

SIMULTANEOUS CONFIDENCE INTERVALS FOR VARIANCE COMPONENTS IN TWO-WAY BALANCED CROSSED CLASSIFICATION RANDOM EFFECTS MODEL WITH INTERACTION

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ABSTRACT

In this work simultaneous confidence intervals for the variance components in the two-way balanced crossed random effects model with interaction have been derived under the usual assumptions of normality and independence of random effects. The intervals are conservative in the sense that the true confidence coefficient is as large as preassigned value. The formulas are illustrated using published data with SAS outputs.

KEY WORDS: Statistical modeling, Estimation, Mixed linear models, Normality and independence assumptions.

MSC 62J10

RESUMEN

En este trabajo se derivan intervalos de confianza simultáneos para los componentes de la varianza en un modelo de efectos aleatorios balanceado de dos vías cruzado con interacción, bajo los supuestos usuales de normalidad e independencia de efectos aleatorios. Los intervalos de confianza son conservadores en el sentido de que el verdadero coeficiente de confianza es tan grande como un valor preasignado. Las fórmulas son ilustradas usando datos publicados y se presentan salidas en SAS.

1. INTRODUCTION

Variance components are the different sources to the variation in an observation. Random effects and mixed linear models are useful in applications that require accounting for components of variability arising from multiple sources. In the study of random and mixed effects models, our interest lies primarily in making inferences about the variance components. Variance components were first employed by Fisher (1918) in connection with genetic research on Mendelian laws of inheritance. Tippett (1931) used variance components to determine a method of optimal sampling design. Daniels (1939) discussed the application of variance components methodology to an investigation of factors that cause unevenness in wool. Early applications of variance components were mainly in genetics and sampling design. Variance components have been of importance in diverse fields of research and applications, for example in animal breeding studies, in biology, in psychology, in industrial applications, behavioral and educational research, and medical research, among others. See, among many others Rao (1997), Raudenbush et al. (2002), Searle et al. (1992), Verbeke and Molenberghs (2000).

Confidence intervals are needed to quantify the uncertainty associated with the point estimates. Confidence interval for variance components have been an important topic of research for over 70 years. Numerous articles have been written on this topic by many authors. See for instance, Arendacká (2005), Bottai and Orsini (2004), Burdick and Graybill (1984), Burch and Iyer (1997), Hartung and Knapp (2000), Taoufik et al. (2007). Most of these papers are concerned with

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developing exact or approximate confidence intervals for specified linear functions of variance components or their ratios. A collection of confidence intervals for different functions of parameters based on the same sample is known as simultaneous confidence intervals (Burdick and Graybill, 1992). The deluge of papers on various aspects of variance components in last several years was in sharp contrast to the trickle of papers dealing with simultaneous confidence intervals of variance components or the variance ratios (See bibliography by Sahai and Khurshid, 2005).

Hartley and Rao (1967) derived a general procedure for the construction of the exact confidence regions for the variance ratios of a general mixed model. The main attribute of their confidence regions is their applicability for a general unbalanced design that includes fixed effects as well as random effects. Their drawback is that these regions are functions of the unknown variance components and are difficult to reduce the general formula to a simpler form even for the simple balanced design models such as the three-stage nested and the two-way crossed classification random effects models.

Simultaneous confidence intervals for the ratios of variance components in balanced random models were proposed by Broemeling (1969 a). Using Kimbal's (1951) inequality Broemeling derived confidence regions for the variance ratios. These regions are polygons and provide simultaneous confidence intervals for the variance ratios. However, the resulting intervals are conservative, that is, the actual confidence level is greater than or equal to the pre-set value. See also Broemeling (1969 b).

The exact confidence coefficients associated with Broemeling's confidence regions were obtained by Sahai and Anderson (1973) in terms of the upper tail of probability integrals of the inverted Dirichlet distribution. Simultaneous confidence intervals for the variance components, excluding the residual component, in two-fold nested and two-way cross-classification random models were derived by Sahai (1974) and are also conservative. Broemeling and Bee (1976), using Kimball's inequality, obtained similar regions for balanced incomplete block random model. Broemeling (1978) extended one-sided simultaneous confidence intervals to two-sided. Tong (1979), however, showed that two-sided simultaneous confidence intervals do not always hold true by giving counter-examples based on mixed linear models. Khuri (1981) proposed a technique for the construction of simultaneous confidence intervals for the values of all continuous functions of the variance components for a general balanced random model. This technique can be easily applied to a wide range of functions that may be of interest to research workers. Confidence intervals obtained by this technique, however, are conservative.

There are two SAS macros to compute simultaneous confidence intervals. The two macros, called *%SimultanTests* and *%SimultanIntervals*, are based on exact evaluations of the underlying multivariate *t* distribution and are extensions of the published macros *%SimTests* and *%SimIntervals*. The macros are widely applicable tools, see Frömke and Bretz (2004).

The description of two-way balanced crossed classification random effects model is available in the literature, see Sahai and Ojeda (2004). Sahai (1974) considered three-stage nested random effects model and two-way crossed classification random effects model without interaction and developed formulas for the simultaneous confidence intervals for variance components excluding the error variance component. The technique is based on combining two or more intervals about the function of parameters from the experimental data. Recently Sahai and Ojeda (2004) considered simultaneous confidence intervals involving two-way crossed classification with and without interaction among others.

The object of present investigation is to develop simultaneous confidence intervals for the variance components of two-way balanced crossed classification random effects model with interaction excluding the error variance component.

2. TWO-WAY BALANCED RANDOM EFFECTS MODEL WITH INTERACTION

The model can be written as

$$Y_{ijk} = \mu + a_i + b_j + \gamma_{ij} + e_{ijk} \quad (i = 1, 2, \dots, p; j = 1, 2, \dots, q \text{ and } k = 1, 2, \dots, r) \quad (1)$$

where $-\infty < \mu < \infty$ is the general constant and a_i, b_j, γ_{ij} and e_{ijk} are mutually independent normal random variables with zero means and respective variances $\sigma_a^2, \sigma_b^2, \sigma_\gamma^2$ and σ_e^2 ($0 \leq \sigma_a^2, \sigma_b^2, \sigma_\gamma^2, \sigma_e^2 < \infty$). The parameters $\sigma_a^2, \sigma_b^2, \sigma_\gamma^2$ and σ_e^2 are known as variance components.

The usual sum of squares S_a^2, S_b^2, S_γ^2 and S_e^2 of ANOVA are independent and further

$$S_a^2 \sim (\sigma_e^2 + r\sigma_\gamma^2 + qr\sigma_a^2)\chi_{[p-1]}^2, \quad (2)$$

$$S_b^2 \sim (\sigma_e^2 + r\sigma_\gamma^2 + pr\sigma_b^2)\chi_{[q-1]}^2, \quad (3)$$

$$S_\gamma^2 \sim (\sigma_e^2 + r\sigma_\gamma^2)\chi_{[(p-1)(q-1)]}^2, \quad (4)$$

$$S_e^2 \sim (\sigma_e^2)\chi_{[pq(r-1)]}^2, \quad (5)$$

where $\chi_{[n]}^2$ is central chi-square distribution with n degrees of freedom (d.f.)

Let

$$\begin{aligned} n_1 &= pq(r-1), n_2 = (p-1)(q-1), n_3 = q-1, n_4 = p-1, \\ \chi_1^2 &= \chi_\ell^2[\alpha_1; n_4], \chi_2^2 = \chi_u^2[\alpha_1; n_4], \chi_3^2 = \chi_\ell^2[\alpha_2; n_2], \\ \chi_4^2 &= \chi_u^2[\alpha_2; n_2], \chi_5^2 = \chi_\ell^2[\alpha_3; n_3], \chi_6^2 = \chi_u^2[\alpha_3; n_3], \\ F_1 &= F_\ell[1-\alpha_1; n_4, n_1], F_2 = F_u[1-\alpha_1; n_4, n_1], \\ F_3 &= F_\ell[1-\alpha_2; n_2, n_1], F_4 = F_u[1-\alpha_2; n_2, n_1], \\ F_5 &= F_\ell[1-\alpha_3; n_3, n_1], F_6 = F_u[1-\alpha_3; n_3, n_1], \\ F_7 &= F_\ell[1-\alpha_4; n_4, n_1], F_8 = F_u[1-\alpha_4; n_4, n_1], \\ F_9 &= F_\ell[1-\alpha_5; n_3, n_2], F_{10} = F_u[1-\alpha_5; n_3, n_2], \end{aligned}$$

where $\chi_\ell^2[\alpha; n]$ and $\chi_u^2[\alpha; n]$ are lower and upper χ^2 -limits which enclose $(1-\alpha)$ probability of the distribution with n d.f. and $F_\ell[1-\alpha; n_2, n_1]$ and $F_u[1-\alpha; n_2, n_1]$ are lower and upper limits of Snedecor's F-distribution with n_2 and n_1 d.f. which enclose $(1-\alpha)$ probability of the distribution.

Below we derive the simultaneous confidence intervals for $(\sigma_a^2, \sigma_\gamma^2)$, $(\sigma_b^2, \sigma_\gamma^2)$ and $(\sigma_a^2, \sigma_b^2, \sigma_\gamma^2)$ for known value of σ_e^2 .

The simultaneous confidence interval for (σ_a^2, σ_b^2) is derived for the known values of σ_e^2 and σ_γ^2 .

2.1 Simultaneous confidence interval for $(\sigma_a^2, \sigma_\gamma^2)$

From (2), (4) and (5) the usual $100(1-\alpha_1)\%$ confidence intervals about parametric functions $(\sigma_e^2 + r\sigma_\gamma^2 + qr\sigma_a^2)$ and $(\sigma_\gamma^2/\sigma_e^2 + q\sigma_a^2/\sigma_e^2)$ are given by

$$\left\{ \frac{S_a^2}{\chi_2^2} \leq (\sigma_e^2 + r\sigma_\gamma^2 + qr\sigma_a^2) \leq \frac{S_a^2}{\chi_1^2} \right\} \quad (6)$$

and

$$\left\{ \frac{1}{r} \left(\frac{n_1 S_a^2}{n_4 S_e^2 F_2} - 1 \right) \leq \left(\frac{\sigma_\gamma^2}{\sigma_e^2} + q \frac{\sigma_a^2}{\sigma_e^2} \right) \leq \frac{1}{r} \left(\frac{n_1 S_a^2}{n_4 S_e^2 F_1} - 1 \right) \right\}. \quad (7)$$

Similarly the $100(1-\alpha_2)\%$ confidence intervals about the parametric functions $(\sigma_e^2 + r\sigma_\gamma^2)$ and $\sigma_\gamma^2/\sigma_e^2$ are

$$\left\{ \frac{S_\gamma^2}{\chi_4^2} \leq (\sigma_e^2 + r\sigma_\gamma^2) \leq \frac{S_\gamma^2}{\chi_3^2} \right\} \quad (8)$$

and

$$\left\{ \frac{1}{r} \left(\frac{n_1 S_\gamma^2}{n_2 S_e^2 F_4} - 1 \right) \leq \frac{\sigma_\gamma^2}{\sigma_e^2} \leq \frac{1}{r} \left(\frac{n_1 S_\gamma^2}{n_2 S_e^2 F_3} - 1 \right) \right\}. \quad (9)$$

From (6) and (8) a set of simultaneous confidence interval about $(\sigma_e^2 + r\sigma_\gamma^2 + qr\sigma_a^2)$ and $(\sigma_e^2 + r\sigma_\gamma^2)$ with confidence coefficient $(1 - \beta_1)$ is

$$\left\{ \frac{S_a^2}{\chi_2^2} \leq (\sigma_e^2 + r\sigma_\gamma^2 + qr\sigma_a^2) \leq \frac{S_a^2}{\chi_1^2}, \quad \frac{S_\gamma^2}{\chi_4^2} \leq (\sigma_e^2 + r\sigma_\gamma^2) \leq \frac{S_\gamma^2}{\chi_3^2} \right\}, \quad (10)$$

where $1 - \beta_1 = (1 - \alpha_1)(1 - \alpha_2)$. The equality holds because the intervals (6) and (8) are statistically independent.

From (7) and (9) a set of simultaneous confidence interval about $\left(\frac{\sigma_\gamma^2}{\sigma_e^2} + q \frac{\sigma_a^2}{\sigma_e^2} \right)$ and $\sigma_\gamma^2 / \sigma_e^2$ with confidence coefficient $(1 - \beta_2)$ is

$$\left\{ \frac{1}{r} \left(\frac{n_1 S_a^2}{n_4 S_e^2 F_2} - 1 \right) \leq \left(\frac{\sigma_\gamma^2}{\sigma_e^2} + q \frac{\sigma_a^2}{\sigma_e^2} \right) \leq \frac{1}{r} \left(\frac{n_1 S_a^2}{n_4 S_e^2 F_1} - 1 \right), \frac{1}{r} \left(\frac{n_1 S_\gamma^2}{n_2 S_e^2 F_4} - 1 \right) \leq \frac{\sigma_\gamma^2}{\sigma_e^2} \leq \frac{1}{r} \left(\frac{n_1 S_\gamma^2}{n_2 S_e^2 F_3} - 1 \right) \right\} \quad (11)$$

where $1 - \beta_2 \geq (1 - \alpha_1)(1 - \alpha_2)$. The inequality using Kimbal's (1951) result holds because the intervals (7) and (9) are not independent. See also Miller (1967, p. 102).

Thus for any fixed value of σ_e^2 , a $100(1 - \beta_1)\%$ confidence region for $(\sigma_a^2, \sigma_\gamma^2)$ is

$$R_1(\sigma_e^2) = \left[(\sigma_a^2, \sigma_\gamma^2) : \frac{1}{r} \left(\frac{S_a^2}{\chi_2^2} - \sigma_e^2 \right) \leq (\sigma_\gamma^2 + q\sigma_a^2) \leq \frac{1}{r} \left(\frac{S_a^2}{\chi_1^2} - \sigma_e^2 \right), \frac{1}{r} \left(\frac{S_\gamma^2}{\chi_4^2} - \sigma_e^2 \right) \leq \sigma_\gamma^2 \leq \frac{1}{r} \left(\frac{S_\gamma^2}{\chi_3^2} - \sigma_e^2 \right) \right].$$

Similarly a $100(1 - \beta_2)\%$ confidence region is

$$R_2(\sigma_e^2) = \left[(\sigma_a^2, \sigma_\gamma^2) : \frac{\sigma_e^2}{r} \left(\frac{n_1 S_a^2}{n_4 S_e^2 F_2} - 1 \right) \leq (\sigma_\gamma^2 + q\sigma_a^2) \leq \frac{\sigma_e^2}{r} \left(\frac{n_1 S_a^2}{n_4 S_e^2 F_1} - 1 \right), \right. \\ \left. \left(\frac{\sigma_e^2}{r} \left(\frac{n_1 S_\gamma^2}{n_2 S_e^2 F_4} - 1 \right) \leq \sigma_\gamma^2 \leq \frac{\sigma_e^2}{r} \left(\frac{n_1 S_\gamma^2}{n_2 S_e^2 F_3} - 1 \right) \right) \right].$$

For given value of σ_e^2 we have

$$P\{(\sigma_a^2, \sigma_\gamma^2) \in R_1(\sigma_e^2) \cap R_2(\sigma_e^2) | \sigma_e^2\} \geq 1 - \beta_1 - \beta_2.$$

Now the intersection of $R_1(\sigma_e^2)$ and $R_2(\sigma_e^2)$ averaged over any distribution of σ_e^2 gives

$$P\{(\sigma_a^2, \sigma_\gamma^2) \in R_1 \cap R_2\} = \int_0^1 P(\sigma_a^2, \sigma_\gamma^2) \in R_1(\sigma_e^2) \cap R_2(\sigma_e^2) \mid \sigma_e^2 dF(\sigma_e^2) \geq 1 - \beta_1 - \beta_2.$$

Thus the simultaneous confidence interval for $(\sigma_a^2, \sigma_\gamma^2)$ determined by $R_1(\sigma_e^2)$ with confidence coefficient $(1 - \beta_1)$ is

$$\left\{ \frac{1}{qr} \left(\frac{S_a^2}{\chi_2^2} - \sigma_e^2 \right) \leq \sigma_a^2 \leq \frac{1}{qr} \left(\frac{S_a^2}{\chi_1^2} - \sigma_e^2 \right), \frac{1}{r} f_1(S_a^2, S_\gamma^2) \leq \sigma_\gamma^2 \leq \frac{1}{r} f_2(S_a^2, S_\gamma^2) \right\} \quad (12)$$

where

$$f_1(S_a^2, S_\gamma^2) = \min \left\{ \left(\frac{S_\gamma^2}{\chi_4^2} - \sigma_e^2 \right), \left(\frac{S_a^2}{\chi_2^2} - \sigma_e^2 \right) \right\}$$

and

$$f_2(S_a^2, S_\gamma^2) = \min \left\{ \left(\frac{S_\gamma^2}{\chi_3^2} - \sigma_e^2 \right), \left(\frac{S_a^2}{\chi_1^2} - \sigma_e^2 \right) \right\}.$$

From $R_2(\sigma_e^2)$ the simultaneous confidence interval for $(\sigma_a^2, \sigma_\gamma^2)$ is

$$\left\{ \frac{\sigma_e^2}{qr} \left(\frac{n_1 S_a^2}{n_4 S_e^2 F_2} - 1 \right) \leq \sigma_a^2 \leq \frac{\sigma_e^2}{qr} \left(\frac{n_1 S_a^2}{n_4 S_e^2 F_1} - 1 \right), \frac{\sigma_e^2}{r} g_1(S_a^2, S_\gamma^2, S_e^2) \leq \sigma_\gamma^2 \leq \frac{\sigma_e^2}{r} g_2(S_a^2, S_\gamma^2, S_e^2) \right\} \quad (13)$$

where

$$g_1(S_a^2, S_\gamma^2, S_e^2) = \min \left\{ \left(\frac{n_1 S_\gamma^2}{n_2 S_e^2 F_4} - 1 \right), \left(\frac{n_1 S_a^2}{n_4 S_e^2 F_2} - 1 \right) \right\},$$

and

$$g_2(S_a^2, S_\gamma^2, S_e^2) = \min \left\{ \left(\frac{n_1 S_\gamma^2}{n_2 S_e^2 F_3} - 1 \right), \left(\frac{n_1 S_a^2}{n_4 S_e^2 F_2} - 1 \right) \right\}.$$

For given sample results $R_1(\sigma_e^2)$ and $R_2(\sigma_e^2)$ will form a region of intersection which will be bounded by the intersection of the upper limits and the intersection of the lower limits of $(\sigma_a^2, \sigma_\gamma^2)$ determined by $R_1(\sigma_e^2)$ and $R_2(\sigma_e^2)$.

Thus the simultaneous confidence interval for $(\sigma_a^2, \sigma_\gamma^2)$ is given by

$$P \left[\frac{n_1 S_a^2 - n_4 S_e^2 F_2}{n_1 qr \chi_2^2} \leq \sigma_a^2 \leq \frac{n_1 S_a^2 - n_4 S_e^2 F_1}{n_1 qr \chi_1^2}, h_1(S_a^2, S_\gamma^2, S_e^2) \leq \sigma_\gamma^2 \leq h_2(S_a^2, S_\gamma^2, S_e^2) \right] \geq 1 - \beta_1 - \beta_2 \quad (14)$$

where

$$h_1(S_a^2, S_\gamma^2, S_e^2) = \min \left\{ \frac{n_1 S_\gamma^2 - n_2 S_e^2 F_4}{n_1 r \chi_4^2}, \frac{n_1 S_a^2 - n_4 S_e^2 F_2}{n_1 r \chi_2^2} \right\},$$

and

$$h_2(S_a^2, S_\gamma^2, S_e^2) = \min \left\{ \frac{n_1 S_\gamma^2 - n_2 S_e^2 F_3}{n_1 r \chi_3^2}, \frac{n_1 S_a^2 - n_4 S_e^2 F_1}{n_1 r \chi_1^2} \right\}.$$

To illustrate the technique of constructing simultaneous confidence interval consider the example from Box and Tiao (1973, p. 337).

We used SAS GLM to compute the Sum of Squares. The relevant portions of the output are displayed in Table 1.

DATA krus; INPUT driver car mile; CARDS;		General Linear Models Procedure					
		Dependent Variable: MILE					
1 1 32.431		Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
1 1 31.709		Model	80	1483.36569583	18.54207120	15.77	0.0001
.....		Error	81	95.24601800	1.17587677		
9 9 27.638		Corrected Total	161	1578.61171383			
9 9 27.385							
;		R-Square		C.V.	Root MSE	MILE Mean	
proc glm;		0.939665		3.912204	1.08437852	27.71783951	
class driver car;		Source	DF	Type I SS	Mean Square	F Value	Pr > F
model mile=driver car		DRIVER	8	1011.65496616	126.45687077	107.54	0.0001
driver*car;		CAR	8	362.09848372	45.26231046	38.49	0.0001
random driver car		DRIVER*CAR	64	109.61224595	1.71269134	1.46	0.0547
driver*car;							
run;		Source	DF	Type III SS	Mean Square	F Value	Pr > F
		DRIVER	8	1011.65496616	126.45687077	107.54	0.0001
		CAR	8	362.09848372	45.26231046	38.49	0.0001
		DRIVER*CAR	64	109.61224595	1.71269134	1.46	0.0547
		General Linear Models Procedure					
		Source	Type III Expected Mean Square				
		DRIVER	Var(Error) + 2 Var(DRIVER*CAR) + 18 Var(DRIVER)				
		CAR	Var(Error) + 2 Var(DRIVER*CAR) + 18 Var(CAR)				
		DRIVER*CAR	Var(Error) + 2 Var(DRIVER*CAR)				

Table 1. SAS Applications: This application illustrates SAS GLM instructions and output.

Here $p=q=9, r=2, S_a^2=362.0985, S_b^2=1011.655, S_\gamma^2=109.6123$ and $S_e^2=95.246$. Thus $n_1=81, n_2=64, n_3=8$ and $n_4=8$, choosing $\alpha_1=\alpha_2=0.02$, and taking equal tail probabilities we get,

$$\chi_1^2 = 1.646, \chi_2^2 = 20.09, \chi_3^2 = 39.984, \chi_4^2 = 94.2709$$

$$\chi_5^2 = 1.646, \chi_6^2 = 20.09, F_1 = 0.20, F_2 = 2.74,$$

$$F_3 = 0.5682, F_4 = 1.70, F_5 = 0.20, F_6 = 2.74,$$

$$F_7 = 0.1987, F_8 = 2.79, F_9 = 0.1987, F_{10} = 2.79.$$

We used SAS VARCOMP and MIXED to estimate the variance components. The relevant portions of the output are displayed in Tables 2 and 3.

DATA krus; INPUT driver car mile; CARDS; 1 1 32.431 1 1 31.709 9 9 27.638 9 9 27.385 ; proc mixed covtest; class driver car; model mile=; random driver car driver*car; run;		The MIXED Procedure				
		Covariance Parameter Estimates (REML)				
		Cov Parm	Estimate	Std Error	Z	Pr > Z
		DRIVER	6.93023219	3.51273113	1.97	0.0485
		CAR	2.41942328	1.25739891	1.92	0.0543
		DRIVER*CAR	0.26840729	0.17734598	1.51	0.1302
		Residual	1.17587677	0.18477121	6.36	0.0001

Table 2. SAS Applications: This application illustrates SAS MIXED instructions and output.

DATA krus; INPUT driver car mile; CARDS; 1 1 32.431 1 1 31.709 9 9 27.638 9 9 27.385 ; proc varcomp; class driver car; model mile=driver car driver*car; run;		Variance Components Estimation Procedure			
		MIVQUE(0) Variance Component Estimation Procedure			
		Source	DRIVER	CAR	DRIVER*CAR
		DRIVER	2592.00000000	0.00000000	288.00000000
		CAR	0.00000000	2592.00000000	288.00000000
		DRIVER*CAR	288.00000000	288.00000000	320.00000000
		Error	144.00000000	144.00000000	160.00000000
		Source	Error	MILE	
		DRIVER	144.00000000	18209.78939089	
		CAR	144.00000000	6517.77270689	
		DRIVER*CAR	160.00000000	2966.73139165	
		Error	161.00000000	1578.61171383	
		Estimate		MILE	
		Variance Component			
		Var(DRIVER)		6.93023219	
		Var(CAR)		2.41942328	
		Var(DRIVER*CAR)		0.26840729	
		Var(Error)		1.17587677	

Table 3. SAS Applications: This application illustrates SAS VARCOMP instructions and output.

Here $\hat{\sigma}_e^2 = 1.1759$, $\hat{\sigma}_\gamma^2 = 0.2684$, $\hat{\sigma}_b^2 = 6.930$ and $\hat{\sigma}_a^2 = 2.4194$.

The simultaneous confidence intervals about the parametric function of σ_a^2 , σ_γ^2 , and σ_e^2 are

$$P\{18.02 \leq (\sigma_e^2 + 2\sigma_\gamma^2 + 18\sigma_a^2) \leq 219.98, 1.16 \leq (\sigma_e^2 + 2\sigma_\gamma^2) \leq 2.74\} = 0.96$$

and

$$P\left\{6.52 \leq \left(\frac{\sigma_\gamma^2}{\sigma_e^2} + 9\frac{\sigma_a^2}{\sigma_e^2}\right) \leq 95.73, 0 \leq \frac{\sigma_\gamma^2}{\sigma_e^2} \leq 0.78\right\} \geq 0.96.$$

The simultaneous confidence interval for $(\sigma_a^2, \sigma_\gamma^2)$ from (14) is

$$P\{0.93 \leq \sigma_a^2 \leq 12.15, 0 \leq \sigma_\gamma^2 \leq 0.83\} \geq 0.92.$$

2.2 Simultaneous Confidence Interval For $(\sigma_b^2, \sigma_\gamma^2)$

The usual $100(1 - \alpha_3)\%$ confidence intervals about the parametric functions of σ_b^2 , σ_γ^2 and σ_e^2 are

$$\left\{\frac{S_b^2}{\chi_6^2} \leq (\sigma_e^2 + r\sigma_\gamma^2 + pr\sigma_b^2) \leq \frac{S_b^2}{\chi_5^2}\right\} \quad (15)$$

And

$$\left\{\frac{1}{r} \left(\frac{n_1 S_b^2}{n_3 S_e^2 F_6} - 1\right) \leq \left(\frac{\sigma_\gamma^2}{\sigma_e^2} + p \frac{\sigma_b^2}{\sigma_e^2}\right) \leq \frac{1}{r} \left(\frac{n_1 S_b^2}{n_3 S_e^2 F_5} - 1\right)\right\} \quad (16)$$

From (8) and (15) a set of simultaneous confidence intervals about the parametric function $(\sigma_e^2 + r\sigma_\gamma^2 + pr\sigma_b^2)$ and $(\sigma_e^2 + r\sigma_\gamma^2)$ with confidence coefficient $(1 - \beta_3)$ is

$$\left\{\frac{S_b^2}{\chi_b^2} \leq (\sigma_e^2 + r\sigma_\gamma^2 + pr\sigma_b^2) \leq \frac{S_b^2}{\chi_5^2}, \frac{S_\gamma^2}{\chi_4^2} \leq (\sigma_e^2 + r\sigma_\gamma^2) \leq \frac{S_\gamma^2}{\chi_3^2}\right\}, \quad (17)$$

where $1 - \beta_3 = (1 - \alpha_3)(1 - \alpha_2)$. Here the equality holds because the intervals (15) and (8) are independent.

Similarly from (9) and (16) a set of simultaneous confidence interval about the parametric functions $\left(\frac{\sigma_\gamma^2}{\sigma_e^2} + p \frac{\sigma_b^2}{\sigma_e^2}\right)$ and $\sigma_\gamma^2 / \sigma_e^2$ with confidence coefficient $(1 - \beta_4)$ is

$$\left\{\frac{1}{r} \left(\frac{n_1 S_b^2}{n_3 S_e^2 F_6} - 1\right) \leq \left(\frac{\sigma_\gamma^2}{\sigma_e^2} + p \frac{\sigma_b^2}{\sigma_e^2}\right) \leq \frac{1}{r} \left(\frac{n_1 S_b^2}{n_3 S_e^2 F_5} - 1\right), \frac{1}{r} \left(\frac{n_1 S_\gamma^2}{n_2 S_e^2 F_4} - 1\right) \leq \frac{\sigma_\gamma^2}{\sigma_e^2} \leq \frac{1}{r} \left(\frac{n_1 S_\gamma^2}{n_2 S_e^2 F_2} - 1\right)\right\} \quad (18)$$

where $1 - \beta_4 \geq (1 - \alpha_3)(1 - \alpha_2)$. The inequality holds because the intervals (9) and (16) are not independent.

Thus for fixed value of σ_e^2 , a $100(1 - \beta_3)\%$ confidence region for $(\sigma_b^2, \sigma_\gamma^2)$ is

$$R_3(\sigma_e^2) = \left[(\sigma_b^2, \sigma_\gamma^2) : \frac{1}{r} \left(\frac{S_b^2}{\chi_6^2} - \sigma_e^2\right) \leq (\sigma_\gamma^2 + p\sigma_b^2) \leq \frac{1}{r} \left(\frac{S_b^2}{\chi_5^2} - \sigma_e^2\right), \frac{1}{r} \left(\frac{S_\gamma^2}{\chi_4^2} - \sigma_e^2\right) \leq \sigma_\gamma^2 \leq \frac{1}{r} \left(\frac{S_\gamma^2}{\chi_3^2} - \sigma_e^2\right) \right].$$

Also a $100(1 - \beta_4)\%$ confidence region is

$$R_4(\sigma_e^2) = \left[(\sigma_b^2, \sigma_\gamma^2) : \frac{\sigma_e^2}{r} \left(\frac{n_1 S_b^2}{n_3 S_e^2 F_6} - 1 \right) \leq (\sigma_\gamma^2 + p \sigma_b^2) \leq \frac{\sigma_e^2}{r} \left(\frac{n_1 S_b^2}{n_3 S_e^2 F_5} - 1 \right), \right. \\ \left. \frac{\sigma_e^2}{r} \left(\frac{n_1 S_\gamma^2}{n_2 S_e^2 F_4} - 1 \right) \leq \sigma_\gamma^2 \leq \frac{\sigma_e^2}{r} \left(\frac{n_1 S_\gamma^2}{n_2 S_e^2 F_3} - 1 \right) \right].$$

Thus the simultaneous confidence interval for $(\sigma_b^2, \sigma_\gamma^2)$ determined by $R_3(\sigma_e^2)$ with confidence $(1 - \beta_3)$ is

$$\left[\frac{1}{pr} \left(\frac{S_b^2}{\chi_6^2} - \sigma_e^2 \right) \leq \sigma_b^2 \leq \frac{1}{pr} \left(\frac{S_b^2}{\chi_5^2} - \sigma_e^2 \right), \frac{1}{r} f_3(S_b^2, S_\gamma^2) \leq \sigma_\gamma^2 \leq \frac{1}{r} f_4(S_b^2, S_\gamma^2) \right] \quad (19)$$

where

$$f_3(S_b^2, S_\gamma^2) = \min \left\{ \left(\frac{S_\gamma^2}{\chi_4^2} - \sigma_e^2 \right), \left(\frac{S_b^2}{\chi_6^2} - \sigma_e^2 \right) \right\},$$

and

$$f_4(S_b^2, S_\gamma^2) = \min \left\{ \left(\frac{S_\gamma^2}{\chi_3^2} - \sigma_e^2 \right), \left(\frac{S_b^2}{\chi_5^2} - \sigma_e^2 \right) \right\}.$$

Similarly for given value of σ_e^2 the simultaneous confidence interval for $(\sigma_b^2, \sigma_\gamma^2)$ from $R_4(\sigma_e^2)$ is

$$\left\{ \frac{\sigma_e^2}{pr} \left(\frac{n_1 S_b^2}{n_3 S_e^2 F_6} - 1 \right) \leq \sigma_b^2 \leq \frac{\sigma_e^2}{pr} \left(\frac{n_1 S_b^2}{n_3 S_e^2 F_5} - 1 \right), \right. \\ \left. \frac{\sigma_e^2}{r} g_3(S_b^2, S_\gamma^2, S_e^2) \leq \sigma_\gamma^2 \leq \frac{\sigma_e^2}{r} g_4(S_b^2, S_\gamma^2, S_e^2) \right\}, \quad (20)$$

where

$$g_3(S_b^2, S_\gamma^2, S_e^2) = \min \left\{ \left(\frac{n_1 S_\gamma^2}{n_2 S_e^2 F_4} - 1 \right), \left(\frac{n_1 S_b^2}{n_3 S_e^2 F_3} - 1 \right) \right\},$$

and

$$g_4(S_b^2, S_\gamma^2, S_e^2) = \min \left\{ \left(\frac{n_1 S_\gamma^2}{n_2 S_e^2 F_3} - 1 \right), \left(\frac{n_1 S_b^2}{n_3 S_e^2 F_2} - 1 \right) \right\}.$$

Now proceeding as in section (2.1) the simultaneous confidence interval for $(\sigma_b^2, \sigma_\gamma^2)$ is

$$P \left[\frac{n_1 S_b^2 - n_3 S_e^2 F_6}{n_1 pr \chi_6^2} \leq \sigma_b^2 \leq \frac{n_1 S_b^2 - n_3 S_e^2 F_5}{n_1 pr \chi_5^2}, h_3(S_b^2, S_\gamma^2, S_e^2) \leq \sigma_\gamma^2 \leq h_4(S_b^2, S_\gamma^2, S_e^2) \right] \geq 1 - \beta_3 - \beta_4 \quad (21)$$

where

$$h_3(S_b^2, S_\gamma^2, S_e^2) = \min \left\{ \frac{n_1 S_\gamma^2 - n_2 S_e^2 F_4}{n_1 r \chi_4^2}, \frac{n_1 S_b^2 - n_2 S_e^2 F_6}{n_1 r \chi_6^2} \right\},$$

and

$$h_4(S_b^2, S_\gamma^2, S_e^2) = \min \left\{ \frac{n_1 S_\gamma^2 - n_2 S_e^2 F_3}{n_1 r \chi_3^2}, \frac{n_1 S_b^2 - n_2 S_e^2 F_5}{n_1 r \chi_5^2} \right\}.$$

To illustrate the technique of constructing the simultaneous confidence intervals, consider the data given in Section 2.1.

The simultaneous confidence interval about the parametric functions $(\sigma_e^2 + 2\sigma_\gamma^2 + 18\sigma_b^2)$ and $(\sigma_e^2 + 2\sigma_\gamma^2)$ are

$$P\{50.35 \leq (\sigma_e^2 + 2\sigma_\gamma^2 + 18\sigma_b^2) \leq 614.61, 1.16 \leq (\sigma_e^2 + 2\sigma_\gamma^2) \leq 2.74\} = 0.96$$

and about the parametric functions $\sigma_\gamma^2/\sigma_e^2 + p\sigma_b^2/\sigma_e^2$ and $\sigma_\gamma^2/\sigma_e^2$ are

$$P\left\{19.12 \leq \left(\frac{\sigma_\gamma^2}{\sigma_e^2} + 9\frac{\sigma_b^2}{\sigma_e^2}\right) \leq 268.35, 0 \leq \frac{\sigma_\gamma^2}{\sigma_e^2} \leq 0.78\right\} \geq 0.96.$$

The simultaneous confidence interval for $(\sigma_b^2, \sigma_\gamma^2)$ formed from the boundary intersections from (21) is

$$P\{2.72 \leq \sigma_b^2 \leq 34.08, 0 \leq \sigma_\gamma^2 \leq 0.83\} \geq 0.92.$$

2.3 Simultaneous Confidence Interval For $(\sigma_a^2, 2\sigma_b^2, \sigma_\gamma^2)$

From (6), (8) and (15) the simultaneous confidence interval about the parametric functions $(\sigma_e^2 + r\sigma_\gamma^2 + qr\sigma_a^2)$, $(\sigma_e^2 + r\sigma_\gamma^2 + pr\sigma_b^2)$ and $(\sigma_e^2 + r\sigma_\gamma^2)$ is

$$\left\{ \frac{S_a^2}{\chi_2^2} \leq (\sigma_e^2 + r\sigma_\gamma^2 + qr\sigma_a^2) \leq \frac{S_a^2}{\chi_1^2}, \frac{S_b^2}{\chi_6^2} \leq (\sigma_e^2 + r\sigma_\gamma^2 + pr\sigma_b^2) \leq \frac{S_b^2}{\chi_5^2}, \right. \\ \left. \frac{S_\gamma^2}{\chi_4^2} \leq (\sigma_e^2 + r\sigma_\gamma^2) \leq \frac{S_1^2}{\chi_3^2} \right\}, \quad (22)$$

where $1 - \beta_5 = (1 - \alpha_1)(1 - \alpha_2)(1 - \alpha_3)$. The equality holds since the intervals (6), (8) and (15) are independent.

Similarly from (7), (9) and (16) a set of simultaneous confidence interval about the parametric functions $(\sigma_\gamma^2/\sigma_e^2 + q\sigma_a^2/\sigma_e^2)$, $(\sigma_\gamma^2/\sigma_e^2 + p\sigma_b^2/\sigma_e^2)$ and $\sigma_\gamma^2/\sigma_e^2$ is

$$\begin{aligned}
\left\{ \frac{1}{r} \left(\frac{n_1 S_a^2}{n_4 S_e^2 F_2} - 1 \right) \leq \left(\frac{\sigma_\gamma^2}{\sigma_e^2} + q \frac{\sigma_a^2}{\sigma_e^2} \right) \leq \frac{1}{r} \left(\frac{n_1 S_a^2}{n_4 S_e^2 F_1} - 1 \right), \right. \\
\left. \frac{1}{r} \left(\frac{n_1 S_b^2}{n_3 S_e^2 F_6} - 1 \right) \leq \left(\frac{\sigma_\gamma^2}{\sigma_e^2} + p \frac{\sigma_b^2}{\sigma_e^2} \right) \leq \frac{1}{r} \left(\frac{n_1 S_b^2}{n_3 S_e^2 F_5} - 1 \right), \right. \\
\left. \frac{1}{r} \left(\frac{n_1 S_\gamma^2}{n_2 S_e^2 F_4} - 1 \right) \leq \frac{\sigma_\gamma^2}{\sigma_e^2} \leq \frac{1}{r} \left(\frac{n_1 S_\gamma^2}{n_2 S_e^2 F_3} - 1 \right) \right\}, \tag{23}
\end{aligned}$$

where $1 - \beta_6 \geq (1 - \alpha_1)(1 - \alpha_2)(1 - \alpha_3)$. The inequality holds since the intervals (7), (9) and (16) are not independent.

Thus for fixed value of σ_e^2 , a 100 $(1 - \beta_5)$ % confidence region for $(\sigma_a^2, \sigma_b^2, \sigma_\gamma^2)$ is

$$\begin{aligned}
R_5(\sigma_e^2) = \left[(\sigma_a^2, \sigma_b^2, \sigma_\gamma^2) : \frac{1}{r} \left(\frac{S_a^2}{\chi_2^2} - \sigma_e^2 \right) \leq (\sigma_\gamma^2 + q\sigma_a^2) \leq \frac{1}{r} \left(\frac{S_a^2}{\chi_1^2} - \sigma_e^2 \right), \right. \\
\left. \frac{1}{r} \left(\frac{S_b^2}{\chi_6^2} - \sigma_e^2 \right) \leq (\sigma_\gamma^2 + p\sigma_b^2) \leq \frac{1}{r} \left(\frac{S_b^2}{\chi_5^2} - \sigma_e^2 \right), \frac{1}{r} \left(\frac{S_\gamma^2}{\chi_4^2} - \sigma_e^2 \right) \leq \sigma_\gamma^2 \leq \frac{1}{r} \left(\frac{S_\gamma^2}{\chi_3^2} - \sigma_e^2 \right) \right].
\end{aligned}$$

Also a 100 $(1 - \beta_6)$ % confidence region is

$$\begin{aligned}
R_6(\sigma_e^2) = \left[(\sigma_a^2, \sigma_b^2, \sigma_\gamma^2) : \frac{\sigma_e^2}{r} \left(\frac{n_1 S_a^2}{n_4 S_e^2 F_2} - 1 \right) \leq (\sigma_\gamma^2 + q\sigma_a^2) \leq \frac{\sigma_e^2}{r} \left(\frac{n_1 S_a^2}{n_4 S_e^2 F_1} - 1 \right), \right. \\
\left. \frac{\sigma_e^2}{r} \left(\frac{n_1 S_b^2}{n_3 S_e^2 F_6} - 1 \right) \leq (\sigma_\gamma^2 + p\sigma_b^2) \leq \frac{\sigma_e^2}{r} \left(\frac{n_1 S_b^2}{n_3 S_e^2 F_5} - 1 \right), \frac{\sigma_e^2}{r} \left(\frac{n_1 S_\gamma^2}{n_2 S_e^2 F_4} - 1 \right) \leq \sigma_\gamma^2 \leq \frac{\sigma_e^2}{r} \left(\frac{n_1 S_\gamma^2}{n_2 S_e^2 F_3} - 1 \right) \right].
\end{aligned}$$

Now, proceeding as in Section 2.1 regardless of the true value of σ_e^2

$$\begin{aligned}
P \left[\frac{n_1 S_a^2 - n_4 S_e^2 F_2}{n_1 q r \chi_2^2} \leq \sigma_a^2 \leq \frac{n_1 S_a^2 - n_4 S_e^2 F_1}{n_1 q r \chi_1^2}, \right. \\
\left. \frac{n_1 S_b^2 - n_3 S_e^2 F_6}{n_1 p r \chi_6^2} \leq \sigma_b^2 \leq \frac{n_1 S_b^2 - n_3 S_e^2 F_5}{n_1 p r \chi_5^2}, h_5(S_a^2, S_b^2, S_\gamma^2, S_e^2) \leq \sigma_\gamma^2 \leq h_6(S_a^2, S_b^2, S_\gamma^2, S_e^2) \right] \geq 1 - \beta_5 \tag{24}
\end{aligned}$$

where

$$h_5(S_a^2, S_b^2, S_\gamma^2, S_e^2) = \min \left\{ \frac{n_1 S_\gamma - n_2 S_e^2 F_4}{n_1 r \chi_4^2}, \frac{n_1 S_b^2 - n_3 S_e^2 F_6}{n_1 r \chi_6^2}, \frac{n_1 S_a^2 - n_4 S_e^2 F_4}{n_1 r \chi_2^2} \right\}$$

and

$$h_6(S_a^2, S_b^2, S_\gamma^2, S_e^2) = \min \left\{ \frac{n_1 S_\gamma - n_2 S_e^2 F_3}{n_1 r \chi_3^2}, \frac{n_1 S_b^2 - n_3 S_e^2 F_5}{n_1 r \chi_5^2}, \frac{n_1 S_a^2 - n_4 S_e^2 F_1}{n_1 r \chi_1^2} \right\}.$$

Equation (24) gives the required simultaneous confidence interval for $(\sigma_a^2, \sigma_b^2, \sigma_\gamma^2)$.

To illustrate the technique of constructing the simultaneous confidence intervals, consider the data given in Section 2.1.

The simultaneous confidence intervals about the parameter functions of $\sigma_a^2, \sigma_b^2, \sigma_\gamma^2$ and σ_e^2 are

$$P\left\{18.02 \leq (\sigma_e^2 + 2\sigma_\gamma^2 + 18\sigma_a^2) \leq 219.98, 50.35 \leq (\sigma_e^2 + 2\sigma_\gamma^2 + 18\sigma_b^2) \leq 614, \right. \\ \left. 1.16 \leq (\sigma_e^2 + 2\sigma_\gamma^2) \leq 2.74\right\} = 0.94$$

and

$$P\left\{6.52 \leq \left(\frac{\sigma_\gamma^2}{\sigma_e^2} + 9\frac{\sigma_a^2}{\sigma_e^2}\right) \leq 95.73, 19.12 \leq \left(\frac{\sigma_\gamma^2}{\sigma_e^2} + 9\frac{\sigma_b^2}{\sigma_e^2}\right) \leq 268.35, 0 \leq \sigma_\gamma^2 / \sigma_e^2 \leq 0.78\right\} \geq 0.94.$$

The simultaneous confidence interval for $(\sigma_a^2, \sigma_b^2, \sigma_\gamma^2)$ is

$$P\left\{0.93 \leq \sigma_a^2 \leq 12.15, 2.72 \leq \sigma_b^2 \leq 34.08, 0 \leq \sigma_\gamma^2 \leq 0.83\right\} \geq 0.88.$$

2.4 Simultaneous Confidence Interval for (σ_a^2, σ_b^2)

The $100(1 - \alpha_4)\%$ confidence intervals about the parametric functions $(\sigma_e^2 + r\sigma_\gamma^2 + qr\sigma_a^2)$ and $\sigma_a^2 / (\sigma_e^2 + r\sigma_\gamma^2)$ are

$$\left\{ \frac{S_a^2}{\chi_2^2} \leq (\sigma_e^2 + r\sigma_\gamma^2 + qr\sigma_a^2) \leq \frac{S_a^2}{\chi_1^2} \right\} \quad (25)$$

and

$$\left\{ \frac{1}{qr} \left(\frac{n_2 S_a^2}{n_4 S_\gamma^2 F_8} - 1 \right) \leq \frac{\sigma_a^2}{\sigma_e^2 + r\sigma_\gamma^2} \leq \frac{1}{qr} \left(\frac{n_2 S_a^2}{n_4 S_\gamma^2 F_7} - 1 \right) \right\}. \quad (26)$$

The $100(1 - \alpha_5)\%$ confidence intervals about the parametric functions $(\sigma_e^2 + r\sigma_\gamma^2 + pr\sigma_b^2)$ and $(\sigma_e^2 + r\sigma_\gamma^2)$ are

$$\left\{ \frac{S_b^2}{\chi_6^2} \leq (\sigma_e^2 + r\sigma_\gamma^2 + pr\sigma_b^2) \leq \frac{S_b^2}{\chi_5^2} \right\} \quad (27)$$

and

$$\left\{ \frac{1}{pr} \left(\frac{n_2 S_b^2}{n_3 S_\gamma^2 F_{10}} - 1 \right) \leq \frac{\sigma_b^2}{\sigma_e^2 + r\sigma_\gamma^2} \leq \frac{1}{pr} \left(\frac{n_2 S_b^2}{n_3 S_\gamma^2 F_9} - 1 \right) \right\}. \quad (28)$$

From (25) and (27) a set of simultaneous confidence interval about the parametric functions $(\sigma_e^2 + r\sigma_\gamma^2 + qr\sigma_a^2)$ and $(\sigma_e^2 + r\sigma_\gamma^2 + pr\sigma_b^2)$ with confidence coefficient $(1 - \beta_7)$ is

$$\left\{ \frac{S_a^2}{\chi_2^2} \leq (\sigma_e^2 + r\sigma_\gamma^2 + qr\sigma_a^2) \leq \frac{S_a^2}{\chi_1^2}, \frac{S_b^2}{\chi_6^2} \leq (\sigma_e^2 + r\sigma_\gamma^2 + pr\sigma_b^2) \leq \frac{S_b^2}{\chi_5^2} \right\} \quad (29)$$

where $1 - \beta_7 = (1 - \alpha_4)(1 - \alpha_5)$. The equality holds because the intervals (25) and (27) are independent.

Similarly from (26) and (28) a set of simultaneous confidence interval for the parametric functions $\sigma_a^2/(\sigma_e^2 + r\sigma_\gamma^2)$ and $\sigma_b^2/(\sigma_e^2 + r\sigma_\gamma^2)$ with confidence coefficient $(1 - \beta_8)$ is

$$\begin{aligned} \left\{ \frac{1}{qr} \left(\frac{n_2 S_a^2}{n_4 S_\gamma^2 F_8} - 1 \right) \leq \frac{\sigma_a^2}{\sigma_e^2 + r\sigma_\gamma^2} \leq \frac{1}{qr} \left(\frac{n_2 S_a^2}{n_4 S_\gamma^2 F_7} - 1 \right), \right. \\ \left. \frac{1}{pr} \left(\frac{n_2 S_b^2}{n_3 S_\gamma^2 F_{10}} - 1 \right) \leq \frac{\sigma_b^2}{\sigma_e^2 + r\sigma_\gamma^2} \leq \frac{1}{pr} \left(\frac{n_2 S_b^2}{n_3 S_\gamma^2 F_9} - 1 \right) \right\}, \end{aligned} \quad (30)$$

where $1 - \beta_8 \geq (1 - \alpha_4)(1 - \alpha_5)$. The inequality holds because the intervals (26) and (28) are not independent.

Now, for fixed value of $(\sigma_e^2 + r\sigma_\gamma^2)$ we construct a $100(1 - \beta_7)\%$ confidence region for $(\sigma_e^2 + \sigma_b^2)$ as

$$\begin{aligned} R_7(\sigma_e^2 + r\sigma_\gamma^2) = \left[(\sigma_a^2, \sigma_b^2) : \frac{1}{qr} \left\{ \frac{S_a^2}{\chi_2^2} - (\sigma_e^2 + r\sigma_\gamma^2) \right\} \leq \sigma_a^2 \leq \frac{1}{qr} \left\{ \frac{S_a^2}{\chi_1^2} - (\sigma_e^2 + r\sigma_\gamma^2) \right\}, \right. \\ \left. \frac{1}{pr} \left\{ \frac{S_b^2}{\chi_6^2} - (\sigma_e^2 + r\sigma_\gamma^2) \right\} \leq \sigma_b^2 \leq \frac{1}{pr} \left\{ \frac{S_b^2}{\chi_5^2} - (\sigma_e^2 + r\sigma_\gamma^2) \right\} \right]. \end{aligned}$$

Similarly a $100(1 - \beta_8)\%$ confidence region for (σ_a^2, σ_b^2) is given by

$$\begin{aligned} R_8(\sigma_e^2 + r\sigma_\gamma^2) = \left[(\sigma_a^2 + \sigma_b^2) : \frac{\sigma_e^2 + \sigma_\gamma^2}{qr} \left(\frac{n_2 S_a^2}{n_4 S_\gamma^2 F_8} - 1 \right) \leq \sigma_a^2 \leq \frac{\sigma_e^2 + r\sigma_\gamma^2}{qr} \left(\frac{n_2 S_a^2}{n_4 S_\gamma^2 F_7} - 1 \right), \right. \\ \left. \frac{\sigma_e^2 + r\sigma_\gamma^2}{pr} \left(\frac{n_2 S_b^2}{n_3 S_\gamma^2 F_{10}} - 1 \right) \leq \sigma_b^2 \leq \frac{\sigma_e^2 + r\sigma_\gamma^2}{pr} \left(\frac{n_2 S_b^2}{n_3 S_\gamma^2 F_9} - 1 \right) \right]. \end{aligned}$$

Regardless of the true value of $(\sigma_e^2 + r\sigma_\gamma^2)$ the simultaneous confidence interval for $(\sigma_e^2 + \sigma_b^2)$ is given by

$$\begin{aligned} P \left[\frac{n_2 S_a^2 - n_4 S_\gamma^2 F_8}{n_2 qr \chi_2^2} \leq \sigma_a^2 \leq \frac{n_2 S_a^2 - n_4 S_\gamma^2 F_7}{n_2 qr \chi_1^2}, \right. \\ \left. \frac{n_2 S_b^2 - n_3 S_\gamma^2 F_{10}}{n_2 pr \chi_6^2} \leq \sigma_b^2 \leq \frac{n_2 S_b^2 - n_3 S_\gamma^2 F_9}{n_2 pr \chi_5^2} \right] \geq 1 - \beta_7 - \beta_8 \end{aligned} \quad (31)$$

To illustrate the technique of constructing this simultaneous confidence intervals consider the data given in Section 2.1.

The simultaneous confidence interval about the parametric functions $(\sigma_e^2 + 2\sigma_\gamma^2 + 18\sigma_a^2)$ and $(\sigma_e^2 + 2\sigma_\gamma^2 + 18\sigma_b^2)$ is

$$P \left\{ 18.02 \leq (\sigma_e^2 + 2\sigma_\gamma^2 + 18\sigma_a^2) \leq 219.98, 50.35 \leq (\sigma_e^2 + 2\sigma_\gamma^2 + 18\sigma_b^2) \leq 614.61 \right\}.$$

Similarly the simultaneous confidence interval about the parameter functions $\sigma_a^2 / (\sigma_e^2 + r\sigma_\gamma^2)$ and $\sigma_b^2 / (\sigma_e^2 + r\sigma_\gamma^2)$ is

$$P \left\{ 0.47 \leq \frac{\sigma_a^2}{(\sigma_e^2 + r\sigma_\gamma^2)} \leq 0.64, 1.41 \leq \frac{\sigma_b^2}{(\sigma_e^2 + r\sigma_\gamma^2)} \leq 20.58 \right\} \geq 0.96.$$

The simultaneous confidence interval for (σ_a^2, σ_b^2) from the boundary intersections is given by

$$P \left\{ 0.89 \leq \sigma_a^2 \leq 12.12, 2.69 \leq \sigma_b^2 \leq 34.05 \right\} \geq 0.92$$

with known values of σ_e^2 and σ_γ^2 .

3. CONCLUSION

We have developed simultaneous confidence intervals for the variance components of two-way balanced crossed classification random effects model with interaction excluding the error variance component under the usual assumptions of normality and independence of random effects. We have developed simultaneous confidence intervals about the parametric function of σ_a^2 , σ_γ^2 , and σ_e^2 , for instance $(\sigma_e^2 + r\sigma_\gamma^2 + pr\sigma_b^2)$, $(\sigma_a^2, \sigma_\gamma^2)$, $\sigma_\gamma^2/\sigma_e^2 + p\sigma_b^2/\sigma_e^2$ and $\sigma_\gamma^2/\sigma_e^2$. The technique has been based on combining intervals about the parametric functions of the parameters. The technique of constructing these simultaneous confidence intervals had been illustrated considering SAS outputs of published data.

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