NON RADIAL EFFICIENCY ANALYSIS. AN APPLICATION TO GARBAGE COLLECTION

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Abstract

This paper proposes a method to identify the shortest path to the efficient subset of the isoquant. To such an end we introduce the concept of input-specific distance as an indirect measure of the shortest path. The objective of the model is to give managers more useful and detailed information oriented towards efficiency achievement. This is done by identifying which of the efficient productive units in the sample share the largest number of similarities with each inefficient productive unit. This information can be used to advise inefficient units about which efficient unit to visit in order to detect its mistakes and to learn how to behave efficiently. The model is extended to identify the input in which reductions would lead more directly to the efficient subset. We propose that a management program should focus mainly on this input. The model is applied to a sample of garbage collection services in Cataluña (Spain) finding the input labor as the one that leads more directly to the efficient subset.

Keywords. Data Envelopment Analysis, non-radial efficiency, management.

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1. Introduction

Technical inefficiency reflects the failure of some firms to obtain the maximum feasible output given the amount of inputs used. Its measurement is crucial to quantify the importance of poor performances in a productive activity. Unfortunately, measurement is not enough. In order to achieve technical efficiency (TE) firms should be able to identify the sources of misperformances and the alternatives available to make better use of their resources.

For example, some common questions posed by inefficient farmers enrolled in farm management programs are How can I become efficient? or What am I doing wrong?. However, current measures of TE do not provide enough information of this kind. The problem is the implicit assumption that inefficient units should behave as those in the best practice frontier. Behavior in this case has two components: how much these units are doing and how they do it. Efficiency measures only inform in terms of “how much”. Therefore the remaining question is how can a unit become efficient in practice?. One way is to obtain information about the “how” part by visiting some of the efficient units in order to find out how they do things. This benchmarking procedure is common to farm management programs and can be extended to other activities such as those carried by the Public Sector. The question here would be to determine which of the efficient units to visit. Even though this is done in real practice in several ways, the purpose of this paper is to provide a method to identify this subset of most relevant units to visit in a more scientific way.

It seems natural to think that an inefficient unit will prefer to visit the efficient unit that is most similar to it rather than an efficient but very different one. We assume that the more similar the efficient unit, the easier it will be for the inefficient unit to detect its own mistakes and therefore to correct them. Now, in order for this idea to be operative, we must find a definition of similarity.

Most empirical studies of TE use radial measures to quantify efficiency. Thus, one can think that the most similar unit is the radial projection of the inefficient unit on the isoquant. Radiality seems to be interesting as a proxy for similarity because all productive processes on the same ray share the same combination of inputs, but at different scales, which in principle should not matter[1]. However, the data available for empirical work usually suffers from severe aggregation problems that question the former interpretation of radiality as similarity. For example, the input capital contains many different types of assets. Therefore, two units with the same input ratio but considerably different amounts of capital cannot, in general terms, be considered similar.

A better criterion for practical purposes may be proximity which can be measured in terms of absolute quantities of inputs. While aware that this is not a scientific criterion, simply it would seem that an inefficient unit will be more interested in visiting a unit that uses more or less the same quantities of inputs than in visiting a unit that is using the same proportion of inputs. The literature on farm management provides some examples of the use of this criterion of similarity. For example, Lund and ∅rum (1997) have developed a computerized efficiency analysis system for management advisory purposes that compares each farm with a reference group composed of the most similar farms in terms of absolute quantities of certain inputs.

To develop an operative way to identify the closest reference group, we introduce the concept of *input-specific distance* as a modified version of the *single-factor efficiency* measure, introduced by Kopp (1981). The input-specific distance computes the length of the movement that is required to reach the efficient subset of the production frontier when the distance to the isoquant is measured along a single input. In this manner, the comparison group on the efficient subset will be composed of efficient units that share the largest number of similarities with the inefficient unit being evaluated. Therefore, this procedure will result in a set of efficient units that inefficient units should visit.

However, this is not the most common way to attack inefficiency in the real world. Actually, in the Public Sector, the government can take a step in helping managers to improve their efficiency levels by developing management programs oriented towards reducing inefficiency. The practical implementation of these programs, usually requires the training of technicians in various aspects of the productive activity. This process is costly and, therefore, it would be helpful to have some indication as to where effort should be placed. For this purpose, we introduce a global measure that is defined as the aggregation of individual input-specific distances. This measure reveals the input that leads more directly to the efficient subset, on average.

The paper is organized as follows. Section 2 reviews the current efficiency measures. In Section 3, the concept of input-specific distance is introduced. An empirical non-parametric model is presented in Section 4. In Section 5, we apply the model to the sector of garbage collection. Finally, concluding remarks are presented in Section 6.

2. Measures of Technical Efficiency

The technology can be characterized by the input requirement set, $\mathbf{u} \to L(\mathbf{u})$, where $L: \Re_+^S \to \Re_+^N$ is a mapping from the output vector $\mathbf{u} \in \Re_+^S$ into the set of input vectors $\mathbf{x} \in \Re_+^N$ that allow to produce $\mathbf{u}$. Throughout the paper, we will assume that $L(\mathbf{u})$ satisfies the properties of convexity, free disposability of inputs and outputs and variable returns to scale.

Koopmans (1951) defines an input-output vector (IOV) as technically efficient if, and only if, increasing any output or decreasing any input is possible only by decreasing some other output or increasing some other input. Based on the previous definition, technical efficiency measures evaluate the performance of a given IOV by comparison to the IOVs on the boundary of $L(\mathbf{u})$.

Two boundary sets are relevant for the measurement of TE. The input isoquant is defined as[2]:

$$Isoq\ L(\mathbf{u}) = \left\{ \mathbf{x} \in \Re_+^{\mathbf{N}} : \mathbf{x} \in L(\mathbf{u}) \wedge l\mathbf{x} \notin L(\mathbf{u}),\ l \in [0,1) \right\} \qquad (1)$$

and the efficient subset of the isoquant is defined as[3]:

$$Eff\ L(\mathbf{u}) = \left\{ \mathbf{x} \in \Re_+^N : \mathbf{x} \in L(\mathbf{u}) \wedge \hat{\mathbf{x}} \leq \mathbf{x}, \hat{\mathbf{x}} \notin L(\mathbf{u}) \right\} \qquad (2)$$

Radial measures of TE carry the comparison along a ray from the origin and are attractive because they maintain the input mix of the IOV onto its projection on the boundary of the input requirement set. Therefore, they have a direct interpretation in terms of proportional cost reduction.

The first radial measure of TE was introduced by Debreu (1951) under the name of *coefficient of resource utilization*. Farrell (1957) generalized the measure to consider also allocative inefficiency as the second component of what he defined as *overall efficiency*. The Debreu-Farrell measure of TE focuses on the maximum equiproportionate reduction in all the inputs that can be achieved holding constant the output vector, and is defined by:

$$DF(\mathbf{x},\mathbf{u}) = \min_{q}\left\{q : q\,\mathbf{x} \in L(\mathbf{u})\ ,\quad q \in \Re_{+}\right\} \qquad (3)$$

where q is a scalar.

However, $DF(\cdot)$ is not always consistent with Koopmans' definition, because the comparison is done with respect to the isoquant and not with respect to the efficient subset. An IOV on the isoquant is considered efficient although it may remain slacks in some inputs.

Non-radial measures of TE avoid this problem by restricting the comparison to the efficient subset. The Russell measure introduced by Färe and Lovell (1978) satisfies this property and is defined as:

$$R(\mathbf{x},\mathbf{u}) = \min_{\boldsymbol{\theta}}\left\{\frac{\sum_{n=1}^{N} q_n}{N} : \left(q_1 x_1, \cdots, q_N x_N\right) \in L(\mathbf{u}),\quad q_n \in [0,1]\ \forall n\right\} \qquad (4)$$

This measure shrinks the input vector not along a ray, but in coordinate directions until an efficient point (in the sense of Koopmans) is reached. The difference between the two measures is illustrated in Figure 1.

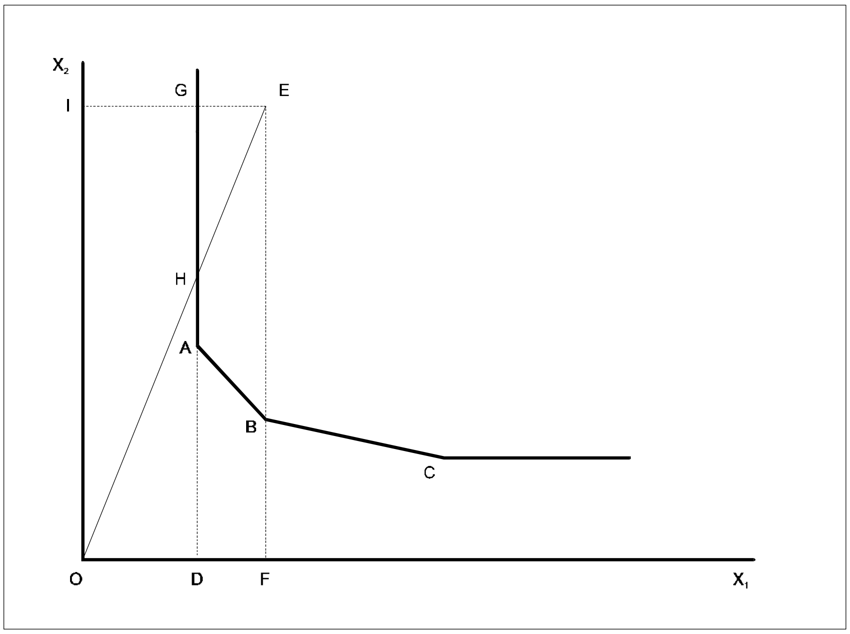

Figure 1

For unit E, the Debreu-Farrell measure of efficiency is given by the ratio OH/OE. The comparison point on the isoquant (H) lies on the same ray of unit E, thus maintaining the proportions in the input mix. To compute the Russell measure, the comparison point must lie on the efficient subset (ABC). By construction, the reference point is A and the measure takes the value 1/2(IG/IE + DA/DG), which represents the maximum average contraction that is feasible.

The indexes discussed above are called by Kopp (1981) *multiple-factor efficiency* measures, as they encompass the efficiency of total factor use[4]. Kopp introduces the notion of *single-factor efficiency* measure as an attempt to understand the individual contribution of each input to inefficiency. The single- factor efficiency measure of input k is given by:

$$K_k(\mathbf{x},\mathbf{u}) = \min_{q_k}\left\{q_k : (x_1,\cdots,q_k x_k,\cdots,x_N) \in L(\mathbf{u})\,,\quad q_k \in \Re_+\right\} \tag{5}$$

Expression (5) gives the contraction in input k needed to reach the isoquant and can be interpreted as the lower bound in the efficiency with which that input is used and the potential for increasing that efficiency. In Figure 1 the Kopp measure takes the value FB/FE for input X_2 and IG/IE for input X_1. It is worth noting that the comparison set is the isoquant and not necessarily the efficient subset.

Single-factor efficiency measures are particularly interesting because they focus on the contraction of the IOV to $Isoq\ L(\mathbf{u})$ along a unique coordinate direction. The highest efficiency index would reflect the input that needs the smallest contraction to reach the isoquant.

3. Input Specific Distance to the Efficient Subset

In Section 2, we have seen three measures of TE that share the common feature of quantifying the relative proportions by which a firm could reduce its inputs. The focus in this section is not on relative contractions but on absolute reductions of the inputs. The magnitude of such reductions is an important factor in designing the best strategy to achieve efficiency. In other words, the task of reducing inputs while maintaining output levels is not trivial and implies considerable efforts (if not, we would not observe inefficient firms). The easiest manner to achieve efficiency may consist in visiting similar units that are efficient. Therefore, a procedure to identify the shortest path to the efficient subset can help the design of a plan towards achieving efficiency.

Definition 1. The smallest distance to the efficient subset is the minimum reduction in the inputs required to reach the efficient subset:

$$D(\mathbf{x},\mathbf{u}) = \min_{\boldsymbol{\theta}}\left\{\sum_{n=1}^{N}(1-q_{\mathbf{n}})\cdot x_n : (q_1 x_1 \cdots, q_N x_N) \in Eff\ \ L(\mathbf{u})\,,\quad q_n \le 1\ \forall n\right\} \quad (6)$$

Obviously, some normalization is needed in order to aggregate input quantities that are expressed in different units of measurement. The adjustment is done by scaling the inputs by their means. A nice feature of this transformation is that it preserves the current measures of technical efficiency discussed in Section 2.

The distance measure defined in (6) determines the shortest path to the efficient subset. For example, in Figure 2, the minimum distance to the efficient subset, for unit A, is given by the segment AB, which implies a reduction only in the input X_2. For unit D, the minimum distance is given by the sum DE+EF, which implies a reduction in input X_1 to reach the isoquant plus a slack reduction in input X_2 to reach the efficient subset[5]. Note that, for example, firm A can use two alternative reference points on the efficient subset, C and B, but B is more similar to A than C. Therefore, firm A could prefer to visit B in order to reach efficiency.

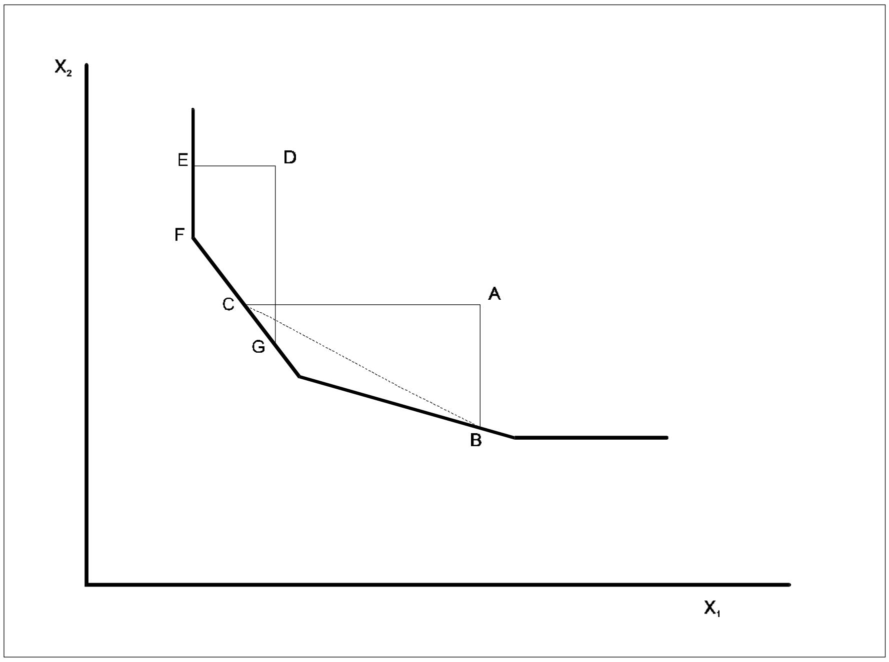

Figure 2

Unfortunately, the smallest distance measure is generally not directly computable because there are infinite different paths to be compared. However, under the hypothesis of convexity of $L(\mathbf{u})$ the measure can be computed indirectly. For this purpose, we introduce the concept of input-specific distance, in which a specific input determines the path for measurement.

Definition 2. (*Input specific distance*) The kth input-specific distance measures the distance to the isoquant along the k-th axis. If the isoquant does not coincide with the efficient subset in the projection of the IOV, the measure considers the slacks in the rest of the inputs as additional distances (to reach the efficient subset):

$$\begin{aligned} D(\mathbf{x},\mathbf{u})_k = \Big\{ \textstyle\sum_{n=1}^{N} (1-q_n)\cdot x_n : (x_1,\cdots,q_k x_k,\cdots,x_N) \in \ Isoq\ L(\mathbf{u}) \wedge \\ \wedge\ (q_1 x_1,\cdots,q_N x_N) \in \ Eff\ \ L(\mathbf{u}) \Big\} \end{aligned} \tag{7}$$

Figure 2 illustrates this concept. The input-specific distance for the input X_1 is DE+EF for unit D and AC for unit A. For the input X_2, the distance is DG for unit D and AB for unit A. Note that in the first case we add the slack distance EF for unit D. Graphically it is clear that the smallest distance defined in (6) coincides with the smallest input-specific distance. This is due to the assumption of convexity of $L(\mathbf{u})$. The distance to any point in the segment joining the two comparison points cannot be strictly smaller than either distance to these points. For instance, the slope of the segment CB is less than one and, therefore, the closer point to A on the efficient subset is B. By convexity of $L(\mathbf{u})$ any point on the efficient subset between C and B must be at least as far from A as the points in the segment CB.

Proposition 1. If L(u) is a convex set the smallest distance to the efficient subset is the smallest input-specific distance[6]:

$$D(\mathbf{x},\mathbf{u}) = \min_n \left\{ D(\mathbf{x},\mathbf{u})_n \quad , \quad n = 1,\dots,N \right\} \tag{8}$$

Another important feature of the input-specific distance is that it allows to obtain conclusions at an aggregate level. Aggregating the measure for the J units in the sample, we obtain a measure of global input-specific distance for each input.

Definition 3. The k-th global input-specific distance is the sum of the k-th input-specific distances of the units in the sample:

$$GD_k = \sum_{j=1}^{J} D(\mathbf{x_j},\mathbf{u_j})_k \tag{9}$$

The measure is the minimum aggregate distance when all units are evaluated with respect to the same input. The smallest global input-specific distance reflects the input in which the inefficient units are closer to the efficient subset. Therefore, it seems that a management program designed to achieve TE should focus mainly in training technicians in the efficient use of that input. However, it might be argued that this criterion is very inflexible because all the units are evaluated along the same input, even though there might be an important number of them that are closer to the efficient subset along a different path. For this reason, it might be convenient to relax the criterion to consider P inputs. In the next section we provide a technique that allows for the inclusion more than one input in the global distance criterion

4. Nonparametric Programming Model

To simplify the notation, we define S and N as the sets of output and input labels respectively, and J as the set of productive units in the sample. The subscript i will denote the IOV that is evaluated and λ the intensity vector, which represents the weights assigned to each unit j entered into the linear combination of units that define the efficient IOV.

The technique of Data Envelopment Analysis (DEA) introduced by Charnes, Cooper, Rhodes (1978) and extended by Banker, Charnes, Cooper (1984) has been widely used to measure the Debreu-Farrell index of technical efficiency[7]. The k-th single-factor measure of efficiency can be computed by solving a transformed version of the original DEA programs. We must take the i-th IOV back to the isoquant along the k-th input. The DEA program must find the linear combination of observed IOVs that minimizes the consumption of input k, constrained to use no more of the rest of the inputs and to produce no less outputs than the i-th IOV. Expression (10) shows the linear program that solves this problem:

$$
\begin{array}{lll}
\min\limits_{\lambda,\theta} & q_k & \\
s.t. & \sum\limits_{j\in J} l_j u_{js} \geq u_{is} & , \quad s \in S \\
 & \sum\limits_{j\in J} l_j x_{jn} \leq x_{in} & , \quad n \in N, n \neq k \\
 & \sum\limits_{j\in J} l_j x_{jk} \leq q_k x_{ik} & \\
 & \sum\limits_{j\in J} l_j = 1 & \\
 & l_j \geq 0 & , \quad j \in J
\end{array}
\tag{10}
$$

However, there are two problems with the solution to this program. First, it can contain slacks in the inputs, that can be interpreted as additional feasible reductions. Second, even if we eliminate the slacks, some additional reductions may be feasible, because the *efficient* IOV used for comparison lies on the isoquant (not necessarily on the efficient subset). An alternative formulation that computes all feasible reductions is given by:

$$
\begin{array}{lll}
\min\limits_{\lambda,\theta} & M \cdot q_k + \sum\limits_{\substack{n\in N \\ n\neq k}} q_n & \\
s.t. & \sum\limits_{j\in J} l_j u_{js} \geq u_{is} & , \quad s \in S \\
 & \sum\limits_{j\in J} l_j x_{jn} \leq q_n x_{in} & , \quad n \in N \\
 & q_n \leq 1 & , \quad n \in N, n \neq k \\
 & \sum\limits_{j\in J} l_j = 1 & \\
 & l_j \geq 0 & , \quad j \in J
\end{array}
\tag{11}
$$

where M is a large enough scalar to force the program to identify input k as the one defining the path to the isoquant. The search for feasible reductions in the rest of the inputs starts after the isoquant is reached, taking the i-th IOV down to the efficient subset. However, the weight assigned to input k also implies that values of $q_{-k} > 1$ are possible if they permit to obtain a lower q_k. Therefore, the constraints $q_n \leq 1$ are necessary to ensure that reductions in input k are not achieved by the small cost (in terms of the objective function) of increasing some other input.

After computing θ in (11), the input-specific distance defined in (7) can be derived as:

$$D(\mathbf{x_i}, \mathbf{u_i})_k = \sum_{n \in N} (1 - q_n) \cdot x_{in} \qquad (12)$$

In empirical applications it can be interesting to decompose the measure in (12) in two terms:

$$D(\mathbf{x_i}, \mathbf{u_i})_k = (1 - q_k) \cdot x_{ik} + \sum_{\substack{n \in N \\ n \neq k}} (1 - q_n) \cdot x_{in} \qquad (13)$$

where the first term represents the distance to the isoquant and the second term the sum of the slacks.

The criterion proposed in the previous section to identify the input that minimizes the global input-specific distance (GD), can be relaxed to consider distances along two or more paths. For the general case of P paths for reduction, the P-global distance can be computed by solving the following program:

$$\begin{aligned}
&\min_{V,T} \sum_{n\in N}\sum_{j\in J} D(\mathbf{x_j},\mathbf{u_j})_n \cdot V_{jn} \\
&s.t. \quad \sum_{n\in N} T_n = P \\
&\qquad \sum_{n\in N} V_{jn} = 1 \quad , \quad j\in J \\
&\qquad V_{jn} \le T_n \quad , \quad j\in J, n\in N \\
&\quad T_n, V_{jn} \in \{0,1\} \quad , \quad j\in J, n\in N
\end{aligned} \tag{14}$$

where T_n is a dichotomous variable that takes the value 1 if the n-th input defines the path for at least one unit and 0 otherwise. Similarly, V_{jn} takes the value 1 if unit j takes the n-th path and 0 otherwise. The constraints state that: 1) only P different paths can be considered for measurement; 2) each unit must consider one (and only one) path; 3) unit j can take the n-th path only if it is one of the P paths allowed. The values of T indicate the P relevant inputs for management purposes. Setting $P=N$ we obtain the smallest aggregate distance to the efficient subset, because the smallest input-specific distances are summed.

5. Empirical Application

In this section, we apply our model to a sample of garbage collection units from 89 municipalities in Cataluña in 1995. Units produce a single output (Tons of garbage) using three inputs (Labor, Containers and Vehicles). All the variables were divided by their means to reduce the problems associated with differences in the units of measurement.

The three measures of TE discussed in Section 2 were estimated using the DEA approach. Table 1 shows some descriptive statistics:

Table 1.- Technical efficiency scores

	MEAN	*MIN*	*EFF*
MULTIPLE-FACTOR	%	%	#
DEBREU-FARRELL	72	20	33
RUSSELL	59	18	12
SINGLE-FACTOR			
KOPP-CONTAINER	40	6	12
KOPP-VEHICLE	58	6	33
KOPP-LABOR	55	8	12

The mean of the Debreu-Farrell measure suggests that it is technically possible to produce the same level of output using 72% of the inputs employed. Thus, there exists a margin for reducing production cost by 28%. The Russell index has a different interpretation, indicating that a maximum average reduction of 41% on the inputs could be achieved. The Russell index takes on lower values than the Debreu-Farrell index, which is a particular case of the former. The Kopp measures provide the lower bounds for the efficiency of each input. According to our results, vehicles and labor are the inputs most efficiently used, while containers is the most problematic one. Note the different interpretation between multiple and single factor indexes: the former measure feasible reductions in all the inputs while the later measure the maximum feasible reduction in each input.

Column EFF shows the number of efficient units for each measure. Differences are due to the different boundary sets used as references. There are 12 efficient units and 33 lie on the isoquant with slacks in the input vehicles.

The global input-specific distances are shown in Table 2. The decomposition shows the critical importance of slack distances to the efficient subset once the isoquant is reached.

Slack distances are larger than the distances to the isoquant in all the cases but labor. The input containers is the input which leads more directly to the isoquant on average. However, once the isoquant is reached the slack distances are the largest. The opposite case corresponds to the input labor which is the one with the largest distance to the isoquant and the smallest distance to the efficient subset. If the relevant comparison set is the efficient subset, then input labor seems to be the input to be considered for management purposes, as it determines the minimum global distance (88).

Table 2.- Decomposition of input-specific distances

PATH	DISTANCE TO THE ISOQUANT (a)	SLACK DISTANCES (b)	GLOBAL INPUT DISTANCE (a+b)
CONTAINERS	48	82	130
VEHICLES	50	72	122
LABOR	51	37	88

To allow for P different paths for reduction, we solved program (13) for P=2,3. The results are shown in Table 3 (numbers in brackets correspond to the number of units for which the smallest input specific distance is in the that input) . The inclusion of additional paths for contraction reduces the global distance to the efficient subset, but only when we include a second input (vehicles). However the difference from the case of focusing only in one input is not significant. Note than in this particular case no gains are obtained by allowing for a third path for contraction (no inneficient unit has the smallest input specific distance in input containers).

Table 3.- Global distance when allowing for P paths for contraction

	P=2	P=3
GLOBAL DISTANCE	87	87
INPUTS INVOLVED	Labor (74)	Labor (74)
	Vehicles (3)	Vehicles (3)
		Containers (0)

Our results would recommend to focus mainly in how units can make a better use of the input Labor. No general indication about what units are doing wrong can be inferred from these results. However, as this is the input in which the majority of the inefficient firms are closest to the efficient subset we believe that improvements in the use of labor would translate into larger efficiency gains than improvements in the use of the other two inputs.

SUMMARY

This paper proposes a model to measure the shortest distance to the efficient subset. For each input k we first compute the distance to the isoquant along that input axis and then we add the remaining slacks that lead to the efficient subset. We refer to this sum as the k^{th} input-specific distance. The smallest input-specific distance reveals the most similar set of efficient units that serve as reference for the inefficient unit evaluated. This information can be incorporated into management programs in order to advise about which efficient units should each inefficient unit visit. The management program can also use this information at an aggregate level to orient the training of technicians in the fields more related with the inputs in which inefficient firms are closest to the efficient subset. This way of designing the program can result in larger gains in efficiency than otherwise.

The model is applied to a sample of garbage collection units. The Debreu-Farrell and the Russell indexes of technical efficiency present means of 72% and 59%, respectively. The input-specific distance model suggest that labor is the input that determines the smallest aggregate distance to the efficient subset. The result would recommend training technicians in fields related with the efficient use of the input labor.

Notes

1. This radial notion of similarity has been analyzed by Day, Lewin and Li (1995) to identify strategic groups in an industry. Using standard DEA, a firm is assigned to the strategic group defined by the firms in its comparison group (where inputs are strategies and output is a measure of performance).

2. For simplicity in the definitions, we will always assume that all the components of the IOVs are strictly positive.

3. We use the standard notation $\hat{\mathbf{x}} \leq \mathbf{x}$ to denote that $\hat{x}_n \leq x_n,\, n = 1 \cdots N \wedge \hat{\mathbf{x}} \neq \mathbf{x}$.

4. More elaborated transformations of these measures are discussed in Zieschang (1984) and Rusell (1985).

5. Note that the distance to any point on the efficient subset between F and G is bigger than DE+EF because the slope on that interval is bigger than one (in absolute value).

6. A formal proof is provided in the appendix.

7. See Färe, Grosskopf and Lovell (1994) for a discussion of this technique.

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APPENDIX

In this appendix we prove that, under the assumption of convexity of $L(\mathbf{u})$, the smallest distance to the efficient subset is the smallest input-specific distance. From Figure 2, it is clear that the smallest distance from A to the efficient subset must be the distance from A to a reference point in the part of the efficient subset below the segment CB, where C and B are the reference points for the measurement of the two input-specific distances. This is the reason for introducing the constraints $q_n \leq 1\ \forall n$, in the definition of the distance function in (6). In the general case of N inputs, and N reference points ($\mathbf{x}^1$,...,$\mathbf{x}^N$) one for each input-specific distance, the smallest distance must be the distance to a point in the part of the efficient subset below the convex hull of ($\mathbf{x}^1$,...,$\mathbf{x}^N$). Note that this must be true if $L(\mathbf{u})$ is a convex set, because in this case the convex hull of ($\mathbf{x}^1$,...,$\mathbf{x}^N$), CB in Figure 2, belongs to $L(\mathbf{u})$ and therefore the efficient subset lies below (or on) it.

Thus, it will suffice to prove that the distance from the input vector ($\mathbf{x}^0$), to any point on the convex hull of ($\mathbf{x}^1$,...,$\mathbf{x}^N$) is bigger or equal to one of the input-specific distances. Let $\mathbf{e}^T$ be a column vector containing N ones. The i^{th} specific input distance would be:

$$D^i = (\mathbf{x^0} - \mathbf{x^i}) \cdot \mathbf{e^T}$$

Take any arbitrary point $\tilde{\mathbf{x}}$ on the convex hull of ($\mathbf{x}^1$,...,$\mathbf{x}^N$):

$$\tilde{\mathbf{x}} = \sum_{j=1}^{N} a_j \cdot \mathbf{x^j} \quad s.t. \quad \sum_{j=1}^{N} a_j = 1 \wedge a_j \geq 0\ \forall j$$

The distance from $\mathbf{x^0}$ to $\tilde{\mathbf{x}}$ is:

$$\tilde{D} = (\mathbf{x^0} - \sum_{j=1}^{N} a_j \cdot \mathbf{x^j}) \cdot \mathbf{e^T}$$

Claim: $\widetilde{D} \geq D^i$ for some i=1...N.

Proof: The proof is by contradiction. Suppose $\widetilde{D} < D^i \quad \forall i$. This implies:

$$(\mathbf{x^0} - \sum_{j=1}^{N} a_j \cdot \mathbf{x^j}) \cdot \mathbf{e^T} < (\mathbf{x^0} - \mathbf{x^i}) \cdot \mathbf{e^T} \qquad \forall i$$

or, simplifying:

$$\mathbf{x^i} \cdot \mathbf{e^T} < \sum_{j=1}^{N} a_j \cdot \mathbf{x^j} \cdot \mathbf{e^T} \qquad \forall i$$

Multiplying both sides by a_i and adding up over i, we get:

$$\sum_{i=1}^{N} a_i \cdot \mathbf{x^i} \cdot \mathbf{e^T} < \sum_{i=1}^{N} a_i \cdot \sum_{j=1}^{N} a_j \cdot \mathbf{x^j} \cdot \mathbf{e^T}$$

and this implies $\sum a_i > 1$, which is a contradiction, because as we know $\sum a_i = 1$. This completes the proof