

The angular momentum in the classical anisotropic Kepler problem



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Abstract

The behavior of the angular momentum of the two dimensional Anisotropic Kepler Problem (AKP) is addressed. We find here ourselves, from the point of view of physics didactics, with a classical mechanics "simple" problem that should be carefully analyzed from the outset. Taking into account that the angular momentum varies with time due to an "inertial torque", we are still allowed to restrict the problem to a two dimensional motion, and then, being the angular momentum in this restricted case, a one-dimensional variable, we study how its behavior can describe the dynamics of this chaotic system. The approach to this problem through the angular momentum, to our knowledge, has not been reported in the literature. We investigate, from a numerical solution of the equations of motion, different features of this quantity and obtain a return plot for the angular momentum, as well as some phase space diagrams for the torque vs. angular momentum, for different values of the anisotropy parameter, by using a Poincare surface section.

Keywords: anisotropic Kepler problem, chaos.

Resumen

En el presente trabajo estudiamos el comportamiento del momento angular en el problema de Kepler anisotrópico (AKP) en dos dimensiones. Nos encontramos aquí, desde el punto de vista de la didáctica de la física, con un problema "sencillo" de la mecánica clásica, que debe ser analizado con cuidado desde el principio. Tomando en cuenta que el momento angular varía con el tiempo debido a una "torca inercial", es aun posible restringir el problema a un movimiento en dos dimensiones, y de esta manera, siendo el momento angular en este caso restringido una variable unidimensional, analizamos cómo su comportamiento puede describir la dinámica de este sistema caótico. El enfoque de este problema a través del momento angular, hasta donde sabemos, no ha sido reportado en la literatura. A partir de una solución numérica de las ecuaciones, se investigan diferentes características de esta variable y obtenemos una "gráfica de retorno" para el momento angular, así como algunos diagramas de espacio fase de la torca contra momento angular, para diferentes valores del parámetro de anisotropía, utilizando una superficie sección de Poincare.

Palabras clave: Problema de Kepler anisotrópico, caos.

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I. INTRODUCTION

The anisotropic Kepler problem (AKP) originated in the description of an electron close to a donor impurity in a Silicon or Germanium semiconductor [1], inspired the classical AKP, and this Hamiltonian system was one of the first "simple" systems for which the chaos was rigorously proved [2, 3]. It has also been used to study the interplay between classical and quantum mechanics [3]. In fact, as Gutzwiller [4] pointed out, the quantum mechanical systems whose classical behavior is chaotic reveal significant differences in the character of their wave functions, the distribution of their energy levels, among other properties. Most of the analysis of the AKP has been done from numerical calculations for the trajectory, both in the coordinates space and in the phase space. There are also mathematical treatments of the classical AKP [5, 6, 7].

II. THE HAMILTONIAN

The Hamiltonian of the AKP can be reduced from three to two degrees of freedom, by taking into account the symmetry around an axis, and appropriate initial conditions¹.

It is expressed in the form

$$H = (p_x^2 / 2\mu) + (p_y^2 / 2\nu) - G / \sqrt{x^2 + y^2}, \quad (1)$$

where μ and ν are the elements of the mass tensor, which means the existence of different mass parameter for each axis. Here it is not considered a centrifugal potential, which stabilizes the trajectories and in which case the hard

¹In the non-restricted case the motion will be in general, in the three dimensional space. This is due to the fact that the angular momentum is not conserved.

chaos is not produced. As we know, in the ordinary Kepler problem this centrifugal potential, is an pseudo-potential which is due to the constant angular momentum.

This Hamiltonian, which is the energy of the particle, is conserved because the potential is independent of the velocity, just the same as in the ordinary Kepler problem, but the angular momentum for the AKP is no longer constant, except of course in the isotropic limit. As in the ordinary case, here the Hamiltonian does not depend explicitly on time, then it means that it is a constant of motion. If we write this Hamiltonian in polar coordinates, in order to see explicitly the behavior of the angular momentum we start from the transformation equations

$$x = r \cos \theta, \quad (2)$$

$$y = r \sin \theta, \quad (3)$$

and taking the derivatives we write

$$\dot{x} = \dot{r} \cos \theta - r \dot{\theta} \sin \theta, \quad (4)$$

$$\dot{y} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta. \quad (5)$$

The momenta p_x and p_y are related to the velocities in the form

$$p_x = \mu \dot{x}, \quad p_y = \nu \dot{y}, \quad (6)$$

and being r the radial coordinate, $r = \sqrt{x^2 + y^2}$, we write from Eq.(1)

$$H = \frac{1}{2} \mu \dot{x}^2 + \frac{1}{2} \nu \dot{y}^2 - \frac{G}{r}, \quad (7)$$

and if we substitute Eqs.(2)-(5) in this expression, we obtain

$$H = 1/2[\dot{r}^2(\mu \cos^2 \theta + \nu \sin^2 \theta) + r^2 \dot{\theta}^2(\mu \sin^2 \theta + \nu \cos^2 \theta) - 2r \dot{\theta} \sin \theta \cos \theta(\mu - \nu)] - G/r. \quad (8)$$

In this expression we see that as the two mass parameters are different, $\mu \neq \nu$, it is not possible to eliminate explicitly the angular variable, θ therefore H is not invariant under an angular translation, which means that the conjugate variable, the angular momentum, is not a constant of motion.²

Yoshida [8] obtained a criterion for the non-integrability of the AKP. Besides the total energy, he proved the non-existence of an additional constant of motion of the problem. Gutzwiller [4] made an analysis of the periodic orbits in the AKP, starting from the one-to-one relation between trajectories and the binary sequences obtained from the sign of the x coordinate of the particle,

taken each time it crosses the x axis. He obtains a formula to fit the action of each periodic orbit, from the numerical data of the binary sequences.

In all the literature mentioned, it seems that the behavior of the angular momentum has not been described. For the two-dimensional Hamiltonian considered, the orbit remains in a plane, then the angular momentum, always directed normally to it, is a one-dimensional variable. In this work we explore the dynamics of the angular momentum, we start by pointing out its time variation, as well as that of the "inertial" torque. Then we make different graphs which involve the angular momentum; those are a return plot from a time series of the variable and some phase space diagrams using a Poincare surface of section, for different values of the anisotropy parameter $\zeta = \mu/\nu$.

III. ANGULAR MOMENTUM

The angular momentum of the particle is by definition

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = (x p_y - y p_x) \mathbf{e}_z, \quad (9)$$

where \mathbf{e}_z is an unitary vector along the z axis. The torque is expressed as

$$\mathbf{N} = d\mathbf{L} / dt = \mathbf{v} \times \mathbf{p} + \mathbf{r} \times \mathbf{F}, \quad (10)$$

the second term of this equation vanishes because the field of force is central, whereas the first term is different from zero, because in this case, being the mass a tensor, then \mathbf{v} and \mathbf{p} are not collinear and the torque is written as

$$\mathbf{N} = (v_x p_y - v_y p_x) \mathbf{e}_z = (\nu - \mu) v_x v_y \mathbf{e}_z, \quad (11)$$

so, this inertial torque is not produced by the force, but by the anisotropy of the mass.³ It is proportional to the difference of the two mass parameters and to the product of the two components of the velocity. From this equation it is expected that near the origin, due to the energy conservation, the velocity is very high compared with regions far away from this point. This gives the torque, as we will see in the numerical calculations, an impulsive character.

The angular momentum will experience variations that can oscillate within certain intervals, it will change between positive and negative values and it will have abrupt and irregular variations.

Suppose that we start the motion of the particle by putting it initially at rest at some radial distance r_0 from the origin. From the Eq. (1) its total energy will be

²We point out here that, if in the last expression for the Hamiltonian, Eq. (8), we put $\mu = \nu$, which corresponds to the isotropic case, then it is immediate to see that θ disappears explicitly, and then the angular momentum is conserved.

³We could think in a similar, but quite different problem where the anisotropy is in the field of force. In this case the field is not conservative (irrotational), the angular momentum and the torque assume a different form, but the equations of motion turn out to be mathematically equivalent. So, there is also one integral of motion which is not the energy.

$$H = E = -\frac{G}{r_0}, \quad (12)$$

which is a negative value because the particle is confined by the potential. From there on the particle will orbit around the origin, and whenever its radial distance be again equal to r_0 , from Eq. (1) its kinetic energy will be zero, which means that it has reached the rest momentarily. Therefore those points for which $r = r_0$ are turning points for the orbit. So we see that the allowed region for the trajectory of the particle is a circle of radius r_0 , which is obtained, according to Eq. (12), as the quotient $-G/E$.

We point out here a qualitative difference between this problem and the ordinary Kepler problem. When we put initially the particle at rest at any point of this boundary, its initial angular momentum and torque are zero, but then as the particle travels toward the origin it acquires angular momentum and will not collide in the very first approach, as it would in the ordinary Kepler problem if the angular momentum is zero.

Then, the angular momentum for this system seems to be a relevant quantity to study.

IV. HAMILTON EQUATIONS

Here we start from the Hamiltonian in the coordinate space xy , Eq. (1), and write the corresponding Hamilton equations [9]

$$\dot{x} = \partial H / \partial p_x = p_x / \mu, \quad (13)$$

$$\dot{y} = \partial H / \partial p_y = p_y / \nu, \quad (14)$$

$$\dot{p}_x = -\partial H / \partial x = -Gx / (x^2 + y^2)^{3/2}, \quad (15)$$

$$\dot{p}_y = -\partial H / \partial y = -Gy / (x^2 + y^2)^{3/2}. \quad (16)$$

Therefore, combining these equations one obtains

$$\ddot{x} = -(G/\mu)x / (x^2 + y^2)^{3/2}, \quad (17)$$

$$\ddot{y} = -(G/\nu)y / (x^2 + y^2)^{3/2}. \quad (18)$$

V. NUMERICAL RESULTS

For the numerical calculations we are using an arbitrary value for the constant G , and we give the system an energy for which the radius of the circular boundary is the unity, so that the x and y coordinates vary between -1 and 1. In all the results and graphs of this work we use a fourth order Runge-Kuta integration for the solution of the equations.

In Figures 1 and 2 we have two examples of paths of the particle, one of them starting in the x axis, with an initial momentum directed along the positive y axis, and the other starts from the rest, at some point P in the circular boundary. Whenever the trajectory gets that boundary, it

should come to rest, and that means a velocity reversal, in which case we may have periodic open orbits; those points reached by the trajectory are turning points of the orbit. In Figure 1 we see that the path is close to a periodic open orbit, which could be found by modifying slightly the initial conditions or the anisotropy parameter.

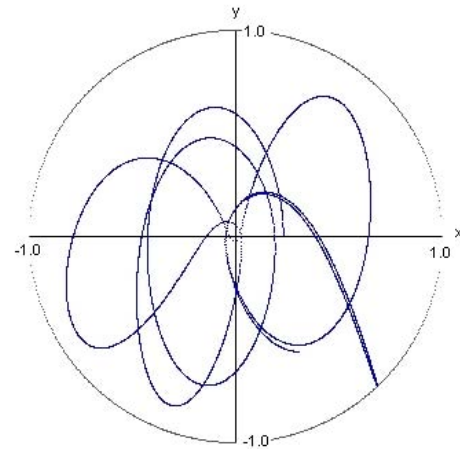


FIGURE 1. Trajectory of the particle starting at the point in the horizontal axis, $x_i = 0.23$, and the asymmetry parameter being $\zeta \equiv \mu/\nu = 2.94$. The unit circle is the boundary of the orbit. We see that in the fourth quadrant the orbit gets close to a turning point.

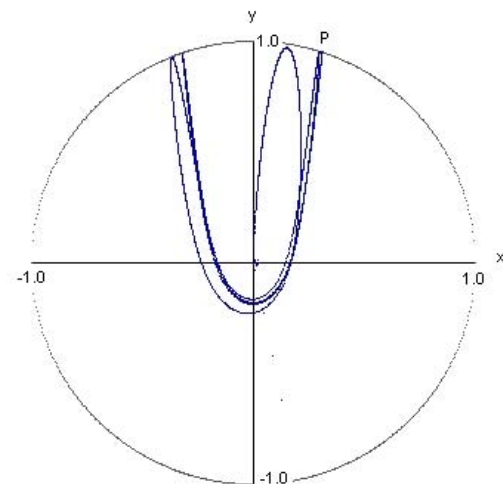


FIGURE 2. Trajectory of the particle starting from the rest at a point P in the boundary, $x_i = 0.31$ and $y_i = \sqrt{1 - x_i^2}$ and $\zeta = 2.94$. As we see, this orbit is close to an open periodic orbit; at the end it goes to a collision with the origin.

In Figure 3 we have a plot for the angular momentum and the torque as a function of time, corresponding to the trajectory of Figure 1 (taking a longer time). With the numerical values of coordinates and momenta we evaluate the expressions (9) and (10). We see here the irregular oscillations of the angular momentum, around the zero

value, and for the torque we appreciate, as we pointed out before, an impulsive character, when the particle approaches the origin, and it goes to a small value when it is far away.

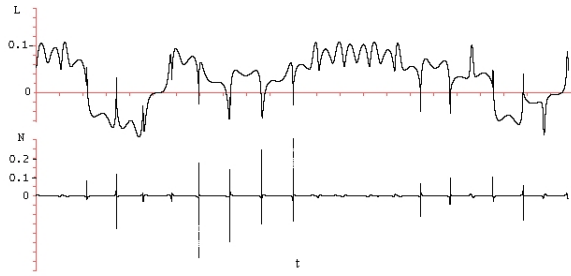


FIGURE 3. At the top we have the angular momentum as a function of time (arbitrary units). This corresponds to the orbit of Figure 1, at a longer elapsed time. The lower graph is the torque in the same time scale.

In order to explore the structure in the variation of the angular momentum, we take a time series of this variable, from its values at a number k , of fixed time intervals, and from it we make in Figure 4, the return plot $L(k+8)$ versus $L(k)$. This shows clearly some isolated fixed points distributed along the diagonal at the angle $\pi/4$, where $L(k+8) = L(k)$. In those fixed points the variation of the angular momentum tends to zero, and as the torque is zero, we see that those points should be near the boundary. This means that near the boundary the angular momentum tends to some fixed values, and those particular values are characteristic of the given path; they will change with the initial conditions and with the ζ parameter.

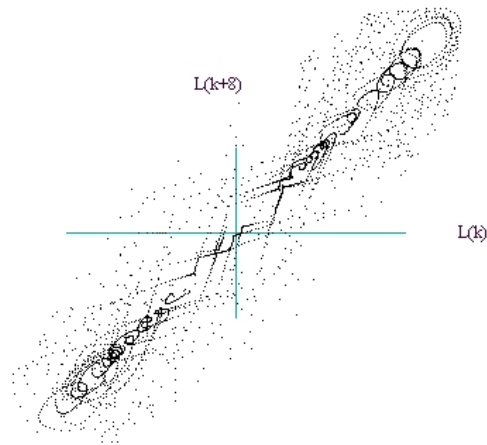


FIGURE 4. A return plot of the angular momentum, obtained from a time series of L , which shows the variation of $L(k+8)$ respect to $L(k)$. We observe several fixed points distributed along the $N = 0$ axis, given by $L(k+8) = L(k)$. The values of the parameters are: $x_i = 0.29$, $y_i = 0$ and $\zeta = 2.94$, the length of each interval was taken as $10^3 \Delta t$, and the number of points is $k = 3500$.

In Figure 5, as in the rest of the diagrams, we use a Poincare surface of section, taken as the x axis, to make a phase space plot for the x coordinate and its respective linear momentum p_x . This diagram describes the projection

of the dynamics of the system on the plane $y = 0$, taking into account that the Hamiltonian, (the energy of the system) is an integral of motion. We find that for $\zeta < 1.15$, where we are close to the isotropic case, the collisions are more frequent, and there is almost no structure in the diagram. There is a defined structure for $1.15 < \zeta < 1.5$. For this plot we are using $\zeta = 1.23$ and for the initial values: $x_i = 0.4$, $y_i = 0$, $p_{ix} = 0$ and $p_{iy} = \sqrt{2vG(1/x_i - 1) - p_{ix}^2/\zeta}$ (so this quantity is known from the energy value, and will be used from here on). We observe two symmetric saddle points lying in the horizontal axis and layers of attractors in the second and fourth quadrants. This diagram is similar to that obtained by Bai and Zheng [7] using spherical coordinates.

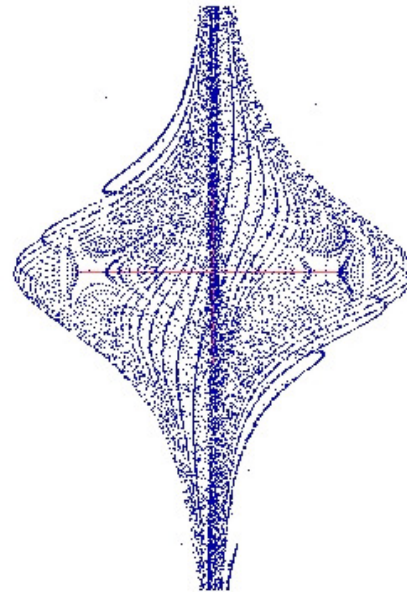


FIGURE 5. A plane phase space for x and p_x , taken a Poincare surface of section as the x axis. The values of the parameters are: $x_i = 0.64$, $y_i = 0$ and $\zeta = 1.21$.

In Figures 6 to 8 we have plots of the phase space diagram for the angular momentum, (the torque versus the angular momentum). This diagram formally gives the same description as a return plot, see Figure 4; but here instead of using a fixed time interval, we are taking the values of the variable from a surface section given by the x axis. This plot shows a complex structure which characterizes this variable. This structure occurs mainly for a region of small torque values, non small angular momentum, and for values of ζ between 1.15 and 1.5. In all these diagrams there is a central vertical region with no structure. In Figure 6 we observe quite clearly, a symmetry in the diagram with respect to the positive-slope diagonal. For values of $\zeta > 1.5$ the width of L values becomes narrower, see Figure 7, the diagram loses structure, and as ζ increases, it tends gradually to a pair of values of L , one negative of the other. That is, for $\zeta > 6$, L tends to a dichotomic behavior, as we appreciate in the Figure 8.

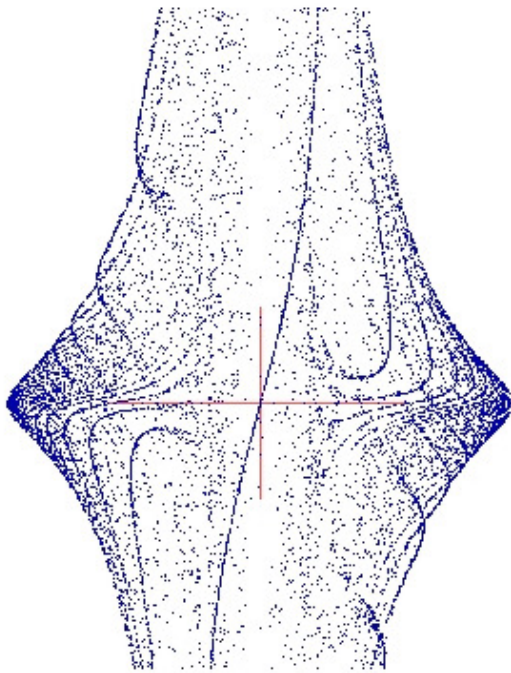


FIGURE 6. A plane phase space for angular momentum L and torque N , using also a Poincare surface of section, as the x axis. The parameters are $x_i = 0.64$, $y_i = 0$ and $\zeta = 1.21$.

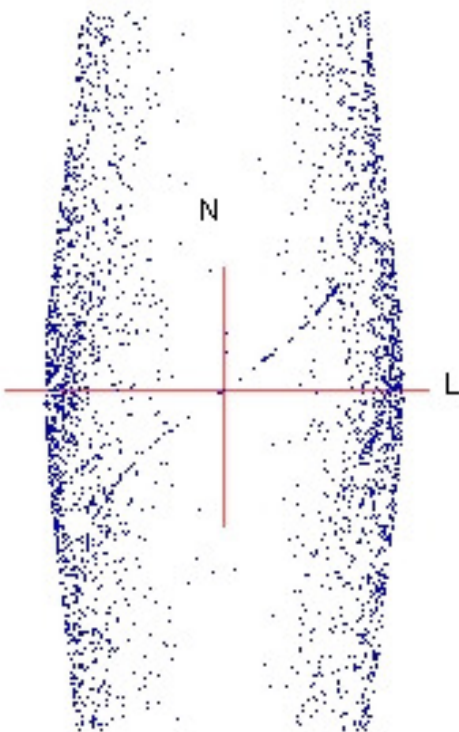


FIGURE 7. A similar plot with the parameters $x_i = 0.64$, $y_i = 0$ and $\zeta = 5.0$.

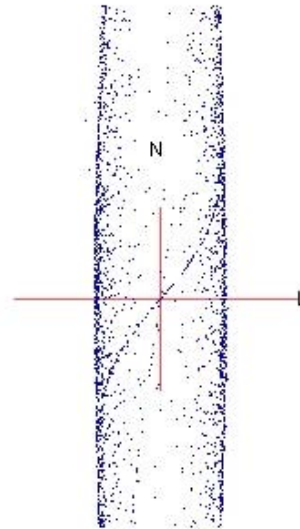


FIGURE 8. A similar plot with the parameters $x_i = 0.64$, $y_i = 0$ and $\zeta = 15.0$.

VI. SOME CONCLUSIONS

We have investigated the behavior of the angular momentum of the two-dimensional classical anisotropic Kepler problem. In spite of having here a central field of force, we observe that the angular momentum varies with time due to the presence of an “inertial torque”. We see how the orbits lie in a circle whose radius depends on the energy value. The boundary of this circle acts as a turning point whenever the particle reaches there. This means that there can be periodic non closed orbits. By means of a numerical solution of the equations of motion, we study the behavior of the angular momentum. We exhibit some return plots where they appear some fixed points, which are characteristic of each particular trajectory. Those fixed points occur at zero torque, which means that near the boundary the angular momentum tends to some fixed values. We also obtain phase space diagrams for the torque and angular momentum within some particular regions for the values of the main parameters of the system, which are the asymmetry parameter and the initial conditions.

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