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On the Structure of Ultraproducts of Real Interpolation Spaces

J.A. López Molina[1], M.E. Puerta[2], M.J. Rivera[1] *

[1] *E.T.S. Ingenieros Agrónomos, Universidad Politécnica de Valencia, Camino de Vera,* 46072 *Valencia, Spain*

[2] *Universidad Eafit, Carrera* 49 7*Sur*-50, 3300 *Medellín, Colombia*

e.mail: jalopez@mat.upv.es, mpuerta@sigma.eafit.edu.co, mjriv@mat.upv.es

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1. Introduction

Dacunha-Castelle and Krivine introduced ultraproducts in Banach space theory, inspired in model theory of Logic, in [4] in 1972. Ultraproducts become a powerful tool in many applications (see for instance [6], [11], [8], [5] and [10]). One of the more important questions is to know the structure of the ultraproduct space, searching the permanence properties of the factor spaces when we form the ultraproduct. For instance, it is known that given a family $\{L^p(\Omega_d, \mathcal{M}_d, \mu_d), d \in \mathfrak{D}\}$, $1 \le p < \infty$ and an ultrafilter $\mathcal{U}$ in $\mathfrak{D}$, the ultraproduct $(L^p(\Omega_d))_{\mathcal{U}}$ is isometric to another $L^p(\Omega, \mathcal{M}, \mu)$ space (see [4]).

However, the proof (based on the classical Bonhenblust-Nakano-Kakutani theorem of characterization of abstract L^p-spaces) avoid all connection with factor measure spaces $(\Omega_d, \mathcal{M}_d, \mu_d)$. In order to circumvent this problem, we would can use the canonical ultraproduct measure space $(\Omega_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}}, \mu_{\mathcal{U}})$ of the family $\{\Omega_d, \mathcal{M}_d, \mu_d), d \in \mathfrak{D}\}$ (see [11] for details), but unfortunately the resulting space $L^p(\Omega_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}}, \mu_{\mathcal{U}})$ does not represent the full ultraproduct space $(L^p(\Omega_d))_{\mathcal{U}}$. However, in several applications the known information about $L^p(\Omega_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}}, \mu_{\mathcal{U}})$ and its relation with factor $L^p(\Omega_d, \mathcal{M}_d, \mu_d)$ spaces is very fruitful (see for instance [7]).

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In this way, in order to study the permanence properties of some type of factor function spaces in ultraproducts of them, two problems naturally arise: the first one is the global structure of the ultraproduct. The second one is the connection between this ultraproduct and the space of the same type than the factor spaces but defined over $(\Omega_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}}, \mu_{\mathcal{U}})$. There are many known results about this second question. For instante, if we start with families $\{L^p(\Omega_d, \mathcal{M}_d, \mu_d), d \in \mathfrak{D}\}$, $1 \leq p < \infty$, the space $L^p(\Omega_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}}, \mu_{\mathcal{U}})$ is complemented in $(L^p(\Omega_d))_{\mathcal{U}}$ (see [4]). Moreover, given a set $\{L^\varphi(\Omega_d, \mathcal{M}_d, \mu_d), d \in \mathfrak{D}\}$ of Orlicz spaces such that φ has the Δ_2 condition, the Orlicz space $L^\varphi(\Omega_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}}, \mu_{\mathcal{U}})$ is also complemented in $(L^\varphi(\Omega_d))_{\mathcal{U}}$ (see [6]).

In this paper *we study these problems for ultraproducts of interpolation spaces (defined by the real interpolation method) between L^p spaces.* Perhaps our results are well known by specialists, but we have not been able to find a written explicit reference containing complete proofs. For instance, our results in the case of order continuous spaces $(1 \leq q < \infty)$ can be deduced without using interpolation theory in a more or less direct way from some general results of Weis [12], but these ones do not cover the case of non order continuous interpolation spaces (case $q = \infty$). On the other hand, the implementation of methods *ad hoc* for this case implies almost the same work than we need for study the general case. Hence we think it can be useful to have in the bibliography a complete direct explicit proof for all cases.

We prove two main results:

1) Given a family of real interpolation spaces $\{(\ell^{p_0}(\Omega_d), \ell^{p_1}(\Omega_d))_{\theta,q}, d \in \mathfrak{D}\}$ of discrete spaces $\ell^{p_j}(\Omega_d), j = 0, 1$ such that every atom has measure 1, $1 < p_0 < p_1 < \infty$, $1 \leq q < \infty$, then the interpolation space $(\ell^{p_0}(\Omega_{\mathcal{U}}), \ell^{p_1}(\Omega_{\mathcal{U}}))_{\theta,q}$ is isomorphic to a complemented subspace of $((\ell^{p_0}(\Omega_d), \ell^{p_1}(\Omega_d))_{\theta,q})_{\mathcal{U}}$.

2) Our second result is the continuous version of 1). Given a family $\{(L^{p_0}(\Omega_d, \mu_d), L^{p_1}(\Omega_d, \mu_d))_{\theta,q}, d \in \mathfrak{D}\}$ where every (Ω_d, μ_d) is a probability space, then the interpolation space $(L^{p_0}(\Omega_{\mathcal{U}}), L^{p_1}(\Omega_{\mathcal{U}}))_{\theta,q}$ is isomorphic to a complemented subspace of $((L^{p_0}(\Omega_d), L^{p_1}(\Omega_d))_{\theta,q})_{\mathcal{U}}$.

We also study the corresponding version of these results in the extreme case $q = \infty$.

Not defined notation is standard. $\ell^p(\Gamma)$, $1 \leq p \leq \infty$ denotes the Lebesgue space L^p defined on a purely atomic measure space (Γ, μ) such that $\mu(x) = 1$ for every atom in Γ. Elements in $\ell^p(\Gamma)$ are denoted by $(x_i)_{i \in \Gamma}$ or (x_i) for short. In such spaces, $\mathbf{e}_k$, $k \in \Gamma$, denotes the family $(x_i)_{i \in \Gamma}$ such that $x_i = 0$ if $i \neq k$ and $x_k = 1$.

We refer the reader to [1] and [3] for the theory of interpolation spaces by the real method. It is known there are several equivalent norms in such spaces. We only explain the used ones on this paper. Given a couple (A_0, A_1) of Banach spaces continuouly embedded in some Hausdorff topological vector space E, we consider on the sum space $A_0 + A_1$ the canonical norm

$$\|x\|_{A_0+A_1} = \inf \{\|a_0\|_{A_0} + \|a_1\|_{A_1} \mid x = a_0 + a_1, a_0 \in A_0, a_1 \in A_1\}.$$

Then, given $0 < \theta < 1$ and $1 \leq q < \infty$, the interpolation space $(A_0, A_1)_{\theta,q}$ is the set of vectors $x \in A_0+A_1$ for which there are sequences $\{x_h\}_{h\in\mathbb{Z}} \subset A_0\cap A_1$ such that $x = \sum_{h\in\mathbb{Z}} x_h$ in the topology of $A_0 + A_1$ and moreover

$$\|x\|_{(A_0,A_1)_{\theta,q}} := \inf \left\{ \max_{j=0,1} \left(\sum_{h\in\mathbb{Z}} e^{(j-\theta)hq} \|x_h\|^q_{A_j} \right)^{\frac{1}{q}}, \quad x = \sum_{h\in\mathbb{Z}} x_h \right\} < \infty,$$

endowed with the norm defined by the function $\|x\|_{(A_0,A_1)_{\theta,q}}$. If $q = \infty$ the definition is analogous but changing the symbol $\sum_{h\in\mathbb{Z}}$ by $\sup_{h\in\mathbb{Z}}$.

The interpolation spaces $(L^{p_0}(\Omega, \mathcal{M}, \mu), L^{p_1}(\Omega, \mathcal{M}, \mu))_{\theta,q}$ have an alternative representation as Lorentz spaces, which we shall use as an auxiliary technical tool. Lorentz spaces are defined in the following way:

Let $(\Omega, \mathcal{M}, \mu)$ be a σ-finite measure space. Let $L^0(\Omega, \mathcal{M}, \mu)$ be the set of $\mathcal{M}$-measurable real or complex functions defined in Ω and finite μ-almost everywhere. The distribution function μ_f of $f \in L^0(\Omega, \mathcal{M}, \mu)$ is defined by

$$\mu_f(\lambda) := \mu \{x \in \Omega \mid |f(x)| > \lambda\}, \ (\lambda \geq 0).$$

and the decreasing rearrangement of f is the function $f^*_\mu : [0, \infty[\longrightarrow [0, \infty]$ defined by the rule

$$f^*_\mu(t) := \inf \{\lambda \geq 0 \mid \mu_f(\lambda) \leq t\}, \ (t \geq 0).$$

In particular, if $f = \sum_{i=1}^n \alpha_i \chi_{A_i}$, and the sets $\{A_i, \ i = 1, 2, ..., n\}$ are pairwise disjoint, the decreasing rearrangement f^* can be computed as follows: Let $\sigma : \{1, 2, ..., n\} \rightarrow \{1, 2, ..., n\}$ be the inyective map defined by

$$|\alpha_{\sigma(1)}| = \max \{|\alpha_k| \mid 1 \leq k \leq n\},$$

and for $i > 1$,

$$|\alpha_{\sigma(i)}| = \max \{|\alpha_k| \mid k \notin \{\sigma(1), \sigma(2), ..., \sigma(i-1)\}\}.$$

Then

$$f^*(t) = |\alpha_{\sigma(1)}| \ \text{ if } \ t \in [0, \mu(A_{\sigma(1)})[\tag{1}$$

and

$$\forall i > 1, \quad f^*(t) = |\alpha_{\sigma(i)}| \quad \text{if} \ t \in \left[\sum_{j=1}^{i-1} \mu(A_{\sigma(j)}), \sum_{j=1}^{i} \mu(A_{\sigma(j)})\right]. \tag{2}$$

Consider now real numbers $1 < p < \infty, 1 \leq q < \infty$. The Lorentz space $L^{p,q}(\Omega, \mathcal{M}, \mu)$ is the set of functions $f \in L^0(\Omega, \mathcal{M}, \mu)$ such that

$$\| f\|_{\mu,p,q} := \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{dt}{t}\right)^{1/q} < \infty. \tag{3}$$

If $q = \infty$, the Lorentz space $L^{p,\infty}(\Omega, \mathcal{M}, \mu)$ is defined as the set of functions $f \in L^0(\Omega, \mathcal{M}, \mu)$ such that

$$\| f\|_{\mu,p,\infty} := \sup_{t>0} t^{\frac{1}{p}} f^*(t) < \infty \tag{4}$$

It can be shown that $\|.\|_{\mu,p,q}$ is a norm when $1 \leq q \leq p$ and a quasi norm in the remaining cases. When $\| f\|_{\mu,p,q}$ is not a norm it can be replaced by another function which holds to be a norm and defines in $L^{p,q}(\Omega, \mathcal{M}, \mu)$ the same topology as the initial quasi norm (see for instance chapter 4, theorem 4.3 in [2]). For our pourposes, it will be enough to compute (3) in some instances. Hence, we refer the interested reader to the quoted book for more detailed explanations.

Then we have

Theorem 1. (Krée, see [9] or [3], theorem 5.2.1) *If $1 \leq q \leq \infty$, the identity map makes $(L^{p_0}(\Omega, \mathcal{M}, \mu), L^{p_1}(\Omega, \mathcal{M}, \mu))_{\theta,q}$ isomorphic to Lorentz space $L^{\overline{p}q}(\Omega, \mathcal{M}, \mu)$, where $\overline{p}$ is defined by the equation $\frac{1}{\overline{p}} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Moreover the equivalence constants between the corresponding norms are independent of (Ω, μ).*

From now on let $\mathfrak{D}$ be a non void index set. Let $0 < \theta < 1$ and $1 \leq q \leq \infty$ be fixed numbers. Assume we have a measure space $(\Omega_d, \mathcal{M}_d, \mu_d)$ for every $d \in \mathfrak{D}$. We are concerned with ultraproducts of interpolation spaces of type $(L^{p_0}(\Omega_d), L^{p_1}(\Omega_d))_{\theta,q}$. By reiteration theorem *we can suppose without lost of generality* $1 < p_0 < p_1 < \infty$.

Let $\mathcal{U}$ be an ultrafilter on $\mathfrak{D}$. All basic material concerning ultraproducts of Banach spaces can be found in [6]. The ultraproduct of the family $\{A_d, \ d \in \mathfrak{D}\}$ of Banach spaces along the ultrafilter $\mathcal{U}$ will be denoted by $(A_d)_{\mathcal{U}}$, and $(x_d)_{\mathcal{U}}$ will be the element in $(A_d)_{\mathcal{U}}$ having $(x_d)_{d\in\mathfrak{D}} \in \Pi_{d\in\mathfrak{D}} A_d$ as a representant.

2. Discrete case

In this Section for every $d \in \mathfrak{D}$ we consider a *discrete* measure space (Ω_d, μ_d) such that $\mu_d(x) = 1$ for every atom $x \in \Omega_d$ and the interpolation space $(\ell^{p_0}(\Omega_d), \ell^{p_1}(\Omega_d))_{\theta,q}$.

The set theoretic ultraproduct of the family $\{\Omega_d, d \in \mathfrak{D}\}$ is the quotient set

$$\mathbf{K} =: ((\Omega_d))_{\mathcal{U}} =: (\Pi_{d\in\mathfrak{D}}\Omega_d)/\mathcal{R}$$

modulo the equivalence

$$\forall (x_d), (y_d) \in \Pi_{d\in\mathfrak{D}}\Omega_d \quad (x_d)\mathcal{R}(y_d) \Longleftrightarrow \{d \in \mathfrak{D} \mid x_d = y_d\} \in \mathcal{U}. \tag{5}$$

The class in $\mathbf{K}$ of an element $(k_d)_{d\in\mathfrak{D}} \in \Pi_{d\in\mathfrak{D}}\Omega_d$ will be denoted by the symbol $(k_d)_{\mathcal{U}}$. If it is not necessary to specify a representant of an element in $\mathbf{K}$, this element will be denoted simply by bold letters $\mathbf{k}$.

As an immediate consequence of this definition and the finite intersection property of members of filters we get next fact which will be freely used without explicit reference: given a *finite* set $\{\mathbf{k}^j\}_{j=1}^n \subset \mathbf{K}$ of different elements, $\mathbf{k}^j = (k_d^j)_{\mathcal{U}}$, $j = 1, 2, ..., n$, there is a set $D \in \mathcal{U}$ such that

$$\forall\, j, h = 1, 2, ..., n \ \ j \neq h \implies \forall\, d \in D, \quad k_d^j \neq k_d^h.$$

Lemma 2. 1) *Let* $1 \le q < \infty$. *The linear mapping*

$$J_{p_0p_1}^q : (\ell^{p_0}(\mathbf{K}), \ell^{p_1}(\mathbf{K}))_{\theta,q} \longrightarrow ((\ell^{p_0}(\Omega_d), \ell^{p_1}(\Omega_d))_{\theta,q})_{\mathcal{U}},$$

defined by

$$\forall\, (\eta_{\mathbf{k}})_{\mathbf{k}\in\mathbf{K}} \in (\ell^{p_0}(\mathbf{K}), \ell^{p_1}(\mathbf{K}))_{\theta,q} \quad J_{p_0p_1}^q((\eta_{\mathbf{k}})) = \sum_{\mathbf{k}\in\mathbf{K}} (\eta_{\mathbf{k}}\mathbf{e}_{k_d})_{\mathcal{U}} \tag{6}$$

is continuous.

2) *Let* $X_{p_0p_1}^\infty$ *be the closure of* $\ell^{p_0}(\mathbf{K})$ *in* $(\ell^{p_0}(\mathbf{K}), \ell^{p_1}(\mathbf{K}))_{\theta,\infty}$. *The linear map*

$$J_{p_0p_1}^\infty : X_{p_0p_1}^\infty \longrightarrow ((\ell^{p_0}(\Omega_d), \ell^{p_1}(\Omega_d))_{\theta,\infty})_{\mathcal{U}}$$

defined by (6) *is continuous.*

Proof. 1) First suppose $(\eta_{\mathbf{k}}) \in \ell^{p_0}(\mathbf{K}) = \ell^{p_0}(\mathbf{K}) \cap \ell^{p_1}(\mathbf{K})$. There is a sequence $\{\mathbf{k}^j\}_{j=1}^\infty \subset \mathbf{K}$ such that $\eta_{\mathbf{k}} = 0$ if $\mathbf{k} \neq \mathbf{k}^j$ for every $j \in \mathbb{N}$ and

$$(\eta_{\mathbf{k}}) = \sum_{j=1}^\infty \eta_{\mathbf{k}^j}\mathbf{e}_{\mathbf{k}^j}$$

in $(\ell^{p_0}(\mathbf{K}), \ell^{p_1}(\mathbf{K}))_{\theta,q}$. The inclusion map $\ell^{p_0}(\mathbf{K}) \subset (\ell^{p_0}(\mathbf{K}), \ell^{p_1}(\mathbf{K}))_{\theta,q}$ is continuous. Hence, given $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that

$$\forall\, m \geq n \geq n_0 \quad \left\| \sum_{j=n}^{m} \eta_{\mathbf{k}^j} \mathbf{e}_{\mathbf{k}^j} \right\|_{(\ell^{p_0}(\mathbf{K}), \ell^{p_1}(\mathbf{K}))_{\theta,q}} \leq \left\| \sum_{j=n}^{m} \eta_{\mathbf{k}^j} \mathbf{e}_{\mathbf{k}^j} \right\|_{\ell^{p_0}(\mathbf{K})} \leq \varepsilon. \quad (7)$$

By definition of the norm in an interpolated space there is a representation

$$\sum_{j=n}^{m} \eta_{\mathbf{k}^j} \mathbf{e}_{\mathbf{k}^j} = \sum_{h\in\mathbb{Z}} \sum_{j=n}^{m} \eta^h_{\mathbf{k}^j} \mathbf{e}_{\mathbf{k}^j} \quad (8)$$

which converges in $\ell^{p_0}(\mathbf{K}) + \ell^{p_1}(\mathbf{K})$ and

$$\begin{aligned} &\sup_{r=0,1} \left(\sum_{h\in\mathbb{Z}} \left\| e^{(r-\theta)h} \sum_{j=n}^{m} \eta^h_{\mathbf{k}^j} \mathbf{e}_{\mathbf{k}^j} \right\|^q_{\ell^{p_r}(\mathbf{K})} \right)^{\frac{1}{q}} \\ &\qquad \leq \left\| \sum_{j=n}^{m} \eta_{\mathbf{k}^j} \mathbf{e}_{\mathbf{k}^j} \right\|_{(\ell^{p_0}(\mathbf{K}), \ell^{p_1}(\mathbf{K}))_{\theta,q}} + \varepsilon. \end{aligned} \quad (9)$$

In particular, (8) implies the convergence of every numerical series

$$\sum_{h\in\mathbb{Z}} \eta^h_{\mathbf{k}^j} = \eta_{\mathbf{k}^j}, \quad n \leq j \leq m. \quad (10)$$

By definition of $\mathbf{K}$ there exists a set $D_\varepsilon \in \mathcal{U}$ such that

$$\forall d \in D_\varepsilon \ \ \forall\, i \neq j, \quad i,j = n, n+1, ..., m \qquad \Longrightarrow \ k^i_d \neq k^j_d.$$

Let us see that for every $d \in D_\varepsilon$ we have

$$\sum_{j=n}^{m} \eta_{\mathbf{k}^j} \mathbf{e}_{k^j_d} = \sum_{h\in\mathbb{Z}} \left(\sum_{j=n}^{m} \eta^h_{\mathbf{k}^j} \mathbf{e}_{k^j_d} \right) \quad (11)$$

in $\ell^{p_0}(\Omega_d) + \ell^{p_1}(\Omega_d)$. In fact, using (8), given $\delta > 0$ there is $r_0 \in \mathbb{N}$ such that for every $r \in \mathbb{Z}, |r| \geq r_0$, we can find $(\alpha_{\mathbf{k}}) \in \ell^{p_0}(\mathbf{K})$ and $(\beta_{\mathbf{k}}) \in \ell^{p_1}(\mathbf{K})$ in such a way that

$$\sum_{|h|\geq|r|} \sum_{j=n}^{m} \eta^h_{\mathbf{k}^j} \mathbf{e}_{\mathbf{k}^j} = \sum_{j=n}^{m} \alpha_{\mathbf{k}^j} \mathbf{e}_{\mathbf{k}^j} + \sum_{j=n}^{m} \beta_{\mathbf{k}^j} \mathbf{e}_{\mathbf{k}^j}$$

and

$$\left(\sum_{j=n}^{m}|\alpha_{\mathbf{k}^j}|^{p_0}\right)^{\frac{1}{p_0}}+\left(\sum_{j=n}^{m}|\beta_{\mathbf{k}^j}|^{p_1}\right)^{\frac{1}{p_1}}\leq\delta. \tag{12}$$

From (12) and (10) we obtain for every $d\in D_\varepsilon$

$$\forall r\in\mathbb{Z},\ |r|\geq r_0,\qquad \sum_{|h|\geq|r|}\sum_{j=n}^{m}\eta^h_{\mathbf{k}^j}\mathbf{e}_{k^j_d}=\sum_{j=n}^{m}\alpha_{\mathbf{k}^j}\mathbf{e}_{k^j_d}+\sum_{j=n}^{m}\beta_{\mathbf{k}^j}\mathbf{e}_{k^j_d}$$

and

$$\left\|\sum_{|h|\geq|r|}\left(\sum_{j=n}^{m}\eta^h_{\mathbf{k}^j}\mathbf{e}_{k^j_d}\right)\right\|_{\ell^{p_0}(\Omega_d)+\ell^{p_1}(\Omega_d)}\leq\left(\sum_{j=n}^{m}|\alpha_{\mathbf{k}^j}|^{p_0}\right)^{\frac{1}{p_0}}+\left(\sum_{j=n}^{m}|\beta_{\mathbf{k}^j}|^{p_1}\right)^{\frac{1}{p_1}}\leq\delta.$$

As a consequence, by definition of each norm in the respective space, by (11), (9) and (7) we have

$$\begin{aligned}\left\|\sum_{j=n}^{m}(\eta_{\mathbf{k}^j}\mathbf{e}_{k^j_d})_{\mathcal{U}}\right\|&=\lim_{d,\mathcal{U}}\left\|\sum_{j=n}^{m}\eta_{\mathbf{k}^j}\mathbf{e}_{k^j_d}\right\|_{(\ell^{p_0}(\Omega_d),\ell^{p_1}(\Omega_d))_{\theta,q}}\\&\leq\lim_{d,\mathcal{U}}\sup_{r=0,1}\left(\sum_{h\in\mathbb{Z}}e^{(r-\theta)hq}\left(\sum_{j=n}^{m}|\eta^h_{\mathbf{k}^j}|^{p_r}\right)^{\frac{q}{p_r}}\right)^{\frac{1}{q}}\\&=\lim_{d,\mathcal{U}}\sup_{r=0,1}\left(\sum_{h\in\mathbb{Z}}e^{(r-\theta)hq}\left\|\sum_{j=n}^{m}\eta^h_{\mathbf{k}^j}\mathbf{e}_{\mathbf{k}^j}\right\|^q_{\ell^{p_r}(\mathbf{K})}\right)^{\frac{1}{q}}\\&\leq\left\|\sum_{j=n}^{m}\eta_{\mathbf{k}^j}\mathbf{e}_{\mathbf{k}^j}\right\|_{(\ell^{p_0}(\mathbf{K}),\ell^{p_1}(\mathbf{K}))_{\theta,q}}+\varepsilon\leq2\varepsilon.\end{aligned}$$

Hence

$$\sum_{j=1}^{\infty}\left(\eta_{\mathbf{k}^j}\mathbf{e}_{k^j_d}\right)_{\mathcal{U}} \tag{13}$$

must be a convergent series in $((\ell^{p_0}(\Omega_d),\ell^{p_1}(\Omega_d))_{\theta,q})_{\mathcal{U}}$ and $J^q_{p_0p_1}$ is well defined in $\ell^{p_0}(\mathbf{K})$. Using the convergence of (13), an argumentation similar to the previous one shows that for every $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$ such that

$$\forall m \geq n_\varepsilon \quad \left\| \sum_{j=1}^{m} \left(\eta_{\mathbf{k}^j} \mathbf{e}_{k_d^j} \right)_{\mathcal{U}} \right\| \leq \left\| \sum_{j=1}^{m} \left(\eta_{\mathbf{k}^j} \mathbf{e}_{k_d^j} \right)_{\mathcal{U}} \right\| + \varepsilon$$

and as a consequence

$$\left\| \sum_{j=1}^{\infty} \left(\eta_{\mathbf{k}^j} \mathbf{e}_{k_d^j} \right)_{\mathcal{U}} \right\| \leq \lim_{m\to\infty} \left(\left\| \sum_{j=1}^{m} \eta_{\mathbf{k}^j} \mathbf{e}_{\mathbf{k}^j} \right\|_{(\ell^{p_0}(\mathbf{K}), \ell^{p_1}(\mathbf{K}))_{\theta,q}} + 2\varepsilon \right)$$

$$= \left\| \sum_{j=1}^{\infty} \eta_{\mathbf{k}^j} \mathbf{e}_{\mathbf{k}^j} \right\|_{(\ell^{p_0}(\mathbf{K}), \ell^{p_1}(\mathbf{K}))_{\theta,q}} + 2\varepsilon = \|(\eta_{\mathbf{k}})\|_{(\ell^{p_0}(\mathbf{K}), \ell^{p_1}(\mathbf{K}))_{\theta,q}} + 2\varepsilon$$

and hence

$$\left\| J^q_{p_0p_1}((\eta_{\mathbf{k}}) \right\| \leq \|(\eta_{\mathbf{k}})\|_{(\ell^{p_0}(\mathbf{K}), \ell^{p_1}(\mathbf{K}))_{\theta,q}} .$$

and we obtain the continuity of the restriction to the subspace $\ell^{p_0}(\mathbf{K})$ of $(\ell^{p_0}(\mathbf{K}), \ell^{p_1}(\mathbf{K}))_{\theta,q}$ of $J^q_{p_0p_1}$. As $\ell^{p_0}(\mathbf{K})$ is dense $(\ell^{p_0}(\mathbf{K}), \ell^{p_1}(\mathbf{K}))_{\theta,q}$ (since $q < \infty$, see for instance proposition 3, chapter 2 in [1]), $J^q_{p_0p_1}$ can be continuously extended to the whole space.

2) The proof is analogous because $X^\infty_{p_0p_1} \subset \ell^{p_1}(\mathbf{K})$ and every element in $X^\infty_{p_0p_1}$ is the limit of its sections in the topology of the space. ∎

Lemma 3. *Let $1 \leq q \leq \infty$. The linear mapping*

$$Q^q_{p_0p_1} : ((\ell^{p_0}(\Omega_d), \ell^{p_1}(\Omega_d))_{\theta,q})_{\mathcal{U}} \longrightarrow (\ell^{p_0}(\mathbf{K}), \ell^{p_1}(\mathbf{K}))_{\theta,q}$$

defined by

$$\forall ((x_i^d))_{\mathcal{U}} \in ((\ell^{p_0}(\Omega_d), \ell^{p_1}(\Omega_d))_{\theta,q})_{\mathcal{U}} \quad Q^q_{p_0p_1}(((x_i^d))_{\mathcal{U}}) = \sum_{\mathbf{k}=(k_d)_{\mathcal{U}} \in \mathbf{K}} (\lim_{d,\mathcal{U}} x^d_{k_d}) \mathbf{e}_{\mathbf{k}}$$

is continuous and surjective.

Proof. 1) First suppose $1 < q \leq \infty$. By theorem 3.7.1 in [3] and by [6] we have the isometric inclusion

$$((\ell^{p_0}(\Omega_d), \ell^{p_1}(\Omega_d))_{\theta,q})_{\mathcal{U}} \subset ((\ell^{p'_0}(\Omega_d), \ell^{p'_1}(\Omega_d))_{\theta,q'})'_{\mathcal{U}}.$$

Hence the transposed map $(J^{q'}_{p'_0p'_1})'$ of the map $J^{q'}_{p'_0p'_1}$ (lemma 2), is well defined and continuous. Let $((x^d_i))_{\mathcal{U}} \in ((\ell^{p_0}(\Omega_d), \ell^{p_1}(\Omega_d))_{\theta,q})_{\mathcal{U}}$ and put $(\eta_{\mathbf{k}})_{\mathbf{k}\in\mathbf{K}} := (J^{q'}_{p'_0p'_1})'(((x^d_i))_{\mathcal{U}})$. Then, for every $\mathbf{k} = (k_d)_{\mathcal{U}} \in \mathbf{K}$ we have

$$\eta_{\mathbf{k}} = \langle (J^{q'}_{p'_0p'_1})'(((x^d_i))_{\mathcal{U}}), \mathbf{e}_{\mathbf{k}}\rangle = \langle (((x^d_i))_{\mathcal{U}}), (\mathbf{e}_{k_d})_{\mathcal{U}}\rangle = \lim_{d,\mathcal{U}} \langle (x^d_i), \mathbf{e}_{k_d}\rangle = \lim_{d,\mathcal{U}} x^d_{k_d}$$

and $Q^q_{p_0p_1}$ turns out to be the restriction to $((\ell^{p_0}(\Omega_d), (\ell^{p_1}(\Omega_d))_{\theta,q})_{\mathcal{U}}$ of $(J^{q'}_{p'_0p'_1})'_q$. Hence $Q^q_{p_0p_1}$ is continuous.

To finish, we check that $Q^q_{p_0p_1}$ is surjective. Given $(\eta_{\mathbf{k}}) \in (\ell^{p_0}(\mathbf{K}), (\ell^{p_1}(\mathbf{K}))_{\theta,q}$, there is a sequence of different elements $\{\mathbf{k}^n\}_{n=1}^{\infty} \subset \mathbf{K}$ such that $(\eta_{\mathbf{k}}) = \sum_{n=1}^{\infty} \eta_{\mathbf{k}^n} \mathbf{e}_{\mathbf{k}^n}$. Since

$$Q^q_{p_0p_1}\left(J^q_{p_0p_1}\left(\sum_{n=1}^{\infty} \eta_{\mathbf{k}^n}\mathbf{e}_{\mathbf{k}^n}\right)\right) = Q^q_{p_0p_1}\left(\sum_{n=1}^{\infty} (\eta_{\mathbf{k}^n}\mathbf{e}^d_{k^n_d})_{\mathcal{U}}\right) \tag{14}$$

$$= \sum_{n=1}^{\infty} \eta_{\mathbf{k}^n}\mathbf{e}_{\mathbf{k}^n} = (\eta_{\mathbf{k}}), \tag{15}$$

we get the surjectivity of $Q^q_{p_0p_1}$.

2) Let $q = 1$. For every $d \in \mathfrak{D}$ let $X^d_{p'_0p'_1}$ be the closure of $\ell^{p'_1}(\Omega_d)$ in $(\ell^{p'_0}(\Omega_d), \ell^{p'_1}(\Omega_d))_{\theta,\infty}$. We remark that the proof given in lemma 2.2 implies $J^{\infty}_{p'_0p'_1}(X^{\infty}_{p'_0,p'_1}) \subset (X^d_{p'_0p'_1})_{\mathcal{U}}$ and by continuity, the range of $J^{\infty}_{p'_0p'_1}$ is actually included in $(X^d_{p'_0p'_1})_{\mathcal{U}}$. By theorem 3.7.1 in [3] and the theorem about duality of ultraproducts we get $((\ell^{p_0}(\Omega_d), \ell^{p_1}(\Omega_d))_{\theta,1})_{\mathcal{U}} \subset (X^d_{p'_0p'_1})'_{\mathcal{U}}$. From now on the proof is analogous to that used in case 1). ∎

THEOREM 4. 1) *If* $1 \le q < \infty$, $(\ell^{p_0}(\mathbf{K}), \ell^{p_1}(\mathbf{K}))_{\theta,q}$ *is isomorphic to a complemented subspace of* $((\ell^{p_0}(\Omega_d), \ell^{p_1}(\Omega_d))_{\theta,q})_{\mathcal{U}}$.

2) *The closure of* $\ell^{p_0}(\mathbf{K})$ *in* $(\ell^{p_0}(\mathbf{K}), \ell^{p_1}(\mathbf{K}))_{\theta,\infty}$ *is isomorphic to a complemented subspace of* $((\ell^{p_0}(\Omega_d), \ell^{p_1}(\Omega_d))_{\theta,\infty})_{\mathcal{U}}$.

Proof. 1) Let $1 \le q < \infty$. By lemma 2.1, $J^q_{p_0p_1}$ is continuous in $(\ell^{p_0}(\mathbf{K}), \ell^{p_1}(\mathbf{K}))_{\theta,q}$. By (15) we get

$$\forall (\eta_{\mathbf{k}}) \in (\ell^{p_0}(\mathbf{K}), \ell^{p_1}(\mathbf{K}))_{\theta,q} \quad \|(\eta_{\mathbf{k}})\| \le \|Q^q_{p_0p_1}\| \|J^q_{p_0p_1}((\eta_{\mathbf{k}}))\|$$

and $J^q_{p_0p_1}$ becomes an isomorphism from $(\ell^{p_0}(\mathbf{K}), \ell^{p_1}(\mathbf{K}))_{\theta,q}$ onto its image. Now let $(\eta_{\mathbf{k}}) \in (\ell^{p_0}(\mathbf{K}), \ell^{p_1}(\mathbf{K}))_{\theta,q}$. There is a sequence of different elements $\{\mathbf{k}^j\}_{j=1}^{\infty} \subset \mathbf{K}$ such that $\eta_{\mathbf{k}} = 0$ if $\mathbf{k} \neq \mathbf{k}^j$ for every $j \in \mathbb{N}$ and

$$(\eta_{\mathbf{k}}) = \sum_{j=1}^{\infty} \eta_{\mathbf{k}^j} \mathbf{e}_{\mathbf{k}^j}.$$

Then, by definition of $\mathbf{K}$, by (14) and (15) we have

$$J^q_{p_0p_1} Q^q_{p_0p_1} \left(J^q_{p_0p_1} \left(\sum_{j=1}^{\infty} \eta_{\mathbf{k}^j} \mathbf{e}_{\mathbf{k}^j} \right) \right) = J^q_{p_0p_1} Q^q_{p_0p_1} \left(\sum_{j=1}^{\infty} (\eta_{\mathbf{k}^j} \mathbf{e}_{k^j_d})_{\mathcal{U}} \right)$$

$$= J^q_{p_0p_1} \left(\sum_{j=1}^{\infty} \eta_{\mathbf{k}^j} \mathbf{e}_{\mathbf{k}^j} \right)$$

and hence, having in mind lemma 3, the map $J^q_{p_0p_1} Q^q_{p_0p_1}$ is a continuous projection from $((\ell^{p_0}(\Omega_d), \ell^{p_1}(\Omega_d))_{\theta,q})_{\mathcal{U}}$ onto $J^q_{p_0p_1}((\ell^{p_0}(\mathbf{K}), \ell^{p_1}(\mathbf{K}))_{\theta,q}$.

2) When $q = \infty$ the proof is analogous by lemma 2.2. ∎

3. Continuous case

In order to study our main problem in the "continuous" case we need an additional hypothesis which will be used in the sequel in an essential way. In this Section, for every $d \in \mathfrak{D}$ we consider a measure space $(\Omega_d, \mathcal{M}_d, \mu_d)$ such that μ_d is *a probability measure* i.e. $\mu_d(\Omega_d) = 1$. We also consider the corresponding spaces $L^{p_0}(\Omega_d, \mathcal{M}_d, \mu_d)$ and $L^{p_1}(\Omega_d, \mathcal{M}_d, \mu_d)$ and the interpolated space $\Lambda^q_d := (L^{p_0}(\Omega_d, \mathcal{M}_d, \mu_d), L^{p_1}(\Omega_d, \mathcal{M}_d, \mu_d))_{\theta,q}$. If there is no risk of confusion, these spaces will denoted by $L^{p_0}(\Omega_d)$, $L^{p_1}(\Omega_d)$ and $(L^{p_0}(\Omega_d), L^{p_1}(\Omega_d))_{\theta,q}$ respectively. Our method actually can be applied if $\sup_{d\in\mathfrak{D}} \mu_d(\Omega_d) < \infty$, but, in order to simplify, we shall deal only with the quoted case.

Under the above assumption, for every $d \in \mathfrak{D}$ we have the continuous inclusions

$$I_d : L^{p_1}(\Omega_d) \longrightarrow (L^{p_0}(\Omega_d), L^{p_1}(\Omega_d))_{\theta,q}$$

and

$$J_d : (L^{p_0}(\Omega_d), L^{p_1}(\Omega_d))_{\theta,q} \longrightarrow L^{p_0}(\Omega_d)$$

with $\|I_d\| \leq 1$ and $\|J_d\| \leq 1$.

Let $\mathcal{U}$ be an ultrafilter on $\mathfrak{D}$. In this section we want to study the structure of the ultraproduct $\Lambda^q_{\mathcal{U}} := ((L^{p_0}(\Omega_d), L^{p_1}(\Omega_d))_{\theta,q})_{\mathcal{U}}$. We consider the set theoretic ultraproduct

$$\Omega_{\mathcal{U}} := (\Omega_d)_{\mathcal{U}} = \Pi_{d\in\mathfrak{D}}\Omega_d/\mathcal{R}$$

where $\mathcal{R}$ is the equivalente relation defined in (5). Given a family $\{A_d, d \in \mathfrak{D}\}$ we define

$$(A_d)_{\mathcal{U}} := \{(x_d)_{\mathcal{U}} \in \Omega_{\mathcal{U}} \ \mid \ x_d \in A_d \ \forall \ d \in \mathfrak{D}\}\,.$$

and we consider the family of subsets of $\Omega_{\mathcal{U}}$

$$\mathcal{F} := \{(A_d)_{\mathcal{U}} \ \mid \ A_d \in \mathcal{M}_d \ \forall \, d \in \mathfrak{D}\}$$

and the set function defined in $\mathcal{F}$ by

$$\forall \ (A_d)_{\mathcal{U}} \in \mathcal{F} \quad \mu_{\mathcal{U}}((A_d)_{\mathcal{U}}) := \lim_{d,\mathcal{U}} \ \mu_d(A_d).$$

It is shown in [11] that $\mathcal{F}$ is an algebra of sets and $\mu_{\mathcal{U}}$ is a measure in $\mathcal{F}$. By the standard Carathéodory procedure, $\mu_{\mathcal{U}}$ can be extended to a measure (which will also be denoted by $\mu_{\mathcal{U}}$) defined in the $\sigma-$algebra $\mathcal{M}_{\mathcal{U}}$ generated by $\mathcal{F}$ in $\Omega_{\mathcal{U}}$. Clearly we have $\mu_{\mathcal{U}}(\Omega_{\mathcal{U}}) = 1$.

Let $\mathcal{S}^q_{p_0p_1} \subset (L^{p_0}(\Omega_{\mathcal{U}}), L^{p_1}(\Omega_{\mathcal{U}}))_{\theta,q}$ be the linear space generated by the characteristic functions of the elements in $\mathcal{F}$ endowed with the induced topology by $(L^{p_0}(\Omega_{\mathcal{U}}), L^{p_1}(\Omega_{\mathcal{U}}))_{\theta,q}$.

LEMMA 5. 1) *Let $1 \leq q < \infty$. There is a continuous linear map*

$$\Psi^q_{p_0p_1} : (L^{p_0}(\Omega_{\mathcal{U}}), L^{p_1}(\Omega_{\mathcal{U}}))_{\theta,q} \longrightarrow \Lambda^q_{\mathcal{U}}$$

such that

$$\forall \ f = \sum_{k=1}^{n} \alpha_k \chi_{(A^k_d)_{\mathcal{U}}} \in \mathcal{S}^q_{p_0p_1}, \quad \Psi^q_{p_0p_1}(f) = \sum_{k=1}^{n} \alpha_k \left(\chi_{A^k_d}\right)_{\mathcal{U}} \tag{16}$$

2) *The map $\Psi^\infty_{p_0p_1} : \mathcal{S}^\infty_{p_0p_1} \longrightarrow \Lambda^\infty_{\mathcal{U}}$ can be continuously extended to the closure of $L^{p_1}(\Omega_{\mathcal{U}})$ in $(L^{p_0}(\Omega_{\mathcal{U}}), L^{p_1}(\Omega_{\mathcal{U}}))_{\theta,\infty}$*

Proof. 1) Let $1 \leq q < \infty$. By theorem 1, $(L^{p_0}(\Omega_{\mathcal{U}}), L^{p_1}(\Omega_{\mathcal{U}}))_{\theta,q}$ is isomorphic to the Lorentz space $L^{\overline{p},q}(\Omega_{\mathcal{U}})$, where $1/\overline{p} = (1-\theta)/p_0 + \theta/p_1$. Define

$$\Psi^q_{p_0p_1} : \mathcal{S}^q_{p_0p_1} \longrightarrow \Lambda_{\mathcal{U}}$$

by

$$\forall f := \sum_{k=1}^{n} \alpha_k \chi_{(A_d^k)_{\mathcal{U}}} \in \mathcal{S}^q_{p_0 p_1}, \quad \Psi^q_{p_0 p_1}(f) = \sum_{k=1}^{n} \alpha_k (\chi_{A_d^k})_{\mathcal{U}}.$$

Assume the sets $\{(A_d^k)_{\mathcal{U}}\ \ k = 1, 2, ..., n\}$ are pairwise disjoint. Let $\sigma : \{1, 2, ..., n\} \longrightarrow \{1, 2, ..., n\}$ be a permutation defining the decresing rearrangement of $\{\alpha_k\}_{k=1}^n$. There is $C > 0$ (independent of the involved measure spaces) such that, using (1), (2) and (3)

$$\|\Psi^q_{p_0 p_1}(f)\|_{\Lambda_{\mathcal{U}}} = \lim_{d,\mathcal{U}} \left\| \sum_{k=1}^{n} \alpha_k \chi_{A_d^k} \right\|_{\Lambda_d} \leq C \lim_{d,\mathcal{U}} \left\| \sum_{k=1}^{n} \alpha_k \chi_{A_d^k} \right\|_{L^{\overline{p}q}(\Omega_d)}$$

$$= C \left(\frac{q}{\overline{p}}\right)^{\frac{1}{q}} \lim_{d,\mathcal{U}} \left(\sum_{k=1}^{n} \alpha_{\sigma(k)} \left(\left(\sum_{j \leq k} \mu_d(A_d^{\sigma(j)}) \right)^{\frac{q}{\overline{p}}} - \left(\sum_{j \leq k-1} \mu_d(A_d^{\sigma(j)}) \right)^{\frac{q}{\overline{p}}} \right) \right)^{\frac{1}{q}}$$

$$= C \left(\frac{q}{\overline{p}}\right)^{\frac{1}{q}} \left(\sum_{k=1}^{n} \alpha_{\sigma(k)} \left(\left(\sum_{j \leq k} \mu_{\mathcal{U}}((A^{\sigma(j)})_{\mathcal{U}}) \right)^{\frac{q}{\overline{p}}} - \left(\sum_{j \leq k-1} \mu_{\mathcal{U}}((A^{\sigma(j)})_{\mathcal{U}}) \right)^{\frac{q}{\overline{p}}} \right) \right)^{\frac{1}{q}}$$

$$= C \left(\int_{\Omega_{\mathcal{U}}} t^{\frac{q}{\overline{p}}-1} f^*(t)^q dt \right)^{\frac{1}{q}}.$$

As a consequence, $\Psi^q_{p_0 p_1}$ is continuous from $\mathcal{S}^q_{p_0 p_1}$ into $\Lambda_{\mathcal{U}}$. But $\mathcal{S}^q_{p_0 p_1}$ being dense in $L^{p_1}(\Omega_{\mathcal{U}})$ and this also being dense in $(L^{p_0}(\Omega_{\mathcal{U}}), L^{p_1}(\Omega_{\mathcal{U}}))_{\theta,q} = L^{\overline{p}q}(\Omega_{\mathcal{U}}, \mathcal{M}_{\mathcal{U}}, \mu_{\mathcal{U}})$, $\Psi^q_{p_0 p_1}$ can be continuously extended to a continuous linear map (again denoted by $\Psi^q_{p_0 p_1}$) from $(L^{p_0}(\Omega_{\mathcal{U}}), L^{p_1}(\Omega_{\mathcal{U}}))_{\theta,q}$ into $\Lambda^q_{\mathcal{U}}$.

2) If $q = \infty$ the proof is very similar with exception of the computation of $\|\Psi^\infty_{p_0 p_1}(f)\|_{\Lambda_{\mathcal{U}}}$. There is now $h \leq n$ such that

$$\begin{aligned} \|f\|_{L^{\frac{p}{1-\theta},\infty}(\Omega_{\mathcal{U}})} &= \alpha_h \left(\sum_{j \leq h} \mu_{\mathcal{U}}((A_d^{\sigma(j)})_{\mathcal{U}}) \right)^{\frac{1-\theta}{p}} \\ &= \lim_{d,\mathcal{U}} \ \alpha_h \left(\sum_{j \leq h} \mu_d(A_d^{\sigma(j)}) \right)^{\frac{1-\theta}{p}}. \end{aligned} \tag{17}$$

On the other hand, since $\mu_d,\ d \in \mathfrak{D}$, are probability measures, for every $d \in \mathfrak{D}$ there is $k_d \in \{1, 2, ..., n\}$ such that

$$\left\| \sum_{k=1}^{n} \alpha_k \chi_{A_d^k} \right\|_{L^{\frac{p}{1-\theta},\infty}(\Omega_d)} = \alpha_{k_d} \left(\sum_{j \le k_d} \mu_d(A_d^{\sigma(j)}) \right)^{\frac{1-\theta}{p}} \le n \max_{j \le n} |\alpha_j|.$$

Hence the limit along the ultrafilter $\mathcal{U}$

$$\lim_{d,\mathcal{U}} \ \alpha_{k_d} \ \left(\sum_{j \le k_d} \mu_d(A_d^{\sigma(j)}) \right)^{\frac{1-\theta}{p}}$$

exists and must be equal to (17). Then we obtain

$$\|\Psi_{p_0p_1}^q(f)\|_{\Lambda_\mathcal{U}} = \lim_{d,\mathcal{U}} \left\| \sum_{k=1}^{n} \alpha_k \chi_{A_d^k} \right\|_{\Lambda_d} \le C \lim_{d,\mathcal{U}} \left\| \sum_{k=1}^{n} \alpha_k \chi_{A_d^k} \right\|_{L^{\frac{p}{1-\theta},\infty}(\Omega_d)}$$

$$= \lim_{d,\mathcal{U}} \alpha_{k_d} \left(\sum_{j \le k_d} \mu_d(A_d^{\sigma(j)}) \right)^{\frac{1-\theta}{p}} = \alpha_h \left(\sum_{j \le h} \mu_\mathcal{U}((A_d^{\sigma(j)})_\mathcal{U}) \right)^{\frac{1-\theta}{p}}$$

$$\le C \|f\|_{L^{\frac{p}{1-\theta},\infty}(\Omega_\mathcal{U})}.$$

∎

We arrive at the main theorem of this section.

THEOREM 6. 1) *If* $1 \le q < \infty$, $\Psi_{p_0p_1}^q$ *is an isomorphism from* $(L^{p_0}(\Omega_\mathcal{U}), L^{p_1}(\Omega_\mathcal{U}))_{\theta,q}$ *onto a complemented subspace of* $\Lambda_\mathcal{U}^q$.

2) $\Psi_{p_0p_1}^\infty$ *is an isomorphism from the closure of* $L^{p_1}(\Omega_\mathcal{U})$ *in* $(L^{p_0}(\Omega_\mathcal{U}), L^{p_1}(\Omega_\mathcal{U}))_{\theta,\infty}$ *onto a complemented subspace of* $\Lambda_\mathcal{U}^\infty$.

Proof. 1) First suppose $1 < q < \infty$. By lemma 5, $\Psi_{p_0p_1}^q$ is continuous from $(L^{p_0}(\Omega_\mathcal{U}), L^{p_1}(\Omega_\mathcal{U}))_{\theta,q}$ into $\Lambda_\mathcal{U}^q$. We shall prove that $\Psi_{p_0p_1}^q$ is an isomorphism onto its image by checking there is $M > 0$ such that $\|\Psi_{p_0p_1}^q(f)\| \ge \|f\|$ whatever be $f \in (L^{p_0}(\Omega_\mathcal{U}), L^{p_1}(\Omega_\mathcal{U}))_{\theta,q}$.

We consider the simple function $f = \sum_{k=1}^{n} \alpha_k \chi_{(A_d^k)_\mathcal{U}} \in (L^{p_0}(\Omega_\mathcal{U}), L^{p_1}(\Omega_\mathcal{U}))_{\theta,q}$. By the duality theorem 3.7.1. in [3], there is $g \in (L^{p_0'}(\Omega_\mathcal{U}),$

$L^{p_1'}(\Omega_{\mathcal{U}}))_{\theta,q'} = (L^{p_0}(\Omega_{\mathcal{U}}), L^{p_1}(\Omega_{\mathcal{U}}))'_{\theta,q}$ such that $\|f\| = \langle f, g\rangle$. By density, we find a bounded sequence of simple functions

$$\{S_m\}_{m=1}^{\infty} := \left\{ \sum_{j=1}^{r_m} \beta_{jm} \chi_{(B_d^j)_{\mathcal{U}}} \right\}_{m=1}^{\infty} \subset (L^{p_0'}(\Omega_{\mathcal{U}}), L^{p_1'}(\Omega_{\mathcal{U}}))_{\theta,q'}$$

such that

$$\|f\| = \langle f, g\rangle = \lim_{m\to\infty} \langle f, S_m\rangle. \tag{18}$$

By lemma 5, we have

$$\forall\, m \in \mathbb{N}, \quad \Psi^{q'}_{p_0' p_1'}(S_m) \in ((L^{p_0'}(\Omega_d, \mu_d), L^{p_1'}(\Omega_d, \mu_d))_{\theta,q'})_{\mathcal{U}}$$

and there is $M > 0$ such that $\|\Psi^{q'}_{p_0' p_1'}(S_m)\| \leq M$. Moreover $(L^{p_0'}(\Omega_d), L^{p_1'}(\Omega_d))_{\theta,q'})_{\mathcal{U}}$ is a weak* dense subspace of $(\Lambda^q_{\mathcal{U}})'$ (see [6]). Then, for every $m \in \mathbb{N}$

$$\begin{aligned}
\|\Psi^q_{p_0 p_1}(f)\| = \lim_{d,\mathcal{U}} \left\| \sum_{k=1}^{n} \alpha_k \chi_{A_d^k} \right\|_{\Lambda_d^q} &\geq \lim_{d,\mathcal{U}} M \left| \left\langle \sum_{k=1}^{n} \alpha_h \chi_{A_d^k}, \sum_{j=1}^{r_m} \beta_{jm} \chi_{B_d^j} \right\rangle \right| \\
&= \lim_{d,\mathcal{U}} M \left| \sum_{k=1}^{n} \sum_{j=1}^{r_m} \alpha_k \beta_{jm} \mu_d(A_d^k \cap B_d^j) \right| \\
&= M \left| \sum_{k=1}^{n} \sum_{j=1}^{r_m} \alpha_k \beta_{jm} \lim_{d,\mathcal{U}} \mu_d(A_d^k \cap B_d^j) \right| \qquad (19) \\
&= M \left| \sum_{k=1}^{n} \sum_{j=1}^{r_m} \alpha_k \beta_{jm} \mu_{\mathcal{U}} \left((A_d^k)_{\mathcal{U}} \cap (B_d^j)_{\mathcal{U}} \right) \right| \\
&= M \left| \left\langle \sum_{k=1}^{n} \alpha_k \chi_{(A_d^k)_{\mathcal{U}}}, \sum_{j=1}^{r_m} \beta_{jm} \chi_{(B_d^j)_{\mathcal{U}}} \right\rangle \right| \qquad (20) \\
&= M |\langle f, S_m\rangle| \qquad (21)
\end{aligned}$$

and by letting $m \to \infty$, we get by (18)

$$\|\Psi(f)\| \geq M\|f\|.$$

Finally, by density of $\mathcal{S}^q_{p_0p_1}$, we obtain the desired inequality for every $f \in (L^{p_0}(\Omega_\mathcal{U}), L^{p_1}(\Omega_\mathcal{U}))_{\theta,q}$.

To finish let us see that $\Psi^q_{p_0p_1}((L^{p_0}(\Omega_\mathcal{U}), L^{p_1}(\Omega_\mathcal{U}))_{\theta,q})$ is complemented in $\Lambda^q_\mathcal{U}$. Since $\Lambda^q_\mathcal{U}$ is a subspace of $(L^{p'_0}(\Omega_d), L^{p'_1}(\Omega_d))'_{\theta,q'}$ (see [6]) we can consider the restriction S to $\Lambda^q_\mathcal{U}$ of the adjoint map $(\Psi^{q'}_{p'_0p'_1})'$ of $\Psi^{q'}_{p'_0p'_1}$. We check that the composition $\Psi^q_{p_0p_1}S$ is a projection from $\Lambda^q_\mathcal{U}$ onto $\Psi^q_{p_0p_1}((L^{p_0}(\Omega_\mathcal{U}), L^{p_1}(\Omega_\mathcal{U})_{\theta,q})$. It is enough to see that

$$\forall\ f \in (L^{p_0}(\Omega_\mathcal{U}), L^{p_1}(\Omega_\mathcal{U})_{\theta,q}, \quad \Psi^q_{p_0p_1}S(\Psi^q_{p_0p_1}(f)) = \Psi^q_{p_0p_1}(f).$$

Once again by density, it suffices to see that

$$\forall\ f = \sum_{i=1}^n \alpha_i \chi_{(A^i_d)_\mathcal{U}} \in \mathcal{S}^q_{p_0p_1}(\Omega_\mathcal{U}), \quad S\Psi^q_{p_0p_1}(f) = f.$$

If $g = \sum_{j=1}^r \beta_j \chi_{(B^j_d)_\mathcal{U}} \in \mathcal{S}^{q'}_{p'_0,p'_1}(\Omega_\mathcal{U})$ we have

$$\langle S\Psi^q_{p_0p_1}(f), g\rangle = \langle \Psi^q_{p_0p_1}(f), \Psi^{q'}_{p'_0p'_1}(g)\rangle$$
$$= \left\langle \sum_{i=1}^n \alpha_i \left(\chi_{(A^i_d)_\mathcal{U}}\right), \sum_{j=1}^r \beta_j \left(\chi_{(B^j_d)_\mathcal{U}}\right) \right\rangle = \langle f, g\rangle$$

(with the same computation that (19), (20) and (21)). By density of $\mathcal{S}^{q'}_{p'_0p'_1}(\Omega_\mathcal{U})$ we obtain $S\Psi^q_{p_0p_1}(f) = f$.

2) Let now $q = \infty$. If X_∞ denotes the closure of $L^{p_1}(\Omega_\mathcal{U})$ in $(L^{p_0}(\Omega_\mathcal{U}), L^{p_1}(\Omega_\mathcal{U}))_{\theta,\infty}$, we have that $\mathcal{S}^\infty_{p_0p_1}$ is dense in X_∞. By lemma 5.2 $\Psi^\infty_{p_0p_1} : X_\infty \longrightarrow \Lambda^\infty_\mathcal{U}$ is continuous. By theorem 3.7.1 in [3], the topological dual X'_∞ is isomorphic to $(L^{p'_0}(\Omega_\mathcal{U}), L^{p'_1}(\Omega_\mathcal{U}))_{\theta,1}$. This theorem with Heinrich's duality theorem ([6]) also yields that $\Lambda^\infty_\mathcal{U}$ is isomorphic to a subspace of $((L^{p'_0}(\Omega_d), L^{p'_1}(\Omega_d))_{\theta,1})'_\mathcal{U}$. Then the proof given in part 1) can be repeated.

3) Finally, let $q = 1$. For every $d \in \mathfrak{D}$, let Y^∞_d be the closure of $L^{p'_0}(\Omega_d)$ in $(L^{p'_0}(\Omega_d), L^{p'_1}(\Omega_d))_{\theta,\infty}$. It is clear that $\Psi^\infty_{p'_0p'_1}(\mathcal{S}^\infty_{p'_0p'_1}) \subset (Y^\infty_d)_\mathcal{U}$ and hence the range of the map

$$\Psi^\infty_{p'_0p'_1} : (L^{p'_0}(\Omega_\mathcal{U}), L^{p'_1}(\Omega_\mathcal{U}))_{\theta,\infty} \longrightarrow \Lambda^\infty_\mathcal{U}$$

is a subset of $(Y^\infty_d)_\mathcal{U}$. Once again by theorem 3.7.1 in [3] and the theorem on duality of ultraproducts, $\Lambda^1_\mathcal{U}$ is isomorphic to a subspace of $(Y^\infty_d)'_\mathcal{U}$ and it can be made a similar argumentation to that given in part 1). ∎

Remark. Of course, by theorem 1 our theorems 4 and 6 can be reformulated in terms of Lorentz spaces, but we omit the details.

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