DECISION MAKING WIH DEMSPTER-SHAFER THEORY AND UNCERTAIN INDUCED AGGREGATION OPERATORS

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RESUMEN

Se desarrolla un nuevo modelo para la toma de decisiones mediante la teoría de la evidencia de Dempster-Shafer. Se analiza situaciones inciertas en donde la información no puede ser tratada mediante números precisos pero sí mediante intervalos de confianza. Para agregar la información, se sugieren diferentes tipos de operadores de agregación inciertos e inducidos tales como el operador *uncertain induced ordered weighted averaging* (UIOWA) y el *uncertain induced hybrid averaging* (UIHA). Como resultado, se obtienen nuevos operadores de agregación tales como el BS-UIOWA y el BS-UIHA. La ventaja de utilizar estos operadores es la posibilidad de modelizar el carácter atitudinal del decisor ante situaciones complejas las cuales no pueden ser tratadas únicamente mediante el grado de optimismo del decisor. Se estudian algunas de sus principales propiedades. También se desarrolla una aplicación del nuevo modelo en un problema de toma de decisiones financieras sobre selección de inversiones.

Palabras clave: Toma de decisiones; Teoría de la evidencia de Dempster-Shafer; Incertidumbre; Operadores de agregación.

ABSTRACT

We develop a new approach for decision making with Dempster-Shafer (D-S) theory of evidence. We focus on a problem where the available information is uncertain and it can be assessed with interval numbers. In order to aggregate the information, we suggest the use of different types of uncertain induced aggregation operators such as the uncertain induced ordered weighted averaging (UIOWA) and the uncertain induced hybrid averaging (UIHA) operator. As a result, we get new types of aggregation operators such as the belief structure – uncertain induced OWA (BS-UIOWA) and the belief structure – uncertain induced hybrid averaging (BS-UIHA) operator. The main advantage of using these operators is the possibility of using complex attitudinal characters in situations where it is not possible to simply use the degree of optimism of the decision maker. We study some of their main properties. We also develop an application of the new approach in a financial decision making problem about selection of investments.

Keywords: Decision making; Dempster-Shafer theory of evidence; Uncertainty; Aggregation operators.

1. INTRODUCTION

The Dempster-Shafer (D-S) theory of evidence (Dempster, 1967; 1968; Shafer, 1976) provides a unifying framework for representing uncertainty because it includes the situations of risk and ignorance as special cases. For further reading on the D-S theory, we recommend for example (Srivastava and Mock, 2002; Yager et al. 1994; Yager and Liu, 2008).

Usually, when using the D-S theory in decision making, it is assumed that the available information are exact numbers (Engemann et al. 1996; Merigó and Casanovas, 2006; 2007a; Yager, 1992; 2004). However, this may not be the real situation found in the decision making problem because often, the available information is vague or imprecise and it is not possible to analyze it with exact numbers. Then, a better approach may be the use of interval numbers. Note that other studies have considered similar approaches by using fuzzy numbers (Casanovas and Merigó, 2007) and linguistic variables (Merigó et al. 2007).

Going a step further, the aim of this paper is to suggest a new approach for uncertain decision making with D-S theory by using uncertain induced aggregation operators. Then, we will be able to use in the same formulation a unifying framework between ignorance and risk, uncertain information assessed with interval numbers and a reordering process in the aggregation step that uses order inducing variables. We will consider different types of uncertain induced aggregation operators such as the uncertain induced ordered weighted averaging (UIOWA) and the uncertain induced hybrid averaging (UIHA) operator.

The main advantage of using these operators is the possibility of considering complex attitudinal characters in situations where it is not possible to use the degree of optimism of the decision maker. Moreover, it is possible to assess the uncertain information by using interval numbers. Then, we are able to represent the uncertain problem considering the best and worst possible scenario. Note that depending on the type of interval number used, it is also possible to consider the most possible scenarios.

These operators provide a parameterized family of aggregation operators that includes the uncertain maximum, the uncertain minimum, the uncertain average and the uncertain OWA (UOWA) operator, among others. By using these aggregation operators, we will be able to create new aggregation methods such as the belief structure – UIOWA (BS-UIOWA) and the belief structure – UIHA (BS-UIHA) operator. We study some of their main properties and we develop different families of UIOWA and UIHA operators that could be used in the analysis such as the step-UIOWA, the S-UIOWA, the centered-UIOWA, the olympic-UIOWA, etc.

In order to do this, the remainder of the paper is organized as follows. In Section 2 we briefly review some basic concepts such as the interval numbers, the D-S theory, the UIOWA and the UIHA operator. Section 3 introduces the new approach when the information is aggregated with the UIOWA operator. In Section 4, we develop a similar approach with the UIHA operator. Finally, in Section 5 we present an illustrative example of the new approach in a financial decision making problem.

2. PRELIMINARIES

In this Section, we briefly review some basic concepts about the interval numbers, the UIOWA operator, the UIHA operator and the D-S theory.

2.1. INTERVAL NUMBERS

The interval number is a very useful and simple technique for representing the uncertainty. It has been used in an astonishingly wide range of applications. For further reading, see for example (Kaufmann and Gil-Aluja, 1987, 1990; Kaufmann et al. 1994; Kaufmann and Gupta, 1985; Moore, 1966).

In the literature, we find different types of interval numbers. For example, if we assume a 4-tuple (a_1, a_2, a_3, a_4), that is to say, a quadruplet; we could consider that a_1 and a_4 represents the minimum and the maximum of the interval number, and a_2 and a_3, the interval with the highest probability or possibility, depending on the use we want to give to the interval numbers. Note that $a_1 \leq a_2 \leq a_3 \leq a_4$. If $a_1 = a_2 = a_3 = a_4$, then, the interval number is an exact number and if $a_2 = a_3$, it is a 3-tuple known as triplet.

In the following, we are going to review some basic interval numbers operations as follows. Let A and B be two triplets, where $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$. Then:

1) $A + B = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$
2) $A - B = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$
3) $A \times k = (k \times a_1, k \times a_2, k \times a_3)$; for $k > 0$.

Note that other operations could be studied (Kaufmann et al. 1985; Moore, 1966) but in this paper we will focus on these ones.

2.2. UNCERTAIN INDUCED OWA OPERATOR

The uncertain induced OWA operator was introduced by Xu (2006a). It is an extension of the OWA operator (Beliakov et al. 2007; Calvo et al. 2002; Merigó 2007; Yager, 1988; 1993; Yager and Kacprzyk, 1997) that uses the main characteristics of two well known aggregation operators: the induced OWA (Merigó and Gil-Lafuente, 2007; Yager, 2003; Yager and Filev, 1999) and the uncertain OWA operator (Xu and Da, 2003). Then, it uses interval numbers for representing the uncertain information and a reordering process that it is based on order inducing variables. It can be defined as follows:

Definition 1. *Let Ω be the set of interval numbers. An UIOWA operator of dimension n is a mapping UIOWA:* $\Omega^n \rightarrow \Omega$ *that has an associated weighting vector W of dimension n such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$*, then*:

$$UIOWA(\langle u_1,\tilde{a}_1\rangle, \ldots, \langle u_n,\tilde{a}_n\rangle) = \sum_{j=1}^{n} w_j b_j \tag{1}$$

where b_j *is the* $\tilde{a}_i$ *value of the UIOWA pair* $\langle u_i, \tilde{a}_i\rangle$ *having the jth largest* u_i, u_i *is the order inducing variable and* $\tilde{a}_i$ *is the argument variable represented in the form of interval numbers.*

From a generalized perspective of the reordering step it is possible to distinguish between descending (DUIOWA) and ascending (AUIOWA) orders. Note that in this case, it is not necessary to compare interval numbers because the reordering step is developed with order inducing variables. The only case where we need to compare interval numbers is in the final result. For doing this, we will use the following criteria. First, we will analyse if there is on order between the interval numbers. If not, we will calculate an average of the interval number. For example, if $n = 2$, $(a_1 + a_2) / 2$; if $n = 3$, $(a_1 + 2a_2 + a_3) / 4$; etc. If there is still a tie, then, we will follow a subjective criterion such as considering only the minimum, the maximum, etc.

Note also that different families of UIOWA operators can be studied by choosing a different weighting vector such as the step-UIOWA operator, the window-UIOWA, the median-UIOWA, the olympic-UIOWA, the centered-UIOWA, the S-UIOWA, etc.

2.3. UNCERTAIN INDUCED HYBRID AVERAGING OPERATOR

The uncertain induced hybrid averaging operator is an extension of the hybrid averaging (Xu, 2006b; Xu and Da, 2003) that uses the weighted average (WA) and the OWA operator, at the same time. It also uses interval numbers for representing the uncertain information and a reordering process based on order inducing variables. It can be defined as follows:

Definition 2. *Let Ω be the set of interval numbers. An UIHA operator of dimension n is a mapping UIHA:* $\Omega^n \rightarrow \Omega$ *that has an associated weighting vector W of dimension n such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$*, then*:

$$UIHA(\langle u_1,\tilde{a}_1\rangle, \ldots, \langle u_n,\tilde{a}_n\rangle) = \sum_{j=1}^{n} w_j b_j \quad (2)$$

where b_j *is the* $\hat{a}_i$ ($\hat{a} = n\omega_i\tilde{a}_i$, $i = 1,2,\ldots,n$) *value of the UIHA pair* $\langle u_i, \tilde{a}_i\rangle$ *having the jth largest* u_i, u_i *is the order inducing variable,* $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ *is the weighting vector of the* $\tilde{a}_i$*, with* $\omega_i \in [0, 1]$ *and the sum of the weights is* 1*, and the* $\tilde{a}_i$ *are interval numbers*.

Note that in this case it is also possible to distinguish between descending (DUIHA) and ascending (AUIHA) orders. Also note that it is only necessary to compare interval numbers in the final result because in the reordering step of the aggregation, this problem is solved by using inducing variables. In this case, we will also follow the same criterion as the one explained for the UIOWA operator.

By using a different manifestation in the weighting vector we are able to develop a wide range of families of UIHA operators. For example, we could obtain the maximum, the minimum, the uncertain average (UA), the uncertain weighted average (UWA), the uncertain OWA, among others. Other families that could be studied are the step-UIHA, the window-UIHA, the median-UIHA, the olympic-UIHA, centered-UIHA, the S-UIHA, etc.

2.4. DEMPSTER-SHAFER THEORY OF EVIDENCE

The D-S theory of evidence (Dempster, 1967; Shafer, 1976) provides a unifying framework for representing uncertainty as it can include the situations of risk and ignorance as special cases. Note that the case of certainty is also included as it can be seen as a particular case of risk or ignorance. Since its appearance, the D-S theory has been applied in a wide range of applications (Reformat and Yager, 2008, Srivastava and Mock, 2002; Yager et al. 1994; Yager and Liu, 2008).

Definition 3. *A D-S belief structure defined on a space X consists of a collection of n nonnull subsets of X,* B_j *for* $j = 1,\ldots,n$*, called focal elements and a mapping m, called the basic probability assignment, defined as,* $m: 2^X \to [0, 1]$ *such that*:

(1) $m(B_j) \in [0, 1]$.

(2) $\sum_{j=1}^{n} m(B_j) = 1$. (3)

(3) $m(A) = 0, \forall A \neq B_j$.

As said before, the cases of risk and ignorance are included as special cases of belief structure in the D-S framework. For the case of risk, a belief structure is called Bayesian belief structure if it consists of n focal elements such that $B_j = \{x_j\}$, where each focal element is a singleton. Then, we can see that we are in a situation of decision making under risk environment as $m(B_j) = P_j = \text{Prob}\ \{x_j\}$.

The case of ignorance is found when the belief structure consists in only one focal element B, where $m(B)$ essentially is the decision making under ignorance environment as this focal element comprises all the states of nature. Thus, $m(B) = 1$. Other special cases of belief structures such as the consonant belief structure or the simple support function are studied in (Shafer, 1976).

3. USING UIOWA OPERATORS IN DECISION MAKING WITH D-S THEORY

In this Section, we describe the process to follow when using UIOWA operators in decision making with D-S theory. We divide it in three subsections. In the first one, we comment the decision process. In the second one, we analyze the aggregation used in the problem. And in the third one, we study different types of UIOWA operators that could be used in the aggregation.

3.1. DECISION MAKING APPROACH

A new approach for decision making with D-S theory is possible by using uncertain induced aggregation operators. The main advantages of using this type of aggregation are: the possibility of dealing with uncertain information, the possibility of using an aggregation that provides a parameterized family of aggregation operators between the maximum and the minimum, and the possibility of using a general formulation in the reordering of the arguments by using inducing variables. Note that in this paper we will focus on the UIOWA and the UIHA operators, but it is also possible to consider other types of uncertain induced aggregation operators by using generalized means and quasi-arithmetic means. The motivation for using interval numbers appear because sometimes, the available information is not clear and it is necessary to assess it with another approach such as the use of interval numbers. Although the information is uncertain and it is difficult to take decisions with it, at least we can represent the best and worst possible scenarios. The decision process can be summarized as follows.

Assume we have a decision problem in which we have a collection of alternatives $\{A_1, \ldots, A_q\}$ with states of nature $\{S_1, \ldots, S_n\}$. $\tilde{a}_{ih}$ is the uncertain payoff, given in the form of interval numbers, to the decision maker if he selects alternative A_i and the state of nature is S_h. The knowledge of the state of nature is captured in terms of a belief structure *m* with focal elements $B_1, \ldots, B_r$ and associated with each of these focal elements is a weight $m(B_k)$. The objective of the problem is to select the alternative which gives the best result to the decision maker. In order to do so, we should follow the following steps:

Step 1: Calculate the uncertain payoff matrix.

Step 2: Calculate the belief function *m* about the states of nature.

Step 3: Calculate the collection of weights, *w*, to be used in the UIOWA aggregation for each different cardinality of focal elements. Note that it is possible to use different methods depending on the interests of the decision maker (Merigó, 2007; Yager, 1988; 1993; 2007; Yager and Filev, 1994).

Step 4: Determine the uncertain payoff collection, M_{ik}, if we select alternative A_i and the focal element B_k occurs, for all the values of *i* and *k*. Hence $M_{ik} = \{a_{ih} \mid S_h \in B_k\}$.

Step 5: Calculate the uncertain aggregated payoff, V_{ik} = UIOWA(M_{ik}), using Eq. (1), for all the values of *i* and *k*.

Step 6: For each alternative, calculate the generalized expected value, C_i, where:

$$C_i = \sum_{r=1}^{r} V_{ik} m(B_k) \tag{4}$$

Step 7: Select the alternative with the largest C_i as the optimal.

3.2. UIOWA OPERATORS IN BELIEF STRUCTURES

Analyzing the aggregation in *Steps* 5 and 6 of the previous subsection, it is possible to formulate in one equation the whole aggregation process. We will call this process the belief structure – UIOWA (BS-UIOWA) aggregation. It can be defined as follows.

Definition 4. *A BS-UIOWA operator is defined by*

$$C_i = \sum_{k=1}^{r} \sum_{j_k=1}^{q_k} m(B_k) w_{j_k} b_{j_k} \quad (5)$$

where w_{j_k} is the weighting vector of the kth focal element such that $\sum_{j=1}^{n} w_{j_k} = 1$ and $w_{j_k} \in [0,1]$, b_{j_k} is the $\tilde{a}_{i_k}$ value of the UIOWA pair $\langle \tilde{a}_{i_k}, \tilde{a}_{i_k} \rangle$ having the j_kth largest u_{i_k}, u_{i_k} is the order inducing variable and the $\tilde{a}_{i_k}$ are interval numbers, and $m(B_k)$ is the basic probability assignment.

Note that q_k refers to the cardinality of each focal element and r is the total number of focal elements. The BS-UIOWA operator is monotonic, commutative, bounded and idempotent.

From a generalized perspective of the reordering step, it is possible to distinguish between descending and ascending orders by using $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the BS-DUIOWA and w^*_{n-j+1} the jth weight of the BS-AUIOWA operator. Then, we obtain the BS-DUIOWA and the BS-AUIOWA operators.

3.3. FAMILIES OF BS-UIOWA OPERATORS

By choosing a different manifestation in the weighting vector of the UIOWA operator, we are able to develop different families of UIOWA and BS-UIOWA operators. As it can be seen in definition 4, each focal element uses a different weighting vector in the aggregation step with the UIOWA operator. Therefore, the analysis needs to be done individually.

For example, it is possible to obtain the uncertain maximum, the uncertain minimum, the UA and the UWA. The uncertain maximum is found if $w_p = 1$ and $w_j = 0$, for all $j \neq p$, and $u_p = \text{Max}\{a_i\}$. The uncertain minimum is obtained if $w_p = 1$ and $w_j = 0$, for all $j \neq p$, and $u_p = \text{Min}\{u_i\}$. The UA is found when $w_j = 1/n$, for all $\tilde{a}_i$ and the UWA is obtained if $u_i > u_{i+1}$, for all a_i.

Other families of UIOWA operators could be used in the BS-UIOWA operator such as the step-UIOWA, the S-UIOWA, the olympic-UIOWA, the window-UIOWA and the centered-UIOWA operator, among others. Note that recently, it is appearing a wide range of papers dealing with the problem of determining OWA weights. In this subsection we simply give a general overview commenting some basic cases that are applicable in the UIOWA operator.

The step-UIOWA operator is found when $w_k = 1$ and $w_j = 0$, for all $j \neq k$ and the window-UIOWA when $w_j = 1/m$ for $k \leq j \leq k + m - 1$ and $w_j = 0$ for $j > k + m$ and $j < k$. Note that k and m must be positive integers such that $k + m - 1 \leq n$.

For the median-UIOWA, we distinguish between two cases. If n is odd we assign $w_{(n+1)/2} = 1$ and $w_j = 0$ for all others, and this affects the argument $\tilde{a}_i$ with the $[(n + 1)/2]$th largest u_i. If n is even we assign, for example, $w_{n/2} = w_{(n/2)+1} = 0.5$, and this affects the arguments with the $(n/2)$th and $[(n/2) + 1]$th largest u_i.

For the weighted median-UIOWA we select the argument $\tilde{a}_i$ that has the kth largest inducing variable u_i, such that the sum of the weights from 1 to k is equal or higher than 0.5 and the sum of the weights from 1 to k – 1 is less than 0.5.

The olympic-UIOWA operator is found if $w_1 = w_n = 0$, and for all others $w_j = 1/(n - 2)$. Note that the olympic-UIOWA is transformed in the olympic-UOWA if $w_p = w_q = 0$, such that $u_p = \text{Max}\{\tilde{a}_i\}$ and $u_q = \text{Min}\{\tilde{a}_i\}$, and for all others $w_j = 1/(n - 2)$.

A further family is the centered-UIOWA operator. This type of aggregation operator is symmetric, strongly decaying and inclusive. It is symmetric if $w_j = w_{j+n-1}$. It is strongly decaying when $i < j \leq (n + 1)/2$, then $w_i < w_j$ and when $i > j \geq (n + 1)/2$ then $w_i < w_j$. It is inclusive if $w_j > 0$. Note that it is possible to consider a softening of the second condition by using $w_i \leq w_j$ instead of $w_i < w_j$ which is known as softly decaying centered-UIOWA operator. Note also the possibility of removing the third condition. Then, we shall refer to this type of aggregation as non-inclusive centered-UIOWA operator.

A further interesting family is the S-UIOWA operator. In this case, we can distinguish between three types: the "orlike", the "andlike", and the "generalized" S-UIOWA operator. The orlike S-UIOWA operator is found when $w_p = (1/n)(1 - \alpha) + \alpha$, $u_p = \text{Max}\{\tilde{a}_i\}$, and $w_j = (1/n)(1 - \alpha)$ for all $j \neq p$ with $\alpha \in [0, 1]$. Note that if $\alpha = 0$, we get the UA and if $\alpha = 1$, we get the uncertain maximum. The andlike S-UIOWA operator is found when $w_q = (1/n)(1 - \beta) + \beta$, $u_q = \text{Min}\{\tilde{a}_i\}$, and $w_j = (1/n)(1 - \beta)$ for all $j \neq q$ with $\beta \in [0, 1]$. Note that if $\beta = 0$ we get the UA and if $\beta = 1$, the uncertain minimum. Finally, the generalized S-UIOWA operator is obtained when $w_p = (1/n)(1 - (\alpha + \beta)) + \alpha$, with $u_p = \text{Max}\{\tilde{a}_i\}$; $w_q = (1/n)(1 - (\alpha + \beta)) + \beta$, with $u_q = \text{Min}\{\tilde{a}_i\}$; and $w_j = (1/n)(1 - (\alpha + \beta))$ for all $j \neq p,q$ where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. Note that if $\alpha = 0$, we get the andlike S-UIOWA and if $\beta = 0$, the orlike S-UIOWA.

Another type of UIOWA operator that we could mention is the EZ-UIOWA weights. In this case, we should distinguish between two classes. In the first class, we assign $w_j = (1/k)$ for $j = 1$ to k and $w_j = 0$ for $j > k$, and in the second class, we assign $w_j = 0$ for $j = 1$ to $n - k$ and $w_j = (1/k)$ for $j = n - k + 1$ to n.

Further families of UIOWA operators that could be used include those that depend on the aggregated objects. For example, we could develop the BADD-UIOWA operator as follows.

$$w_j = \frac{b_j^{\alpha}}{\sum_{j=1}^{n} b_j^{\alpha}} \tag{6}$$

where $\alpha \in (-\infty, \infty)$, and b_j is the $\tilde{a}_i$ value of the UIOWA pair $\langle u_i, \tilde{a}_i\rangle$ having the jth largest u_i. Note that the sum of the weights is 1 and $w_j \in [0,1]$. Also note that if $\alpha = 0$, we get the UA and if $\alpha = \infty$, we get the uncertain maximum. In this operator, it appears the problem of how to deal with interval numbers. For simplicity, we recommend to use the average of the interval as the value $\tilde{a}_i$ to be used in the calculation of the weights.

Other families of UIOWA operators that depend on the aggregated objects could be developed by using $(1 - b_j)^{\alpha}$, $(1/ b_j)^{\alpha}$, etc., instead of b_j^{α}. Note that these families were developed for the OWA operator in (Yager, 1993).

A further useful method for obtaining the weighting vector is the functional method known as basic interval monotonic function (BUM) (Yager, 1996). Let f be a function f: $[0, 1] \rightarrow [0, 1]$ such that $f(0) = f(1)$ and $f(x) \geq f(y)$ for $x > y$. Using this BUM function we obtain the UIOWA weights w_j for $j = 1$ to n as

$$w_j = f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right) \tag{7}$$

It is easy to see that the weights w_j satisfy that the sum of the weights is 1 and $w_j \in [0,1]$.

Finally, if we assume that all the focal elements use the same weighting vector, then, we can refer to these families as the BS-uncertain maximum, the BS-uncertain minimum, the BS-UA, the BS-UWA, the BS-step-UIOWA, the BS-S-UIOWA, the BS-olympic-UIOWA, , the BS-centered-UIOWA, etc.

4. USING UIHA OPERATORS IN D-S THEORY

In some situations, the decision maker could prefer to use another type of uncertain aggregation operator such as the UIHA operator. The main advantage of this operator is that it uses the characteristics of the UWA and the UIOWA in the same aggregation. Then, if we introduce this operator in decision making with D-S theory, we are able to develop a unifying framework that includes in the same formulation probabilities, UWAs and UIOWAs.

In order to use this type of aggregation in D-S framework we should consider that now in *Step* 3, when calculating the collection of weights to be used in the aggregation, we are using two weighting vectors because we are mixing in the same problem the UWA and the UIOWA.

In *Step* 5, when calculating the uncertain aggregated payoff, we should use the UIHA operator instead of the UIOWA operator by using Eq. (2).

In this case, it is also possible to formulate in one equation the whole aggregation process. We will call it the BS-UIHA operator.

Definition 5. *A BS-UIHA operator is defined by*

$$C_i = \sum_{k=1}^{r} \sum_{j_k=1}^{q_k} m(B_k) w_{j_k} b_{j_k} \tag{8}$$

where w_{j_k} *is the weighting vector of the kth focal element such that* $\sum_{j=1}^{n} w_{j_k} = 1$ *and* $w_{j_k} \in [0,1]$, b_{j_k} *is the* $\hat{a}_i$ ($\hat{a}_i = n\omega_i\tilde{a}_i$, $i = 1,2,\ldots,n$) *value of the UIHA pair* $\langle u_{i_k}, \tilde{a}_{i_k}\rangle$ *having the* j_k*th largest* u_{i_k}, u_{i_k} *is the order inducing variable* $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ *is the weighting vector of the* $\tilde{a}_i$, *with* $\omega_i \in [0, 1]$ *and the sum of the weights is* 1, *and the* $\tilde{a}_{i_k}$ *are interval numbers, and* $m(B_k)$ *is the basic probability assignment.*

As we can see, the focal weights are aggregating the results obtained by using the UIHA operator. Note that if $\omega_i = 1/n$ for all i, then, Eq. (8) is transformed in Eq. (5).

In this case, we could also study different properties and particular cases of the BS-UIHA operator, in a similar way as it has been explained for the BS-UIOWA operator such as the distinction between descending (BS-DUIHA) and ascending (BS-AUIHA) orders.

When aggregating the collection of uncertain payoffs of each focal element, it is also possible to consider a wide range of families of UIHA operators. For example, we could mention the uncertain hybrid maximum, the uncertain hybrid minimum, the uncertain Hurwicz hybrid criteria, the UA, the UWA and the UIOWA operator. These operators are obtained in a similar way as it has been explained in subsection 3.3 excepting for the UWA and the UIOWA. Note that the UWA is found when $w_j = 1/n$, for all j, and the UIOWA operator when $\omega_i = 1/n$, for all i, respectively.

Other families of UIHA operators that could be used are the step-UIHA operator, the window-UIHA, the olympic-UIHA, the S-UIHA, the EZ-UIHA, the median-UIHA, the centered-UIHA, the BADD-UIHA, etc. Note that these families follow a similar methodology as it has been explained for the UIOWA operator.

Finally, if we use the same family of UIHA operator for all the focal elements, then, we can refer to the aggregation as the BS-uncertain hybrid maximum, the BS-uncertain hybrid minimum, the Hurwicz BS-uncertain hybrid criteria, the BS-step-UIHA, the BS-window-UIHA, the BS-olympic-UIHA, the BS-S-UIHA, the BS-centered-UIHA, etc.

5. APPLICATION IN FINANCIAL DECISION MAKING

In the following, we are going to develop an application of the new approach in a decision making problem. We will develop an application in the selection of financial strategies. Note that other decision making applications could be developed such as the selection of investments, financial products, human resources, assets, etc.

We will develop the example considering a wide range of uncertain induced aggregation operators such as the UA, the UWA, the UOWA, the UIOWA and the UIHA operator.

Assume a company is planning its financial strategy for the next year and they consider 5 possible financial strategies to follow.

- A_1 = Financial strategy 1.
- A_2 = Financial strategy 2.
- A_3 = Financial strategy 3.
- A_4 = Financial strategy 4.
- A_5 = Financial strategy 5.

In order to evaluate these financial strategies, the company uses a group of experts. They consider that the key factor is the economic situation of the company for the next year. After careful analysis, the experts have considered five possible situations that could happen in the future: S_1 = Very bad, S_2 = Bad, S_3 = Normal, S_4 = Good, S_5 = Very good.

Depending on the uncertain situations that could happen in the future, the experts establish the uncertain payoff matrix. As the available information about the future benefits of the company is very imprecise, the experts use interval numbers to assess the information. The results are shown in Table 1.

Table 1: Uncertain payoff matrix

	S_1	S_2	S_3	S_4	S_5
A_1	(10,20,30)	(40,50,60)	(70,80,90)	(40,50,60)	(50,60,70)
A_2	(50,60,70)	(30,40,50)	(20,30,40)	(60,70,80)	(40,50,60)
A_3	(70,80,90)	(40,50,60)	(30,40,50)	(30,40,50)	(40,50,60)
A_4	(30,40,50)	(50,60,70)	(20,30,40)	(50,60,70)	(60,70,80)

After careful analysis of the information, the experts have obtained some probabilistic information about which state of nature will happen in the future. This information is represented by the following belief structure about the states of nature.

Focal element

$B_1 = \{S_2, S_3, S_4\} = 0.3$

$B_2 = \{S_1, S_2, S_5\} = 0.3$

$B_3 = \{S_1, S_2, S_3, S_4\} = 0.4$

The attitudinal character of the company is very complex because it involves the opinion of different members of the board of directors. Therefore, the experts use order inducing variables for analysing the attitudinal character of the enterprise. The results are shown in Table 2.

Table 2: Order inducing variables

	S_1	S_2	S_3	S_4	S_5
A_1	30	22	16	35	26
A_2	12	18	24	20	30
A_3	16	11	21	33	25
A_4	30	26	12	18	24

The experts establish the following weighting vectors for both the UWA and the UIOWA operator.

Weighting vector

$W_3 = (0.3, 0.3, 0.4)$

$W_4 = (0.2, 0.2, 0.3, 0.3)$

$W_5 = (0.1,0.2, 0.2, 0.2, 0.3)$

With this information, we can obtain the aggregated payoffs. The results are shown in Table 3.

Table 3: Uncertain aggregated payoffs

	UA	*UWA*	*UOWA*	*UIOWA*	*UIHA*
V_{11}	(50,60,70)	(49,59,69)	(49,59,69)	(52,62,72)	(52,62,72)
V_{12}	(33.3,43.3,53.3)	(35,45,55)	(31,41,51)	(34,44,54)	(40,50,60)
V_{13}	(40,50,60)	(43,53,63)	(37,47,57)	(37,47,57)	(42,51,60)
V_{21}	(36.6,46.6,56.6)	(39,49,59)	(35,45,55)	(36,46,56)	(36,46,56)
V_{22}	(40,50,60)	(40,50,60)	(39,49,59)	(41,51,61)	(37,46.5,56)
V_{23}	(40,50,60)	(40,50,60)	(37,47,57)	(37,47,57)	(32.5,41,49.5)
V_{31}	(33.3,43.3,53.3)	(33,43,53)	(33,43,53)	(34,44,54)	(34,44,54)
V_{32}	(50,60,70)	(49,59,69)	(49,59,69)	(49,59,69)	(44.5,54.5,64.5)
V_{33}	(42.5,52.5,62.5)	(40,50,60)	(40,50,60)	(40,50,60)	(34.5,43,51.5)
V_{41}	(40,50,60)	(41,51,61)	(38,48,58)	(38,48,58)	(38,48,58)
V_{42}	(46.6,56.6,66.6)	(48,58,68)	(45,55,65)	(48,58,68)	(55.5,66,76.5)
V_{43}	(37.5,47.5,57.5)	(37,47,57)	(35,45,55)	(35,45,55)	(34,43,52)

Once we have the aggregated results, we have to calculate the uncertain generalized expected value. The results are shown in Table 4.

Table 4: Uncertain generalized expected value

	UA	*UWA*	*UOWA*	*UIOWA*	*UIHA*
A_1	(41,51,61)	(42.4,52.4,62.4)	(38.8,48.8,58.8)	(40.6,50.6,60.6)	(44.4,54,63.6)
A_2	(39,49,59)	(39.7,49.7,59.7)	(37,47,57)	(37.9,47.9,57.9)	(34.9,44.15,53.4)
A_3	(42,52,62)	(40.6,50.6,60.6)	(40.6,50.6,60.6)	(40.9,50.9,60.9)	(37.35,46.75,56.15)
A_4	(41,51,61)	(41.5,51.5,61.5)	(38.9,48.9,58.9)	(39.8,49.8,59.8)	(41.65,51.4,61.15)

As we can see, depending on the uncertain aggregation operator used, the results and decisions may be different. A further interesting issue is to establish an ordering of the financial strategies. Note that this is very useful when the decision maker wants to consider more than one alternative. The results are shown in Table 5.

Table 5: Ordering of the financial strategies

	Ordering		*Ordering*
UA	$A_3 \succ A_1 = A_4 \succ A_2$	*UIOWA*	$A_3 \succ A_1 \succ A_4 \succ A_2$
UWA	$A_1 \succ A_4 \succ A_3 \succ A_2$	*UIHA*	$A_1 \succ A_4 \succ A_3 \succ A_2$
UOWA	$A_3 \succ A_4 \succ A_1 \succ A_2$		

As we can see, depending on the aggregation operator used, the results and the decisions may be different. With the UA, the UOWA and the UIOWA the optimal choice is A_3. And with the UWA and the UIHA, the best result is A_1.

6. CONCLUSIONS

We have studied the D-S theory of evidence in decision making with uncertain information assessed with interval numbers. By using interval numbers, we can represent uncertain situations where the results are not clear but it is possible to consider the best and worst possible scenarios and the most possible ones. We have also used uncertain induced aggregation operators because it gives more flexibility in the attitudinal character of the decision maker in order to assess complex situations such as the decisions taken by the board of directors of an enterprise. Mainly, we have focussed on the UIOWA and the UIHA operators. Then, we have obtained two new aggregation operators: the BS-UIOWA and the BS-UIHA operator. We have analysed some of the main properties and different particular cases.

We have also developed an application of the new approach in a business decision making problem about selection of financial strategies. We have seen the usefulness of this approach about using probabilities, UWAs and UIOWAs in the same problem. We have also seen that depending on the aggregation operator used, the results and decisions may be different.

In future research, we expect to develop further extensions to this approach by adding new characteristics in the problem and applying it to other decision making problems.

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