 \leq w \leq 1$ and $\alpha > (p-1)$.

The boundary condition holds

$$\overline{u}(t,x)_{|_{(0,\infty)\times\partial\Omega}} = \Phi(t) w_{|_{\partial\Omega}} = 0.$$

The comparison between the initial data

$$u_0(x) \leq M w(x), \quad x \in \Omega$$

trivially holds by taking M big enough (recall the assumption (19) on u_0). The construction of $\underline{u} \leq 0$ is similar. ∎

Remark 5. The above statement can be improved in many different directions (but with longer proofs). For instance, in the case of $p = 2$ the homogeneity assumed on b is not needed. More precisely, in G. Díaz and J.I. Díaz [18], the finite extinction time property was established for the problem

$$\left\{ \begin{array}{ll} b(u)_t - \Delta u = f(x,t), & x \in \Omega, t > 0, \\ u = 0, & x \in \partial\Omega, t > 0, \\ u(0,x) = u_0(x), & x \in \Omega, \end{array} \right. \tag{24}$$

by assuming

$$\int_{0^+} \frac{ds}{b^{-1}(s)} < +\infty \tag{25}$$

and the existence of T_f such that $f(t,x) \equiv 0$, for $t > T_f$ and $x \in \Omega$. Notice that now $p = 2$ and that if $b(s) = |s|^{\alpha-1}s$ then (25) if and only if $\alpha > 1$, i.e., the same condition than (20). In fact, in this paper it is also shown that condition (25) is also necessary for the existence of a finite extinction time.

Remark 6. Notice that the *finite extinction time* can not be satisfied (in case of the general formulation of (P)) each time that the *strong maximum principle* holds (see, e.g., Nirenberg [37]) or the *unique continuation property* is verified (see, e.g., Ghidaglia [29] and its references).

When condition (20) holds, it is said that we have a *fast diffusion* (in fact, this term is more appropriate when talking on the balance between the accumulation and the diffusion terms). It is very easy to see that if we assume (20) then the conclusion of the above theorem remains true under the presence of a nondecreasing absorption term as, for instance,

$$\left(|u|^{\alpha-1}u\right)_t - \triangle_p u + |u|^{q-1}u = 0$$

for any $q > 0$. The finite extinction time property also occurs due to suitable balance between the accumulation and absorption terms. It is the so called *strong absorption case.*

THEOREM 6. *Let u satisfying*

$$(P_{\alpha,p,q}) \begin{cases} \left(|u|^{\alpha-1}u\right)_t - \triangle_p u + |u|^{q-1}u = 0, & t \in (0,\infty)\,,\ x \in \Omega, \\ u = 0, & t \in (0,\infty)\,,\ x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

with

$$u_0 \in L^\infty(\Omega)\,. \tag{26}$$

Assume

$$\mu > 0 \quad \textit{and} \quad 0 < q < \alpha \quad \textit{with } p > 1 \quad \textit{arbitrary.} \tag{27}$$

Then the finite extinction time property holds.

Proof. It is easy to see that the function $\overline{u}(x,t) = \Phi(t)$, with Φ the (unique) solution of the ODE

$$\begin{cases} \left(\Phi^\alpha\right)_t + \mu\Phi^q = 0, & t \in (0,\infty)\,, \\ \Phi(0) = M, \end{cases} \tag{28}$$

(compare it with (22)) is a supersolution once that $M \geq \|u_0\|_{L^\infty(\Omega)}$. The assumption (27) implies that Φ vanishes after some finite time T_Φ. ∎

Remark 7. A general survey containing many references on this property is due to Kalashnikov [32].

3.3. The finite extinction time via the mass symmetrized comparison principle. Thanks to the *mass symmetrized comparison principle* it is possible to extend the last two theorems to more general equations for which the construction of super and subsolutions can be very difficult (specially in the case of the first of the theorems).

Theorem 7. *Let u be the solution of (P) with $f \equiv 0, h \equiv 0, u_0 \in C_c(\Omega)$, $u_0 \geq 0$ and assume $b(u) = |u|^{\alpha-1} u$, (14) and (9). We also suppose that one of the two following conditions holds:*

$$\left\{ \begin{array}{l} (p-1) < \alpha \text{ and} \\ \varphi(\eta) := \widehat{g}(|\eta|^{\frac{1}{\alpha}-1}\eta) = \varphi_1(\eta) + \varphi_2(\eta), \eta \in \mathbb{R} \\ \text{with } \varphi_1 \text{ (resp. } \varphi_2) \text{ nondecreasing and convex} \\ \text{(resp. nondecreasing and concave),} \end{array} \right. \tag{29}$$

or

$$\left\{ \begin{array}{l} \widehat{g}(\eta) = \mu |\eta|^{q-1}\eta \text{ with } \quad \mu > 0 \text{ and} \\ \qquad q < \alpha. \end{array} \right. \tag{30}$$

Then the finite extinction time property is verified. More precisely, if we define as $T_{0,\Omega}$ the first extinction time (in which $\|u(T_0,\cdot)\|_{L^1(\Omega)} \equiv 0$) then

$$T_{0,\Omega} \leq T_{0,\Omega^*},$$

where T_{0,Ω^*} is the first extinction time for the symmetrized problem (P^*).

Proof. By the mass symmetrized comparison principle and the result by Hardy, Littlewood and Polya mentioned in the above Section we have that

$$\|b(u(t,\cdot))\|_{L^1(\Omega)} \leq \|b(U(t,\cdot))\|_{L^1(\Omega)}$$

for any $t > 0$. Assumption (29) (resp (30)) allows to apply Theorem 5 (resp Theorem 8) which proves the result. ∎

Remark 8. Notice that the general structure of $\mathbf{A}(x,u,\xi)$ may be the origin of very complicated behaviors of the solution of the associated eigenvalue problem

$$\left\{ \begin{array}{ll} -\operatorname{div} \mathbf{A}(x,w,\nabla w) = \lambda w^{p-1} & \text{in} \quad \Omega, \\ w = 0 & \text{on} \quad \partial\Omega. \end{array} \right.$$

So that the arguments of the proof of Theorem 5 do not apply directly to problem (P).

3.4. THE FINITE EXTINCTION TIME VIA AN ENERGY METHOD. A method which do not use any comparison principle can be applied to the study of this property. The following is merely a special version of the method:

THEOREM 8. *Let u be the solution of (P) with $h \equiv 0$,*

$$\begin{cases} f \in L^\infty((0,\infty)\times\Omega) \text{ such that } \exists T_f > 0 \text{ with} \\ f(t,x) \equiv 0 \text{ a.e. } t \geq T_f \text{ and a.e. } x \in \Omega, \end{cases} \tag{31}$$

$u_0 \in L^\infty(\Omega)$, $b(u) = |u|^{\alpha-1}u$, $\alpha > 0$, $\mathbf{A}$ *satisfying* (14) *and*

$$g(x,\eta) \geq 0 \quad \forall \eta \in \mathbb{R}. \tag{32}$$

Assume that (20) holds (i.e., $p-1 < \alpha$). Then the finite extinction property holds.

Proof. We take as *test function* $v = |u|^{k-1}u$ (which we shall write, for simplicity, as $v = u^k$) with $k > 0$ to be determined later. We also write u^α instead of $|u|^{\alpha-1}u$ by simplicity in the notation (nevertheless, it is *not* required that $u \geq 0$). Integrating on the open (bounded) set Ω in each term of the equation we get :

$$\begin{aligned} \int_\Omega \frac{\partial u^\alpha}{\partial t} u^k dx &= \int_\Omega \alpha u^{(\alpha-1)+k} u_t dx \\ &= \frac{\alpha}{(\alpha+k)} \frac{d}{dt}\left(\int_\Omega u^{\alpha+k} dx\right) \end{aligned}$$

(the justification of the final formula for u weak solution of (P), i.e., without the condition $(u^\alpha)_t \in L^1(\Omega)$, is due to Alt and Luckhaus [2]),

$$\begin{aligned} -\int_\Omega \operatorname{div} \mathbf{A}(x,u,\nabla u) u^k dx &= k\int_\Omega \mathbf{A}(x,u,\nabla u)\cdot\nabla u u^{k-1} dx \\ &\geq k\int_\Omega |\nabla u|^p u^{k-1} dx. \end{aligned}$$

So, using (31) and (32) we get that, if $t > T_f$, then

$$\frac{\alpha}{(\alpha+k)} \frac{d}{dt}\int_\Omega u^{\alpha+k}(x,t) dx + k\int_\Omega |\nabla u|^p u^{k-1} dx \leq 0.$$

We need the following *interpolation* result

LEMMA 2. *Let $p \geq 1$ and $k \geq 1$. There exists a constant $C = C(m(\Omega), p, N, k)$ such that if $w \in W_0^{1,1}(\Omega)$ and $\int_\Omega |\nabla w|^p |w|^{k-1} dx < +\infty$ we have that*

$$\left(\int_\Omega |w|^s dx \right)^{\frac{p+k-1}{s}} \leq C k^p \int_\Omega |\nabla w|^p |w|^{k-1} dx$$

with

$$\begin{array}{ll} 1 \leq s \leq \frac{N(p+k-1)}{N-p} & \text{if} \quad p < N, \\ 1 \leq s \leq \infty & \text{if} \quad p = N, \\ s = \infty & \text{if} \quad p > N. \end{array}$$

Idea of the proof of the Lemma. Define $z(x) = |w(x)|^{\frac{p+k-1}{p}} sign(w(x))$. Then

$$\int_\Omega |\nabla z|^p dx = \left(\frac{p+k-1}{p} \right)^p \int_\Omega |\nabla w|^p |w|^{k-1} dx$$

and the conclusion follows from the application of the Poincaré-Sobolev and Hölder inequalities. ∎

Continuation of the proof of Theorem 8. By the above lemma we have

$$\frac{\alpha}{(\alpha+k)} \frac{d}{dt} \left(\int_\Omega u^{\alpha+k}(t,x) dx \right) + C \left(\int_\Omega u^s(t,x) dx \right)^{\frac{p+k-1}{s}} \leq 0$$

for $t > T_f$. Applying Hölder inequality we get

$$\left(\int_\Omega u^{\alpha+k}(t,x) dx \right)^{\frac{1}{\alpha+k}} \leq C(n(\Omega)) \left(\int_\Omega u^s(t,x) dx \right)^{\frac{1}{s}}$$

(take $k = 1$ if $p \geq N$ and $k \geq \frac{N}{p}(\alpha - (p-1)) - \alpha$ if $p < N$). Then if we define

$$Y(t) := \int_\Omega u^{\alpha+k}(t,x) dx$$

we have that

$$\begin{cases} Y'(t) + C Y(t)^\gamma \leq 0 \quad \text{on} \quad (T_f, \infty), \quad \gamma = \frac{p+k-1}{\alpha+k} \in (0,1), \\ Y(T_f) = Y_f > 0. \end{cases}$$

So, again, $\exists T_0 > T_f$ such that $Y(t) \equiv 0$ if $t \geq T_0$ and the conclusion holds. ∎

Remark 9. Some similar energy method can be applied to the case of *strong absorption* (see, e.g., Tsutsumi [40]).

Remark 10. Under some extra decay assumptions on $f(t,\cdot)$, near T_f, it is possible to show something unexpected: $T_0 = T_f$ (see Antontsev and Díaz [4]).

Remark 11. Similar energy methods applied to *higher order quasilinear parabolic equations* can be found in Bernis [14], [15].

Remark 12. One of pioneering applications of this type of energy methods was concerning the case $p = 2$ and $\Omega = \mathbb{R}^N$. In that case the condition for the existence of a finite extinction time is

$$\alpha > \frac{N}{N-2},$$

stronger than $\alpha > 1$ correspondent to bounded domains (see Benilan and Crandall [12]).

As a final and global remark we point out that the three methods used in this section can be also applied to the study of other different qualitative properties, as for instance, the existence of a *finite blow-up time* T_∞ (such that $||b(u(t,\cdot))||_{L^r(\Omega)} \to +\infty$ as $t \to +\infty$, for some $r \in [1, +\infty]$). Obviously, this property requires completely different assumptions on $\mathbf{A}, b$ and g. The connection between the finite extinction time and the finite blow-up time properties for a couple of different nonlinear equations has been considered in Kawohl and Peletier [34].

4. The finite speed of propagation property

4.1. Introduction. The formulation of problem (P) is very general. It includes not only the linear heat equation

$$u_t - \Delta u = 0 \tag{33}$$

but many other cases in which the behavior of the correspondent solutions is very different to the one of the solution of the linear heat equation (remember the remarks concerning the finite extinction time property as peculiar of fast diffusion or strong absorption and opposite to properties as the strong maximum principle or the unique continuation property which holds for the linear equation).

Another qualitative property typical of some suitable *nonlinear* models concerns the *finite speed of propagation of disturbances*: if the initial datum

u_0 vanishes on a positively measured set of Ω (i.e., $\operatorname{supp}(u_0) \subset \Omega$) then $\operatorname{supp} u(t,\cdot) \subset \Omega$, for any $t \in (0,t^*)$, for some $t^* > 0$.

This behavior (typical of the linear *wave* equation) fails for the linear heat equation (this can be illustrated in many ways: the strong maximum principle, the explicit representation formula for $\Omega = \mathbb{R}^N$, etc). It is said that the linear heat equation has an *infinite* speed of propagation.

When the finite speed of propagation holds then

$$\operatorname{supp}(u(t,\cdot)) := \overline{\{x \in \Omega: \quad u(t,x) \neq 0\}} \subset \Omega$$

(at least for some small times t) and so some hypersurfaces $(0,\infty) \times \mathbb{R}^N)$

$$\mathcal{F} = \bigcup_{t>0} \mathcal{F}(t), \quad \mathcal{F}(t) = \partial(\operatorname{supp} u(t,\cdot)) - \partial\Omega$$

are formed. Those hypersurfaces are called as *free boundaries* (since they are not a priori determined) and play a very important role in the study of the model (usually is in those free boundaries where are located the singularities of the gradient and/or the second derivatives of the solutions).

The main goal of this section is to illustrate how the two comparison principles can be applied to the study of the occurrence of this property. As in the previous section, a third method (involving different *energy arguments*) will be also presented.

4.2. The finite speed of propagation via the pointwise comparison principle. As in the Subsection 3.2, the main idea will be to construct suitable super and subsolutions (now vanishing locally in some subdomains). In fact, those functions use to be constructed by modifying *special solutions* of the equation (so this task is closer to an quantitative study of pde's than the usual approach to pde's by methods of *functional analysis*).

To start with, let us consider the nonlinear equation

$$\left(|u|^{\alpha-1}\, u\right)_t - \Delta_p u = 0, \quad \alpha > 0, \quad p > 1. \tag{34}$$

Although we remain interested in the Cauchy-Dirichlet problem $(P_{\alpha,q})$, it is useful to start by considering the pure Cauchy problem (i.e., $\Omega = \mathbb{R}^N$). A very important family of *exact solutions* is the one given by

$$U_M(t,x) = \frac{1}{t^\lambda}\left[C - k\frac{\sum_{i=1}^N |x_i|^{p'}}{t^{\beta p'}}\right]_+^{(p-1)/(p-1-\alpha)} \tag{35}$$

which arises when

$$(p-1) > \alpha \tag{36}$$

(notice that the *fast diffusion* was $(p-1) < \alpha$), where

$$p' = \frac{p}{p-1}, \quad \beta = \frac{\alpha}{(\alpha+1)(p-N)+(N-1)p},$$
$$\lambda = \frac{N\beta}{\alpha} \text{ and } k = \beta^{(p'-1)}\frac{p-1-\alpha}{p},$$

($C > 0$ arbitrary). Such solutions were obtained, by first time, by G. I. Barenblatt in 1952 for the case $p = 2$ (also in the case, they were refound by R. E. Pattle in 1959). The case $p \neq 2$ was found by A. Bamberger in 1975. We point out that when $p \neq 2$ the solution U_M is not radially symmetric with respect to the usual Euclidean norm of $\mathbb{R}^N$. Nevertheless, it is possible to find other exact solutions with free boundaries and symmetry (although they are not so explicit as U_M). Many references on this topic can be found in the surveys by Kalashnikov [32] and [42]. We also point out that:

$$\int_{\mathbb{R}^N} U_M(t,x)dx = M, \qquad M = M(C,\alpha,p,N),$$
$$U(t,\cdot) \to M\delta_0(x),$$

and that the free boundary generated by U_M is explicitly given by the equation

$$\sum_{i=1}^{N} |x_i|^{p'} = \frac{C}{k}t^{\beta p'}.$$

A simple result is the following.

THEOREM 9. *Let u satisfying*

$$\begin{cases} (|u|^{\alpha-1}u)_t - \Delta_p u = 0, & t \in (0,\infty), x \in \Omega, \\ u = 0, & t \in (0,\infty), x \in \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases}$$

with

$$\begin{cases} u_0 \in C_c(\Omega) \quad \textit{such that} \\ \operatorname{supp} u_0 \subset B(x_0,R_0) \subset \Omega. \end{cases} \tag{37}$$

Assume that

$$(p-1) > \alpha. \tag{38}$$

Then the finite speed of propagation holds.

Proof. As in Theorem 5, we can apply the pointwise comparison principle thanks to the result by Benilan [11]. By choosing M big enough and thanks to the assumption (37) we have that

$$u_0(x) \leq U_M(\tau, x - \widehat{x}_0) \qquad \forall x \in \Omega,$$

for some $\tau > 0$. Since the function $\overline{u}(t,x) := U_M(t+\tau, x - \widehat{x}_0)$ satisfies that

$$\begin{cases} \left(|\overline{u}|^{\alpha-1}\,\overline{u}\right)_t - \Delta_p \overline{u} = 0, & t \in (0,\infty), x \in \Omega, \\ \overline{u} \geq 0, & t \in (0,\infty), x \in \partial\Omega, \\ \overline{u}(0,x) \geq u_0(x) & x \in \Omega, \end{cases}$$

we conclude that

$$u(t,x) \leq \overline{u}(t,x) \quad t > 0, x \in \Omega.$$

By taking (if needed) different values of M and τ we get, similarly that

$$-U_{M'}(t+\tau', x - \widehat{x}_0) \leq u(t,x) \quad x \in \Omega, t > 0.$$

Thus, at least for $t \in [0, t^*)$ with t^* small enough, we conclude that

$$u(t,x) \equiv 0 \qquad \text{a.e.} \quad x \in \Omega - B(\widehat{x}_0, R(t))$$

for some function $R(t)$ and the result follows. ∎

Remark 13. Again, the above statement can be improved in many different directions. For instance, in the case $p = 2$ we can replace $b(u) = |u|^{\alpha-1}u$ by a general nondecreasing function satisfying that

$$\int_{0^+} \frac{ds}{b(s)} < +\infty \tag{39}$$

and the finite speed of propagation holds (see Díaz [19]). Notice that if $p = 2$ and $b(u) = |u|^{\alpha-1}u$ then (39) holds if and only if $\alpha < 1$, i.e., same condition than (38). If $N = 1$ (and $p = 2$) it was proved by A.S. Kalashnikov (and independently by L. A. Peletier) in 1974, that condition (39) is also necessary.

Remark 14. Once that the free boundary exists it becomes interesting to study its dynamics: how fast it starts near $t = 0$ (in some cases there is a *waiting time)*, how it behaves for $t \to +\infty$), the regularity of the free boundary,etc.). Many of those questions remain still open (see the survey Kalashnikov [32]).

When assumption (38) holds it is said that we have a *slow diffusion*. It is easy to see that if (38) holds then the finite speed of propagation remains true under the presence of nondecreasing absorption term as, for instance,

$$(|u|^{\alpha-1}\,u)_t - \Delta_p u + \mu|u|^{q-1}u = 0, \quad \mu > 0,$$

for any $q > 0$. The finite speed of propagation also occurs when the balance between the diffusion and absorption is suitable (called again as the *strong absorption case*). We can consider, even, the case of nonhomogeneous boundary conditions.

THEOREM 10. *Let u satisfying*

$$(P_{\alpha,p,q})\begin{cases} (|u|^{\alpha-1}\,u)_t - \Delta_p u + \mu\,|u|^{q-1} = 0, & t \in (0,\infty), x \in \Omega, \\ u = h, & t \in (0,\infty), x \in \partial\Omega, \\ u(0,x) = u_0(x) & x \in \Omega, \end{cases}$$

with

$$h \in L^\infty\left((0,\infty)\times\Omega\right) \cap L^p_{loc}\left(0,\infty : W^{1,p}(\Omega)\right), \quad h \geq 0 \text{ on } (0,\infty)\times\partial\Omega, \tag{40}$$

$$u_0 \in L^\infty(\Omega), \quad u_0 \geq 0 \text{ on } \Omega. \tag{41}$$

Assume

$$\mu > 0 \quad \text{and} \quad 0 < q < p-1. \tag{42}$$

Then the finite speed of propagation holds. More precisely: a) There exists a positive constant $L > 0$ such that the null set of $u(t,\cdot)$ is not empty assumed that the set

$$\Omega - \left(\operatorname{supp}(u_0) \bigcup (\cup_{\tau>0} \operatorname{supp}(h(\tau,\cdot)))\right)$$

is big enough, i.e.,

$$N(u(t,\cdot)) := \{x \in \Omega\colon u(t,x) = 0\} \supset \left\{x \in \Omega;\, d(x, \operatorname{supp}(u_0) \bigcup (\cup_{\tau>0} \operatorname{supp}\,(h(\tau,\cdot))) \geq L\right\}$$

for any $t > 0$. b) If we assume, in addition, that

$$q < \alpha \leq 1 \tag{43}$$

then there exists $t_0 \geq 0$ such that for every $t \geq t_0$

$$N(u(t,\cdot)) \supset \left\{x \in \Omega : d(x, \cup_{\tau>0} \operatorname{supp}(h(\tau,\cdot)) \geq \widehat{L}\right\}$$

for some $\widehat{L} > 0$.

Proof. We recall a result of Díaz [20] proving that the function

$$w_\lambda(x) = C^*_\lambda |x - x_0|^{\frac{1}{p-1-q}},$$

$$C^*_\lambda = \left[\frac{\lambda (p-1-q)^p}{p^{(p-1)} \left(pq + N(p-1-q) \right)} \right]^{\frac{1}{p-1-q}}$$

satisfies that

$$-\Delta_p w_\lambda + \lambda |w_\lambda|^{q-1} w_\lambda = 0,$$

assumed that (42) holds, i.e., $\lambda > 0$ and $q < p-1$. Let us prove a). Let $x_0 \in \Omega - (\operatorname{supp} u_0 \bigcup \cup_{\tau>0} \operatorname{supp} h(\tau, \cdot))$, and let $R = d(x, (\operatorname{supp} u_0 \bigcup \cup_{\tau>0} \operatorname{supp} h(\tau, \cdot)))$. Consider $\Omega(x_0) := B(x_0, R) \cap \Omega$. Then $\overline{u}(t,x) := W_\mu(x)$ is a *local supersolution*, i.e., a supersolution on $\Omega(x_0)$ since

$$\begin{array}{lll} (|\overline{u}|^{\alpha-1}\,\overline{u})_t - \Delta_p \overline{u} + \mu |\overline{u}|^{q-1} \overline{u} = 0 & \text{on} & (0,\infty) \times \Omega(x_0), \\ \overline{u}(0,x) \geq 0 = u_0(x), & \text{on} & \Omega(x_0) \\ \overline{u}(t,x) \geq 0 = h(t,x) & \text{on} & (0,\infty) \times \Omega(x_0) \cap \partial\Omega, \end{array}$$

and the condition

$$\overline{u}(t,x) \geq u(t,x) \quad \text{on} \quad (0,\infty) \times \partial\Omega(x_0) - \partial\Omega,$$

is satisfied if, for instance,

$$C_\mu R^{\frac{p}{p-1-q}} \geq ||u||_{L^\infty((0,\infty)\times\Omega)} \quad (\geq u(t,x) \quad \text{a.e.}(t,x)),$$

i.e., if

$$R \geq \left[\frac{||u||_{L^\infty((0,\infty)\times\Omega)}}{C_\mu} \right]^{\frac{p-1-q}{p}}$$

(notice that $||u||_{L^\infty((0,\infty)\times\Omega)} < \infty$ thanks to the assumptions on h and u_0, as we can prove in many ways: for instance by using a suitable global supersolution). Then by the pointwise comparison principle on $(0,\infty) \times \Omega(x_0)$ we obtain that

$$0 \leq u(t,x) \leq C^*_\mu |x - x_0|^{\frac{p}{p-1-q}}$$

and so $u(t, x_0) = 0$ (even if u is not necessarily continuous).

To prove part b) we take as *local supersolution* the function

$$\overline{u}(t,x) := w_{\mu/2} + V(t)$$

with $V(t)$ satisfying

$$\begin{cases} \frac{d}{dt}\left(|V|^{\alpha-1}V\right) + \frac{\mu}{2}|V|^{q-1}V = 0, \\ V(0) = ||u_0||_{L^\infty(\Omega)}, \end{cases} \tag{44}$$

i.e.,

$$V(t) = \left[||u_0||^{\alpha-q}_{L^\infty(\Omega)} - \frac{\mu(\alpha-q)}{2\alpha}t\right]_+^{\frac{1}{\alpha-q}}. \tag{45}$$

Then

$$(|\overline{u}|^{\alpha-1}\overline{u})_t = \alpha\left(w_{\mu/2}(x) + V(t)\right)^{\alpha-1}\dot{V} \geq \frac{d}{dt}\left(|V|^{\alpha-1}V\right),$$

$$\Delta_p\overline{u} = \Delta_p w_{\mu/2},$$

$$\mu|\overline{u}|^{q-1}\overline{u} \quad \geq \tfrac{\mu}{2}|w_{\mu/2}|^{q-1}w_{\mu/2} + \tfrac{\mu}{2}|V|^{q-1}V \ ,$$

and so

$$\left(|\overline{u}|^{\alpha-1}\overline{u}\right)_t - \Delta_p\overline{u} + \mu|\overline{u}|^{q-1}\overline{u} \geq 0.$$

Moreover

$$\overline{u}(0,x) = w_{\mu/2} + V(0) \geq ||u_0||^{\alpha-q}_{L^\infty(\Omega)} \geq u_0(x).$$

Finally, taking

$$t_0 = \frac{2\alpha}{\mu(\alpha-q)}||u_0||^{\alpha-q}_{L^\infty(\Omega)}$$

we get that $V(t) \equiv 0 \quad \forall t \geq t_0$ and the conclusion follows as in part a). ∎

Remark 15. The above result is taken from Díaz and Hernández [23] where other, and more general, results can be found.

Remark 16. In the model of chemical reactions, the *null set* $N(u(t,\cdot))$ is called as *dead core*. In that model usually $h(t,x) \equiv 1$ and so $N(u(t,\cdot))$ only occurs at the interior of Ω.

Remark 17. Notice that if $h \equiv 0$ part b) shows the *extinction in finite time*. Notice also that assumptions (42) (in addition to (43)) implies the *formation of dead core for t large* even for $h \equiv 1$ and $u_0 > 0$. This property has a similar nature to the so called *instantaneous shrinking of the support* established by Brezis and Friedman in 1976, or by Evans and Knerr in 1979, both for the case of $\Omega = \mathbb{R}^N$ and $u_0 > 0$ such that $\lim_{|x|\to\infty} u_0(x) = 0$ (see references in the survey Kalashnikov [32]).

4.3. The finite speed of propagation via the mass symmetrized comparison principle. The above method requires the construction of sophisticated supersolutions. This is possible only for simple nonlinear operators. The application of the mass symmetrized comparison principle show us how important is to have symmetry conditions on the partial differential equation in order to have solutions with *small* support.

Theorem 11. *Let u be the solution of (P) with $f \equiv 0$, $h \equiv 0$, $u_0 \in C_c(\Omega)$, $u_0 \geq 0$ and assume $b(u) = |u|^{\alpha-1}u$, (14) and (9). We also suppose the following conditions*

$$\begin{cases} (p-1) > \alpha, \\ \phi(\eta) := \widetilde{g}\left(|\eta|^{\frac{1}{\alpha}-1}\eta\right) = \varphi_1(\eta) + \varphi_2(\eta), \quad \eta \in \mathbb{R} \\ \text{with } \varphi_1 \text{ (resp. } \varphi_2\text{) nondecreasing convex} \\ \text{(resp. nondecreasing concave),} \end{cases}$$

and

$$\int_\Omega b(u(t,x))dx = \int_{\Omega^*} b(U(t,x))dx, \quad \forall t \geq 0, \tag{46}$$

where U denotes the solution of the symmetrized problem. Then the support of $u(t,\cdot)$ satisfy

$$m\,(\,\mathrm{supp}\,u(t,\cdot)) \geq m\,(\,\mathrm{supp}\,U(t,\cdot)) \tag{47}$$

for any $t > 0$.

Proof. By using the mass symmetrized comparison principle, (46) and that

$$\int_\Omega b(u(t,x)dx = \int_0^{m(\Omega)} b(\widetilde{u}(t,\sigma))d\sigma$$

we have

$$\begin{aligned} \int_s^{m(\Omega)} b(\widetilde{u}(t,\sigma))d\sigma &= \int_0^{m(\Omega)} b(\widetilde{u}(t,\sigma))d\sigma - \int_0^s b(\widetilde{u}(t,\sigma))d\sigma \\ &\geq \int_0^{m(\Omega)} b(\widetilde{U}(t,\sigma))d\sigma - \int_0^s b(\widetilde{U}(t,\sigma))d\sigma. \end{aligned}$$

Let

$$\begin{aligned} &\text{support of } \widetilde{u} = [0, R_u(t)], \quad 0 < R_u(t) \leq m(\Omega) \\ &\text{support of } \widetilde{U} = [0, R_U(t)], \quad 0 < R_U(t) \leq m(\Omega) \end{aligned}$$

(recall that $\widetilde{u}$ and $\widetilde{U}$ are nondecreasing functions). Then, necessarily $R_u(t) \geq R_U(t)$ since otherwise we would deduce that

$$\int_{R_u(t)}^{m(\Omega)} b(\widetilde{u}(t,\sigma))d\sigma \geq \int_{R_u(t)}^{R_U(t)} b(\widetilde{U}(t,\sigma))d\sigma > 0$$

which is a contradiction. Finally, it suffices to remark that

$$\widetilde{u}(t,\cdot) = [0, m(\operatorname{supp} u(t,\cdot))]$$

(analogously for U) and the conclusion holds. ∎

Remark 18. Notice that by (47) if $\operatorname{supp} U(t^*,\cdot) = \Omega$, for some $t^* > 0$, then $\operatorname{supp} u(t^*,\cdot) = \Omega$.

Remark 19. Assumption (46) is satisfied, for instance, when the *conservation of the mass* holds, i.e.,

$$\int_\Omega b(u(t,x))dx = \int_\Omega b(u_0(x))dx, \quad \forall t > 0.$$

In that case $\int_\Omega b(u_0(x))dx = \int_{\Omega^*} b(U_0(x))dx = \int_{\Omega^*} b(U(t,x))dx$ and (46) is verified. The *conservation of the mass* is typical of pure diffusion processes (i.e., when $g = \widetilde{g}$). It can be shown (see Díaz [21]) that assumption (46) is also verified when, besides the Dirichlet condition $u(t,x) = 0$, $t > 0$, $x \in \partial\Omega$, we have the additional information that

$$\frac{\partial u}{\partial \mathbf{n}}(t,x) = 0 \quad \text{for} \quad t \in (0,\widehat{T}), \quad x \in \partial\Omega,$$

for some $\widehat{T} > 0$ (in that case the conclusion (47) holds at least for $t \in [0,\widehat{T})$).

In the case of strong absorption we can allow a nonzero Dirichlet condition

THEOREM 12. *Let u be the solution of (P) with $f \equiv 0$ and*

$$h(t,x) \equiv h, \quad \text{a positive constant.} \tag{48}$$

Let $u_0 \in L^\infty(\Omega)$ with

$$0 \leq u_0(x) \leq h \qquad \text{a.e. } x \in \Omega. \tag{49}$$

Assume $b(u) = |u|^{\alpha-1}u$, (14), (9) *and*

$$\widehat{g}(\eta) = \mu|\eta|^{q-1}\eta \quad \textit{with } \mu > 0 \textit{ and } q < (p-1).$$

Then the supports of $u(t,\cdot)$ *and* $U(t,\cdot)$ *satisfy that*

$$m\,(\,\mathrm{supp}\,u(t,\cdot)) \geq m\,(\,\mathrm{supp}\,U(t,\cdot)) \quad \textit{for} \quad t > 0. \tag{50}$$

Idea of the proof. By introducing the change of variables $v(t,x) = h - u(t,x)$ and $V(t,x) = h - U(t,x)$ we can apply the mass symmetrized comparison principle to v and V. Finally, it suffices to apply the result by Hardy, Littlewood and Polya for an appropriate choice of convex function Φ (see Díaz [21]). ∎

Remark 20. Estimates (47) and (50) allows to compare the *waiting times* (when arising) for u and U.

Remark 21. Estimate (50) shows that the *dead core* has a bigger measure under radially symmetric conditions. That was first observed in Bandle and Stakgold [7].

4.4. The finite speed of propagation via an energy method. The study of the finite speed of propagation (and other qualitative properties) can be carried out by using some energy arguments which, in contrast with the ones of Section 3, now have a local character.

Theorem 13. *Let* $\mathbf{A}$ *satisfying* (4) *and*

$$|\mathbf{A}(x,u,\xi)| \leq C|\xi|^{p-1}.$$

Let $g(x,u)$ *such that*

$$g(x,\eta)\eta \geq 0 \qquad \forall \eta \in \mathbb{R}.$$

Assume

$$\alpha < (p-1)$$

and let u *be a local solution of the equation*

$$\left(|u|^{\alpha-1}u\right)_t - \mathrm{div}\,\mathbf{A}(x,u,\nabla u) + g(x,u) = 0 \quad \textit{on} \quad (0,\infty)\times B(x_0,R)$$

(for some $x_0 \in \mathbb{R}^N$, $R > 0$) such that

$$u(0,x) = 0 \quad a.e. \quad x \in B(x_0, \rho_0), \qquad \rho_0 < R.$$

Then there exists $t^* > 0$ and $\rho : [0, t^*] \longrightarrow [0, \rho_0]$ nondecreasing such that

$$u(t,x) = 0 \quad a.e. \quad x \in B(x_0, \rho(t)).$$

Idea of the proof. By multiplying by u and integrating by parts we get

$$\frac{\alpha}{\alpha+1}\int_{B_\rho} |u(t,x)|^{\alpha+1}dx + \int_0^t \int_{B_\rho} \mathbf{A}(x,u,\nabla u)\cdot \nabla u dx ds \leq \int_0^t \int_{\partial B_\rho} u\mathbf{A}(x,u,\nabla u)\cdot \mathbf{n} d\Gamma ds$$

(this can be rigorously justified from the notion of bounded weak *local* solution). Here $B_\rho = B(x_0, \rho)$. We introduce the *local energies*

$$E(t,\rho) := \int_0^t \int_{B_\rho} \mathbf{A}(x,u,\nabla u)\cdot \nabla u dx ds$$

and

$$b(t,\rho) := \operatorname{ess\,sup}_{s\in(0,t)} \left(\frac{\alpha}{\alpha+1}\int_{B_\rho} |u(s,x)|^{\alpha+1}dx \right).$$

Using Hölder inequality we get that

$$b + E \leq \frac{1}{pc}\left(\int_0^t \int_{B_\rho} |u|^p dx ds\right)\left(\frac{\partial E}{\partial \rho}\right)^{\frac{p-1}{p}}$$

where we used that

$$\frac{\partial E}{\partial \rho}(t,\rho) = \int_0^t \int_{\partial B_\rho} \mathbf{A}(x,u,\nabla u)\cdot \nabla u d\Gamma ds.$$

We need the following

LEMMA 3. (Interpolation-trace inequality) *For any $\sigma \in [0, p-1]$ there exist $C > 0$ and $\theta \in [0,1]$ such that for any $w \in W^{1,p}(G)$, G open bounded set of $\mathbb{R}^N$, we have*

$$||w||_{L^p(\partial\Omega)} \leq C\left(||\nabla w||_{L^p(G)} + ||w||_{L^{\sigma+1}(G)}\right)^\theta \left(||w||_{L^{\sigma+1}(G)}\right)^{1-\theta}.$$

Applying the Lemma and Young inequality we obtain that

$$E^{\gamma} \leq C t^{\frac{1-\theta}{p-1}} \left(\frac{\partial E}{\partial \rho} \right)$$

for some exponent $\gamma \in (0,1)$. This implies the result. ∎

Remark 22. Notice that the result holds without making explicit the boundary conditions. It has a *local* nature.

Remark 23. The first local energy method was due to S.N. Antontsev, in 1981. A rigorous justification of his arguments, containing also several improvements, was made in Díaz and Veron [27].

Remark 24. Other qualitative properties (as the *formation of dead cores*, t*he instantaneous shrinking of the support*, etc) can be proved by this type of local energy arguments. See, e.g., Antontsev, Díaz and Shmarev [5]. Those authors are preparing a book containing many other applications.

Remark 25. For the application of this type of arguments to *higher order equations* see Bernis [14], [15] and their references.

As a global, and final, remark we mention that the finite speed of propagation, the finite extinction time and other qualitative properties can be analyzed for hyperbolic first order equations of the type

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \Phi_i(u) + g(x,u) = f(t,x)$$

see Díaz and Veron [26] and Díaz and Kruhzkov [24].

Acknowledgements

The present article corresponds to a series of lectures that the author prepared for the *Ecole CIMPA-UNSA-UNESCO-BURKINA: Equations d'evolution et applications*, Ouagadougou (Burkina Faso), July 13-31, 1998. The author thanks the organizers of the *Ecole* for their kind invitation which unfortunately was not possible. In spite it, the author received the great privilege of to be represented there by Philippe Benilan (prematurely died on February 2001) who read (and without any doubt, improved) the transparencies prepared by the author to that end.

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