EXTRACTA MATHEMATICAE Vol. 12, Núm. 1, 19 – 23 (1997)

When a Composition Algebra is Barrelled?

Jorge Bustamante González and Raúl Escobedo Conde

B. Univ. Autónoma de Puebla, FCFM, Avda. San Claudio y 14 *sur,*
CP 72570, *Puebla, PUE, México, e-mail: jbusta fcfm.buap.mx*

(Research paper presented by J.M.F. Castillo)

AMS *Subject Class.* (1991): 46A08, 46H20, 54C10 *Received April* 24, 1997

In this paper, X and Y will denote completely regular Hausdorff spaces, $C(X)$ is the family of all real-valued continuous maps $f : X \to \mathbb{R}$. A composition algebra A_φ on X is given by a continuous map $\varphi\colon X \to Y$ in such a way that

$$A_\varphi = \{g \circ \varphi\colon g \in C(Y)\}.$$

All functional spaces are endowed with the compact-open topology.

Here we are concerned with the following question: given X, Y and φ, when A_φ in a barrelled space?, that is, when the barrels (closed absorbent absolutely convex sets) in A_φ are a base of neighborhoods of the null function? If $X = Y$ and φ is the identity map, this problem was solved independently by L. Nachbin and T. Shirota (see [1], Theorem 2.5.-1).

Recall that a subset $Q \subset Y$ is C-embedded (in Y) if every $f \in C(Q)$ can be extended to a function in $C(Y)$.

Given a family A of functions $f\colon X \to \mathbb{R}$, and $Q \subset X$, Q is said to be A-bounding if for each $f \in A$, $f(Q)$ is bounded in $\mathbb{R}$. X is called a NS (Nachbin-Shirota) space if each $C(X)$-bounding set is relatively compact.

A map $\varphi\colon X \to Y$ is semiproper if for each compact subset $H \subset Y$, there exists a compact subset $K \subset X$ such that $\varphi(K) = H \bigcap \varphi(X)$. Recall that $P \subset X$ is φ-saturated if $P = \varphi^{-1}(\varphi(P))$ and φ is superproper if for each φ-saturated and A_φ-bounding subset $Q \subset X$, there exists a compact subset $K \subset X$ such that $\varphi(Q) \subset \varphi(K)$. We know that not all semiproper map are superproper and if $\varphi(X)$ is C-embedded in Y and is φ superproper, then φ is semiproper (see remark bellow).

Our main result state as follows:

THEOREM. *Let X and Y be completely regular Hausdorff spaces, $\varphi\colon X \to Y$ a continuous map such that $\varphi(X)$ is C-emmbedded in Y. The following assertions are equivalent:*

(i) *A_φ is barrelled;*

(ii) *φ is a superproper map;*

(iii) *$\varphi(X)$ is a NS space, $\varphi(X)$ is closed in Y and φ is semiproper.*

Remark. Under the hypothesis of the theorem above define a homomorphism

$$A\colon C(Y) \to A_\varphi$$

by $Ag = g \circ \varphi$. J.G. Llavona and J.A. Jaramillo ([3], 1.19) proved that if X is Lindelöf, $\varphi(X)$ is C-embedded in Y and A_φ is barrelled, then $\varphi\colon X \to Y$ is semiproper and $\varphi(X)$ is closed in Y. They also proved ([3], 1.13) that A is an open map if, and only if, $\varphi(X)$ is closed in Y and $\varphi\colon X \to Y$ is semiproper. We see the relevance of the Nachbin-Shirota condition by comparing this last result with the theorem above, because if we take $X = Y$ and φ the identity map then, of course, A is open, but $A_\varphi = C(X)$ need not to be barrelled.

The proof of (ii) implies (iii) is inspired in proposition 1.19 of [3]. The proof of (ii) implies (i) is a modification of the ideas used in [1] in order to prove the Nachbin-Shirota theorem.

Proof. [(i) implies (ii)] Fix a φ-saturated and A_φ-bounding subset $Q \subset X$. The set

$$V_Q = \{f \in A_\varphi\colon f(Q) \subset [-1,1]\}$$

is a barrel (in A_φ). Then, there exist a compact set $K \subset X$ and $0 < \epsilon < 1$, such that

$$\{f \in A_\varphi\colon f(K) \subset (-\epsilon,\epsilon)\} \subset V_Q \quad . \tag{1}$$

Therefore, $\varphi(Q) \subset \varphi(K)$. Indeed, suppose to the contrary that there exists $q_0 \in Q$ such that $\varphi(q_0) \notin \varphi(K)$, then there exists $g \in C(Y)$ such that

$$g(\varphi(K)) \subset (-\epsilon,\epsilon) \quad \text{and} \quad g(\varphi(q_0)) > 1.$$

Then, for $f = g \circ \varphi$, we have that $f \in A_\varphi$, $f(K) \subset (-\epsilon,\epsilon)$ and $f \notin V_Q$. This gives a contradition with (1).

Thus, φ is a superproper map.

[(ii) implies (i)] Let V be a barrel in A_φ. For any subset $Q \subset X$ denote $V_Q = \{f \in A_\varphi \colon f(Q) \subset [-1,1]\}$ and

$$Z_Q = \{f \in A_\varphi \colon Q \subset Z(f)\},$$

where $Z(f) = \{x \in X \colon f(x) = 0\}$.

Following the proof of Nachbin-Shirota theorem in [1] pp. 94-96, it is not difficult to find some $d > 0$ such that $dV_X \subset V$ and $\frac{d}{2}V_S \subset V$ for every $S \subset X$ such that $Z_S \subset V$.

Now, for each $\lambda \in A_\varphi^*$ (the topological dual of A_φ) set $\hat{\lambda} = \lambda \circ A \in C(Y)^*$, that is $\hat{\lambda}(g) = \lambda(g \circ \varphi)$. Then $\hat{\lambda}$ is a continuous linear functional on $C(Y)$ with support $\mathrm{Supp}\,(\hat{\lambda})$ (see [1], 2.4-8). Set $S_\lambda = \varphi^{-1}(\mathrm{Supp}\,(\hat{\lambda}))$. It is easy to prove that S_λ is a φ-saturated set in X such that:

(a) If $f \in A_\varphi$ and $S_\lambda \subset Z(f)$, then $\lambda(f) = 0$, and

(b) If $Q \subset X$ and $\lambda(Z_Q) = \{0\}$, then $S_\lambda \subset Q$.

Let $V^\circ = \{\lambda \in A_\varphi^* \colon |\lambda(f)| \leq 1,\ f \in V\}$ be the polar of V and $K_V = \mathrm{cl}_X(\bigcup_{\lambda \in V^\circ} S_\lambda)$, then $Z_{K_V} \subset V$ (we have used the bipolar theorem).

Let us prove that K_V is an A_φ-bounding set. Suppose to the contrary that there exists $f \in A_\varphi$ such that $f(K_V)$ is an unbounded set in $\mathbb{R}$. For $n = 1, 2, ...$ set $U_n = \{x \in X \colon |f(x)| > n\}$. It is clear that $\{U_n\}$ is a non increasing family of open subsets of X such that $\bigcap_n U_n = \emptyset$. Moreover, for $n = 1, 2, ...,$ $K_V \bigcap U_n \neq \emptyset$. Thus, for $n = 1, 2...,$ there exists $\lambda_n \in V^\circ$ such that $U_n \bigcap S_{\lambda_n} \neq \emptyset$. Taking into account property (b) above, if $\lambda_n(Z_{X \setminus U_n}) = \{0\}$, then $S_{\lambda_n} \subset X \setminus U_n$, which leads to a contradiction. Therefore, for $n = 1, 2, ..,$ there exists $f_n \in A_\varphi$ such that $(X \setminus U_n) \subset Z(f_n)$ and $\lambda_n(f_n) \neq 0$.

There exists $g \in C(Y)$ such that $f = g \circ \varphi$. Set

$$T_n = \{y \in Y \colon |g(y)| \geq n\}, \quad n = 1, 2, ...$$

For $m = 1, 2, ...,$ $\varphi(S_{\lambda_m})$ is a compact set, then

$$\{n \in \mathbb{N} \colon T_n \bigcap \varphi(S_{\lambda_m}) \neq \emptyset\}$$

is a finite set. On the other hand, since $\varphi(\mathrm{cl}_X U_n \bigcap S_{\lambda_m}) \subset T_n \bigcap \varphi(S_{\lambda_m})$, it holds that for $m = 1, 2, ...,$

$$\{n \in \mathbb{N} \colon \mathrm{cl}_X U_n \bigcap S_{\lambda_m} \neq \emptyset\}$$

is finite set. Hence, for some subsequences which we still denote U_n and S_{λ_m} we may suppose that, for $m = 1, 2, ...$, and $n > m$, $\text{cl}_X U_n \bigcap S_{\lambda_m} = \emptyset$. Fix a sequence of real numbers $\{\alpha_n\}$ such that

$$\lambda_m(f) = \sum_{n=1}^{\infty} \alpha_n \lambda_m(f_n) = \alpha_m + \sum_{n=1}^{m-1} \alpha_n \lambda_m(f_n) = m.$$

There exits $\alpha > 0$ such that $f \in \alpha V$ (V is an absorbent set). Taking into account that $\lambda_m \in V^\circ$, we have that

$$\frac{1}{\alpha} m = \lambda_m \left(\frac{f}{\alpha} \right) \geq 1, \quad \text{for } m \geq 1.$$

The inequality above gives a contradiction ($\alpha^{-1} f \in V$ and $\lambda_m \in V^\circ$). We have proved that K_V is an A_φ-bounding set.

Since K_V is a φ-saturated A_φ-bounding subset of V such that $\frac{d}{2} V_{K_V} \subset V$, then there exists a compact subset $H \subset X$, such that $\varphi(K_V) \subset \varphi(H)$. It follows that

$$\left\{ f \in A_\varphi \colon f(H) \subset \left[-\frac{d}{2}, \frac{d}{2} \right] \right\} \subset \frac{d}{2} V_{K_V} \subset V.$$

Then, V is a neighborhood of zero in A_φ.

[(ii) implies (iii)] Denote by νY the Hewitt-Nachbin realcompatification of Y (see [2], 8). We can consider that $\varphi(X) \subset Y \subset \nu Y$. Set $M = \text{cl}_{\nu Y} \varphi(X)$. Since $\varphi(X)$ is C-embedded in Y, $M = \nu(\varphi(X))$. Set $\hat{\varphi} = i \circ \varphi$, where $i \colon \varphi(X) \to M$ is the natural embedding. Notice that φ and $\hat{\varphi}$ give the same saturated sets in X and that $A_\varphi = A_{\hat{\varphi}}$. Thus $\hat{\varphi}$ is a superproper map. Define $\hat{A} \colon C(M) \to A_\varphi = A_{\hat{\varphi}}$ by $\hat{A} = g \circ \hat{\varphi}$.

If $H \subset M$ is a compact set and

$$V_H = \{ f \in C(M) \colon \sup_{x \in H} |f(x)| \leq 1 \},$$

then $\hat{A}(V_H)$ is a barrel in $A_{\hat{\varphi}}$. Therefore, $\hat{A}(V_H)$ a neighborhood of zero in $A_{\hat{\varphi}}$ (see (ii) implies (i)). Thus, $\hat{A}$ is an open map. Since φ is a continuous map, it is easy to prove that $\hat{A}$ is continuous map. That is, $\hat{A}$ is a topological isomorphism.

Now, we know that $\hat{\varphi}(X) = M$ and $\hat{\varphi} \colon X \to M$ is semiproper map (see [3], 1.14). Therefore, $\varphi(X)$ is closed in Y and φ is semiproper.

On the other hand, if $J \subset \varphi(X)$ is a $C(\varphi(X))$-bounding set and $K = \varphi^{-1}(J)$, then K is a φ-saturated A_φ-bounding set. Therefore, there exists a compact set $H \subset X$ such that $\varphi(K) \subset \varphi(H)$. Since φ is continuous map, $\varphi(H)$ is compact subset of Y. Taking into account that $\varphi(X)$ is closed in Y and $J \subset \varphi(H)$, then J is relatively compact subset of $\varphi(X)$.

The arguments above say that $\varphi(X)$ is a Nachbin-Shirota space.

[(iii) implies (ii)] Fix a φ-saturated and A_φ-bounding subset $Q \subset X$. Denote the closure of $\varphi(Q)$ in $\varphi(X)$ by H. Taking into account that $\varphi(X)$ is a Nachbin-Shirota space and $\varphi(Q)$ is a $C(\varphi(X))$-bounding set, it follows by the Nachbin-Shirota's theorem that H is a compact subset of $\varphi(X)$. Therefore, H is a compact subset of Y. On the other hand, since φ is a semiproper map, then there exists a compact set $K \subset X$ such that $\varphi(K) = H \bigcap \varphi(X) = H$. Since $\varphi(Q) \subset H = \varphi(K)$, φ is a superproper map. ∎

References

[1] Beckenstein, E., Narici, L. and Suffel, C., "Topological Algebras", North Holland Math. Studies, 24, Amsterdam, 1977.

[2] Gillman, L., Jerison, M., "Rings of Continuous Functions", Van Nostrand, New Jersey, 1960.

[3] Llavona, J.G., Jaramillo, J.A., Homomorphisms between algebras of continuous functions, *Can. J. Math.*, Vol. **LXI** (1) (1989), 132–162.