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Exhaustively Axiomatizing $S3^o_{\rightarrow}$ and $S4^o_{\rightarrow}$

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RESUMEN

$S3^o_{\rightarrow}$ y $S4^o_{\rightarrow}$ son las restricciones con la *Conversa de la Propiedad Ackermann* de los fragmentos implicativos de las lógicas $S3$ ($S3_{\rightarrow}$) y $S4$ ($S4_{\rightarrow}$) de Lewis, respectivamente. El objetivo de este artículo es proporcionar todas las axiomatizaciones posibles con axiomas independientes de $S3^o_{\rightarrow}$ y $S4^o_{\rightarrow}$ que pueden formularse con una modificación de la "lista fuerte y natural de implicaciones válidas" de Anderson y Belnap.

ABSTRACT

$S3^o_{\rightarrow}$ and $S4^o_{\rightarrow}$ are the restrictions with the *Converse Ackermann Property* of the implicative fragments of Lewis' $S3$ ($S3_{\rightarrow}$) and $S4$ ($S4_{\rightarrow}$) respectively. The aim of this paper is to provide all possible axiomatizations with independent axioms of $S3^o_{\rightarrow}$ and $S4^o_{\rightarrow}$ that can be formulated with a modification of Anderson and Belnap's "strong and natural list of valid entailments".

I. INTRODUCTION

An implicative logic L has the *Converse Ackermann Property* (CAP) if $(A\rightarrow B)\rightarrow C$ is unprovable in L whenever C is a propositional variable. The CAP can intuitively be interpreted as the non-derivability of non-necessitive propositions from necessitive ones (A is *necessitive* iff A is of the form $\Box B$).

The question about which systems do possess the CAP is first proposed in Anderson and Belnap (1975), §8.12. In Méndez (1987) it is answered for implicative and positive logics, in Méndez (1988) for logics with a so-called semi-classical negation, and in Kamide (2002) for logics with a strong negation. Syntactically speaking, logics with the CAP are defined by restricting *contraction*

$$(A\rightarrow (A\rightarrow B))\rightarrow(A\rightarrow B)$$

and *assertion*

$$A\to((A\to B)\to B),$$

to the case in which B is an implicative formula (A is implicative iff A is of the form $B\to C$). Thus, logics with the CAP are contractionless logics. Two interesting logics of this type are $S3^o_\to$, and $S4^o_\to$. These ones can intuitively be understood as the restrictions with the CAP of $S3_\to$ and $S4_\to$ respectively (i.e., the implicative fragments of Lewis' S3 and S4 as they were axiomatized by Hacking [see Hacking (1963) and Méndez (1988)]. This paper is a sequel to Méndez (1987), Méndez (1988), Salto, Robles and Méndez (1999), Salto, Robles and Méndez (2001) and especially, Robles and Méndez (2002). We exhaustively axiomatize $S3^o_\to$ and $S4^o_\to$ with a modification of Anderson and Belnap's "strong and natural list of valid entailments".

II. List of Characteristic Theses

Anderson and Belnap's list is the following (see Anderson and Belnap (1975), §8.15)

1. $A\to A$
2. $(A\to B)\to((B\to C)\to(A\to C))$
3. $(B\to C)\to((A\to B)\to(A\to C))$
4. $(A\to(A\to B))\to(A\to B)$
5. $(A\to(B\to C))\to((A\to B)\to(A\to C))$
6. $(A\to B)\to((A\to(B\to C))\to(A\to C))$
7. $(D\to B)\to((A\to(B\to C))\to(A\to(D\to C)))$
8. $(C\to D)\to((A\to(B\to C))\to(A\to(B\to D)))$
9. $(A\to(B\to C))\to(A\to((D\to B)\to(D\to C)))$
10. $(A\to(B\to C))\to(A\to((C\to D)\to(B\to D)))$
11. $(A\to((B\to C)\to D))\to((B\to C)\to(A\to D))$
12. $(B\to C)\to((A\to((B\to C)\to D))\to(A\to D))$
13. $(A\to B)\to(((A\to B)\to C)\to C)$
14. $((A\to A)\to B)\to B$

For $S3^o_{\to}$ we have added

15. $(B{\to}C){\to}(A{\to}A)$
16. $(A{\to}B){\to}((C{\to}D){\to}(A{\to}B))$

and for $S4^o_{\to}$

17. $B{\to}(A{\to}A)$
18. $(A{\to}B){\to}(C{\to}(A{\to}B))$

[see Méndez (1988)].

In order to preserve the CAP [see Méndez (1987)] 4, 5 and 6 must be restricted to:

4'. $(A{\to}(A{\to}(B{\to}C))){\to}(A{\to}(B{\to}C))$

5'. $(A{\to}(B{\to}(C{\to}D))){\to}((A{\to}B){\to}(A{\to}(C{\to}D)))$

6'. $(A{\to}B){\to}((A{\to}(B{\to}(C{\to}D))){\to}(A{\to}(C{\to}D)))$

and 11, 12, 13 and 14 to:

11'. $(A{\to}((B{\to}C){\to}(D{\to}E))){\to}(((B{\to}C){\to}(A{\to}(D{\to}E)))$

12'. $(B{\to}C){\to}((A{\to}((B{\to}C){\to}(D{\to}E))){\to}(A{\to}(D{\to}E)))$

13'. $(A{\to}B){\to}(((A{\to}B){\to}(C{\to}D)){\to}(C{\to}D))$

14'. $((\mathrm{A}{\to}\mathrm{A}){\to}(\mathrm{B}{\to}\mathrm{C})){\to}(\mathrm{B}{\to}\mathrm{C})$

In other words, 4 and 14 are restricted to the case in which B is an implicative formula; 5, 6 and 13 to the case in which C is an implicative formula and 11 and 12 to the case in which D is an implicative formula.

Now, $S3_{\to}$ can be axiomatized with 1, 3, 5 and 16, and $S4_{\to}$ with 1, 5 and 18 with *modus ponens* as the sole rule of inference. Then, $S3^o_{\to}$ is axiomatized with 1, 3, 5′ and 16. Nevertheless, 1, 5′ and 18 do not axiomatize $S4^o_{\to}$ (cfr, Matrix VI). Therefore, $S4^o_{\to}$ is axiomatized with 1, 3, 5′, and 18. On the other hand, 1–16 are theorems of $S3_{\to}$ and 1–18 are theorems of $S4_{\to}$. Consequently, 1–3, 4′, 5′, 6′, 7–10, 11′, 12′, 13′, 14′, 15 and 16 are theorems of $S3^o_{\to}$, and for $S4^o_{\to}$ we add to these 17 and 18.

In what follows we exhaustively axiomatize $S3^o_{\to}$ and $S4^o_{\to}$ with these theorems.

III. SYNTACTIC LEMMAS

The first one (elementary propositional axiomatics) reads:

LEMMA III.1

i) 3 is derivable from 1 and 9; 3 is derivable from 5′ (6′), 8 and 15 (17).

ii) 2 is derivable from 1 and 10; 2 is derivable from 4′ (5′, 6′), 7 and 15 (17).

iii) 16 is derivable from 2 and 15; 18 is derivable from 2 and 17

iv) 16 is derivable from 3, 5′ (6′) and 15; 18 is derivable from 3, 5′ (6′) and 17.

v) 5′ is derivable from 2 (3), 6′ and 16 (18).

vi) 3 is derivable from 2, 5′ (6′) and 16 (18).

vii) 13 is derivable from 2, 4′ and 16 (18).

viii) *Trans*. is derivable from 2 (3, 7, 8, 9, 10) and 14′.

ix) As regards 1, 15 and 17:

   a. 1 is derivable from 15 (17) and any thesis of the form $A\to B$.

   b. 15 is derivable from 1 and 16.

   c. 1 is derivable from 14′, 16 (18) and *Trans*.

   d. 15 is derivable from 14′, 16 (18) and *Trans*.

   e. 17 is derivable from 1 and 18.

In (i)–(ix) the numerals refer to the theses in the list. The sole rule of inference is *modus ponens*. The rule *Trans*. (Transivity) is:

$$\text{If } A\to B \text{ and } \vdash B\to C, \text{ then } \vdash A\to C$$

Proof of Lemma III.1 is left to the reader. Now we remark a result of Robles and Méndez (2001) that can be useful.

LEMMA III.2 (*Exhaustive axiomatization of* $E^0_\to$ *with* 1–3, 4′, 5′, 6′, 7–10, 11′, 12′, 13′ *and* 14′). $E^o_\to$ may be axiomatized (with modus ponens) using any selection that includes one (and only one) thesis from each set in the following groups:

a: {1}, {2,3,7,8,9,10}, {4′ ,5′ ,6′},{11′ ,12′ ,13′}.

b: {14′}, {3,8,9}, {4′, 5′, 6′}, {11′, 12′, 13′}.

c: {14′}, {2,7,10}, {4′, 5′, 6′}.

It is clear from Lemma III.1 that $E^o_{\rightarrow}$ plus 15 or 16 is $S3^o_{\rightarrow}$, and that $E^o_{\rightarrow}$ plus 17 or 18 is $S4^o_{\rightarrow}$. In the following lemma we exhaustively axiomatize $S3^o_{\rightarrow}$.

LEMMA III.3 $S3^o_{\rightarrow}$ may be axiomatized (with modus ponens) using any selection that includes one (and only one) thesis of each set in (a), (b), (c), (d), (e) and (f) below

a: {5′, 6′}, {2, 3, 7, 8, 9, 10}, {15}.

b: {1, 14′}, {5′, 6′}, {2, 3, 7, 8, 9, 10}, {16}.

c: {4′}, {2, 7, 10}, {15}.

d: {1, 14′}, {4′}, {2, 7, 10}, {16}.

e: {4′}, {3, 8, 9}, {11′, 12′, 13′}, {15}.

f: {1, 14′}, {4′}, {3, 8, 9}, {11′, 12′, 13′}, {16}.

Proof: We recall that $S3^o_{\rightarrow}$ is axiomatized with 1, 3, 5′ and 16 (see §II), and that $E^o_{\rightarrow}$ plus 16 (15) is $S3^o_{\rightarrow}$. On the other hand, in the proof to follow, (i), (ii), etc. always refer to the items of Lemma III.1. We prove that all selections in each group are equivalent and that they all are equivalent to the standard axiomatization of $S3^o_{\rightarrow}$.

Group (a): First note that 1 is always present in each selection (ix.a), so by (iv), 5′, 3 and 15, it is an axiomatization of $S3^o_{\rightarrow}$. Now, (i) shows that 3 can be replaced with 8 or 9; (iii), (vi) show that 2 can substitute 3; and (ii), with (iii) or (vi), that 7 or 10 are interchangeable with 3. (Notice that these changes can be made either in the presence of 5′ or in the presence of 6′). Finally, by (iv), (v), 5′ can be replaced with 6′.

Group (b): by (viii), (ix.c), 1 is always present and by (ix.b), so is 15. Now apply the results on group (a).

Group (c): by (ix.a), 1 is always present and by (ii), 2 is always present as well. By (iii), 16 is derivable. Next, by (vii), 13′ is derivable. Now, 1, 2, 4′ and 13′ axiomatize $E^o_{\rightarrow}$.

Group (d): inmediate: results on group (c) and (ix.b), (ix.d), and (viii).

Group (e): by (ix.a), 1 is in each one of the selections. Now, {1}, {4′}, {3, 8, 9}, and {11′, 12′, 13′} axiomatize $E^{o}_{\to}$.

Group (f): by (ix.b), (ix.d), 1 is in each one of this selections. Now, apply the results on group (e).

LEMMA III.4 $S4^{o}_{\to}$ may be axiomatized (with modus ponens) using any selection that includes one (and only one) thesis from the sets in (a), (b), (c), (d), (e) and (f) below:

a: {5′, 6′}, {2, 3, 7, 8, 9, 10}, {17}.

b: {1, 14′}, {5′, 6′}, {2, 3, 7, 8, 9, 10}, {18}.

c: {4′}, {2, 7, 10}, {17}.

d: {1, 14′}, {4′}, {2, 7, 10}, {18 }.

e: {4′}, {3, 8, 9}, {11′, 12′, 13′}, {17}.

f: {1, 14′}, {4′}, {3, 8, 9}, {11′, 12′, 13′}, {18}.

Proof: We recall that $S4^{o}_{\to}$ is axomatized with 1, 5′, 3 and 18 (see §II), and that $E^{o}_{\to}$ plus 17 (18) is $S4^{o}_{\to}$. Now, the proof of Lemma III.4 is like that of Lemma III.3.

IV MATRICES

We provide six matrices to be used in the independence proofs of section 5. Designated values are starred.

Matrix I

$\to$	0	1	2
0	2	2	2
1	0	0	2
2^*	0	0	2

VERIFIES: 2, 3, 4′–6′, 7–10, 11′–13′, 16 and 18.
FALSIFIES: 1 (with *A*=1), 14′ (with *A*=*B*=*C*=1), 15 (with *A*=*B*=1 and *C*=2) and 17 (with *A*=*B*=1).

Matrix II

$\rightarrow$	0	1	2
0	2	2	2
1	1	2	2
2^*	0	1	2

VERIFIES: 1–3, 7–10, 11′–14′, 15 and 18.
FALSIFIES: 4′ (with A=1, B=2 and C=0), 5′ (with A=B=1, C=2 and D=0), and 6′ (with A=B=1, C=2 and D=0).

Matrix III

$\rightarrow$	0	1	2	3
0	2	2	2	2
1	0	2	2	2
2^*	0	1	2	2
3^*	0	1	1	2

VERIFIES: 1–3, 4′–6′, 7–10, 11′–14′, 15 and 16.
FALSIFIES: 17 (with A=B=3) and 18 (with A=B=C=1).

Matrix IV

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	2	0	3
2^*	0	0	2	3
3^*	0	0	0	3

VERIFIES: 1–3, 4′–6′, 7–10, and 11′–14′.
FALSIFIES: 15 (with A=1 and B=C=3), 16 (with A=B=2 and C=D=3), 17 (with A=B=1) and 18 (with A=B=C=1).

Matrix V

$\to$	0	1	2	3
0	3	2	2	3
1	3	3	3	3
2	0	1	3	3
3^*	0	1	2	3

VERIFIES: 1, 3, 4′, 8, 9, 14′ and 15–18
FALSIFIES: 2 (with *A*=2, *B*=0 and *C*=1), 5′ (with *A*=2, *B*=0, *C*=2 and *D*=1), 6′ (with *A*=2, *B*=0, *C*=2 and *D*=1), 7 (with *A*=2, *B*=0, *C*=1 and *D*=3), 10 (with *A*=*B*=2, *C*=0 and *D*=1), 11′ (with *A*=*B*=2, *C*=0, *D*=2 and *E*=1), 12′ (with *A*=*B*=2, *C*=0, *D*=2, and *E*=1), and 13′ (with *A*=2, *B*=0, *C*=2 and *D*=1).

Matrix VI

$\to$	0	1	2	3	4
0	4	4	4	4	4
1	0	3	3	3	4
2	0	2	3	3	4
3^*	0	0	2	3	4
4^*	0	0	2	3	4

VERIFIES: 1, 4′–6′, 11′–14′ and 15–18.
FALSIFIES: 2 (with *A*=3, *B*=2, and *C*=1), 3 (with *A*=3, *B*=2 and *C*=1), 7 (with *A*=1, *B*=2, *C*=1 and *D*=3), 8 (with *A*=1, *B*=3, *C*=2 and *D*=1), 9 (with *A*=1, *B*=2, *C*=1 and *D*=3) and 10 (with *A*=1, *B*=3, *C*=2 and *D*=1).

Regarding these matrices, we comment the following facts:

1) Matrices I, II, IV and VI show that at least one thesis in the sets {1, 14′, 15}, {2, 3, 7, 8, 9, 10}, {4′, 5′, 6′}, {15, 16} has to be included to axiomatize $S3^o_\to$. If 15 is chosen, we have group (a) (and the equivalent group (b)) and group (c) (and the equivalent group (d)) because 1, 3, 4′, 8, 9, 14′, 15 and 16 do not axiomatize $S3^o_\to$ (Matrix V).

2) Given that 1, 3, 4′, 8, 9, 14′, 15 and 16 do not axiomatize $S3^o_\to$, we add any thesis from the set {11′, 12′, 13′} which gives us group (e) and the equivalent group (f).

3) 1–3, 4′–6′, 7–10, 11′–14′ do not axiomatize $S3^o_\to$ (Matrix IV).

4) Comments on (i) and (ii) are *mutatis mutandis* applicable to $S4^o_\to$ (just change 15 and 16, whenever present, for 17 and 18, respectively).

5) 1–3, 4′–6′, 7–10, 11′–14′, 15 and 16 do not axiomatize $S4^o_\to$ (Matrix III).

V. Exhaustively Axiomatizing $S3^o_\to$ and $S4^o_\to$

Finally, we prove

THEOREM V.1. $S3^o_\to$ may be axiomatized (with modus ponens) using any selection that includes one (and only one) thesis from each set in the following groups:

a: {5′, 6′}, {2, 3, 7, 8, 9, 10}, {15}.

b: {1, 14′}, {5′, 6′}, {2, 3, 7, 8, 9, 10}, {16}.

c: {4′}, {2, 7, 10}, {15}.

d: {1, 14′}, {4′}, {2, 7, 10}, {16}.

e: {4′}, {3, 8, 9}, {11′, 12′, 13′}, {15}.

f: {1, 14′}, {4}, {3 ,8, 9}, {11′, 12′, 13′}, {18}.

The 72 resulting selections (the order of the axioms is not taken into account) are the only axiomatizations of $S3^o_\to$ that can be formulated with the sixteen first theses in the list.

THEOREM V.2. Replace 15 and 16 in groups (a)–(f) above with 17 and 18 respectively. Then, $S4^o_\to$ may be axiomatized (with modus ponens) using any selection that includes one (and only one) thesis from each set in these groups. The 72 resulting selections (the order of the axioms is not taken into account) are the only axiomatizations of $S4^o_\to$ that can be formulated with the eighteen theses in the list.

In order to prove that all selections in theorems 1, 2 have independent axioms, use the matrices in section IV as follows:

Independence of 1 and 14′: Matrix I.

Independence of 4′, 5′ and 6′: Matrix II.

Independence of 2, 3, 7, 8, 9 and 10: Matrix VI.

Independence of 11′, 12′ and 13′: Matrix V.

Independence of 15 and 16: Matrix IV.

Independence of 17 and 18: Matrix III.****

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Notes

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